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RESEARCH ARTICLE

OCTAGONAL – PENTAGONAL NUMBERS AND AN INTERESTING PUZZLE

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ABSTRACT

The study of polygonal numbers has been a fascinating topic of research and recreation for more than two millennia. Several mathematicians had investigated and proved exceptional results regarding these numbers. In this paper, I will formally introduce the polygonal numbers of order k, from which we get pentagonal and octagonal numbers. The main purpose of this paper is to determine positive integers which are pentagonal as well as octagonal. Finally, I pose an interesting puzzle whose solution depends on the calculations pertaining to determining octagonal – pentagonal numbers.

Keywords: Polygonal Numbers, Pentagonal Numbers, Octagonal Numbers, Quadratic Diophantine Equation, Recursive Equations

1. Introduction

The followers of Pythagoras known as Pythagoreans were probably the first to unite the concept of geometry with numbers. In particular, they discovered several numerical relationships by arranging numbers which forms polygons of different sorts. Such polygonal arrangement of numbers was traditionally called as Polygonal Numbers. In this paper, I will define the polygonal numbers of order k and determine the set of all positive integers which are both pentagonal and octagonal. Using these calculations, I will solve an interesting puzzle.

2. Definitions

The polygonal numbers of order *k* for $k \geq 3$ are defined by $P_k(n) = \frac{n}{2}\left[(k-2)n-(k-4)\right]$ (1)

For *k* = 5, we obtain pentagonal numbers which are given by $P_5(n) = \frac{n(3n-1)}{2}$ (2)

For *k* = 8, we obtain octagonal numbers which are given by $P_8(n) = n(3n-2)$ (3)

3. Describing the Problem

The chief objective of this paper is to determine all positive integers which are simultaneously pentagonal and octagonal. From (2) and (3), we need to find positive integers *n*, *m* such that $P_5(n) = P_8(m)$.

That is, for some positive integers *n*, *m* we require $\frac{n(3n-1)}{2} = m(3m-2)$ (4)

Multiplying both sides by 24 and simplifying we get

$$36n^2 - 24n = 72m^2 - 48m \Rightarrow (6n-1)^2 = 1 + 8(3m-1)^2 - 8$$

$$(6n-1)^2 - 2\left[2(3m-1)^2\right] = -7$$

$$x^2 - 2y^2 = -7 \quad (5); \qquad x = 6n-1,\ y = 2(3m-1)$$

Equation (5) is called Quadratic Diophantine Equation. We now need to solve equation (5).

To do this, let us first solve the equation $u^2 - 2v^2 = 1$ (6)

4. Solving the Quadratic Diophantine Equation

To solve the equation $u^2 - 2v^2 = 1$ we consider the following computations

$$(1-\sqrt{2}) \times (1+\sqrt{2}) = -1 \Rightarrow 1-\sqrt{2} = \frac{-1}{1+\sqrt{2}} = \frac{-1}{2-(1-\sqrt{2})} = \frac{-1}{2+\cfrac{1}{2-(1-\sqrt{2})}}$$

$$1-\sqrt{2} = \frac{-1}{2+\cfrac{1}{2+\cfrac{1}{2-(1-\sqrt{2})}}} = \cdots = \frac{-1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}}$$

$$\sqrt{2} = 1 + \frac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}} \qquad (7)$$

If we now consider the successive convergents from the continued fraction expansion (7), then we get $\frac{1}{1},\frac{3}{2},\frac{7}{5},\frac{17}{12},\frac{41}{29},\frac{99}{70},\frac{239}{169},\frac{577}{408},\frac{1393}{985},\frac{3363}{2378},\ldots$ (8)

If we consider second, fourth, sixth, eighth, tenth, in general even indexed convergents from (8), and considering their numerators as *u*, denominators as *v*, we notice that $(u,v)=(3,2);(17,12);(99,70);(577,408);(3363,2378);(19601,13860);(114243,80782);$ (665857, 470832);(3880899,2744210); . . . (9) are solutions to (6) namely $u^2-2v^2=1$. We also note that $(u,v)=(1,0)$ is a trivial solution to $u^2-2v^2=1$. We notice that but for the first two solutions, for any $k\geq 0$, all other subsequent solutions of $u^2-2v^2=1$ were given by the recursive equations

$$\left.\begin{aligned} u_{k+2} &= 6u_{k+1}-u_k \\ v_{k+2} &= 6v_{k+1}-v_k \end{aligned}\right\} \quad (10)$$

Now the solutions of (5) were given by $x=u+4v, y=2u+v$ (11), were (u,v) are solutions to (6). Thus using (9) (including the trivial solution) in (11), we see that the solutions of (5) namely $x^2-2y^2=-7$ are given by $(x,y)=(1,2);(11,8);(65,46);(379,268);(2209,1562);(12875,9104);(75041,53062);$(2549185, 1802546); (14857739, 10506008); . . . (12)

The other possible solutions to (5) were given by $x=|-u+4v|, y=|2u-v|$ (13). Thus using (9) in (13), we see that the solutions of (5) namely $x^2-2y^2=-7$ are given by

$(x,y)=(5,4);(31,22);(181,128);(1055,746);$(6149,4348);(35839,25342);(208885,147704); (1217471,860882); (7095941,5017588); . . . (14)

Thus the complete solutions to $x^2-2y^2=-7$ were given by the union of ordered pairs described by the collections (12) and (14). In fact, but for the two initial solutions in (12) and (14), for any $k\geq 0$ each subsequent solution can be obtained by the recursive equations

$$\left.\begin{aligned} x_{k+2} &= 6x_{k+1}-x_k \\ y_{k+2} &= 6y_{k+1}-y_k \end{aligned}\right\} \quad (15)$$

5. Octagonal – Pentagonal Numbers

While deriving (5), we observed that $n=\frac{x+1}{6}, m=\frac{y+2}{6}$ (16). Thus, using (12) and (14), we can list the numbers which are pentagonal as well as octagonal. In doing so, we find that only for the ordered pairs (*x*, *y*) = (5,4); (65,46); (6149, 4348); (75041,53062); (7095941, 5017588); … , we get the integer values of *n*, *m* given by (*n*, *m*) = (1, 1), (11, 8), (1025, 725); (12507, 8844); (1182657, 836265); . . . (17).

Hence, the positive integers which are both octagonal and pentagonal are given by 1, 176, 1575425, 234631320, 2098015778145, . . . (18)

6. Posing and Solving the Puzzle

In this section, I will pose an interesting puzzle and using the ideas described above, I will solve it. The puzzle is the following:

"Find all positive integers such that if one is added to its twice we get a perfect square, as well as, if four is added to it we still get a perfect square." (19)

To solve the puzzle as described in (19), if we assume that *N* is such a number, then we must have $2N+1=x^2, N+4=y^2 \quad (20)$.

Eliminating *N* in the two equations from (20), we obtain $x^2-2y^2=-7$ which is precisely equation (5) derived in section 3. The ordered pairs in (12) and (14) provide the solutions to $x^2-2y^2=-7$. Moreover from (20), we have $N=\dfrac{x^2-1}{2}=y^2-4$.

Hence using (12) and (14) except the order pair (1,2), the solutions to the given puzzle are given by the numbers 12,60,480,2112,16380,71820,556512,2439840,18905100,82882812, . . . (21)

7. Conclusion

The primary objective of this paper is to determine all positive integers which are both pentagonal and octagonal. These being special cases of polygonal numbers, I had introduced polygonal numbers in (1). If we consider values of *k* as 5 and 8, then we get pentagonal and octagonal numbers respectively. In section 3, equating these two types of numbers we arrived at the Quadratic Diophantine Equation $x^2-2y^2=-7$ as described in (5).

In order to solve (5), I considered another Quadratic Diophantine Equation $u^2-2v^2=1$ as in (6). The solutions of (6) were arrived by considering the even indexed convergents of the continued fraction in (8). In fact all the possible solutions of (6) were described in (9). Using the transformed equations (11) and (13), I had obtained all possible solutions of (5) as mentioned in (12) and (14). All subsequent solutions except the first two in (9), (12) and (14) can be obtained through similar recursive equations mentioned in (10) and (15). Thus, using the ordered pairs from (12) and (14), I had met the primary objective of determining all possible positive integers which are both pentagonal and octagonal as described in (18).

Finally, an interesting puzzle is posed through (19), whose solution has direct connection with that of solutions of (5), which were already known through (12) and (14). Using this information, I had described the numbers in (21), forming the solutions to the puzzle. Thus in this paper, I had obtained numbers which are both pentagonal and octagonal and using this information, I had solved an interesting puzzle.

REFERENCES

[1]. Isabella G. Bashmakova, Diophantus and Diophantine Equations, The Mathematical Association of America, 1998.

[2]. Sprindzuk Vladimir G., Classical Diophantine Equations, Springer-Verlag Berlin and Heidelberg GmbH & Company, 1993.

[3]. Wolfgang M. Schmidt, Diophantine Approximations and Diophantine Equations, Springer; 1996,

[4]. R. Sivaraman, Recognizing Ramanujan's House Number Puzzle, German International Journal of Modern Science, 22, November 2021, pp. 25 – 27.

[5]. R. Sivaraman, On Solving Brahmagupta's Puzzle, International Journal of Scientific Research and Modern Education, Volume 7, Issue 1, 2022.

[6]. R. Sivaraman, Square – Pentagonal Numbers, International Journal of Multidisciplinary Research and Modern Education, Volume 8, Issue 1, 2022.

[7]. R. Sivaraman, Summing Through Integrals, Science Technology and Development, Volume IX, Issue IV, April 2020, pp. 267 – 272.