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RESEARCH ARTICLE

SUM OF TWO SIXTH POWERS EQUAL TO DIFFERENCE OF TWO FOURTH POWERS

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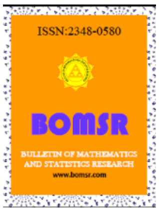

ABSTRACT

In this paper we examine the diophantine equation $a^6 + b^6 = c^4 - d^4$. We arrive at a parametric solution by algebraic means and also obtain a solution by elliptic curve method. We also provide solutions for the equation $\boldsymbol{a}^n + \boldsymbol{b}^n = \boldsymbol{c}^4 - \boldsymbol{d}^4$, where $\boldsymbol{n}$ is positive integer non-divisible by $\mathbf{4}$. Note that Izadi [1] has given solutions to a similar diophantine equation, where difference of two sixth powers equals to difference of two cubes, obtained by elliptic curve method.

1. Parametric solution to $a^6 + b^6 = c^4 - d^4$

Assuming in the equation

$$a^6 + b^6 = c^4 - d^4 \tag{1}$$

$c = x + y$ and $d = x - y$ we get

$$c^4 - d^4 = 8x^3y + 8xy^3. \tag{2}$$

Attempting to match right hand side summands of the equation (2) to a^6 and b^6 respectively, we arrive at

$$a^6 = 8x^3y \qquad \text{and} \qquad b^6 = 8xy^3, . \tag{3}$$

which is satisfied e.g. by

$$x = 8p^6 \qquad \text{and} \qquad y = q^6. \tag{4}$$

This produces the following parametric solution of the equation (1):

a = 4p3q, b = 2pq3, c = 8p6 + q6 and d = 8p6 − q6. (5)

Note that the solution (5) satisfies additional condition $GCD(a, b, c, d) = 1$ whenever q is odd and p, q are relatively prime.

To obtain a numerical example we take $p = 2$ and $q = 1$, which produces (6)

$$a = 32,\ b = 4,\ \ c = 513 \text{ and } \ d = 511. \tag{6}$$

2. Parametric solution to $a^n + b^n = c^4 - d^4$

Let n be positive integer non-divisible by 4. By modifying the equations (3) we get

$$a^n = 8x^3y \quad \text{and} \quad b^n = 8xy^3 \tag{7}$$

This will be satisfied by

$$x = 2w^p n \quad \text{and} \quad y = 2v^q n \tag{8}$$

provided $3 + 3w + v$ and $3 + w + 3v$ are both divisible by n. Assuming more general case $n = 2k$ with odd k we find a solution

$$w = \frac{7k-3}{4} \quad \text{and} \quad v = \frac{3k-3}{4} \qquad case\ k \equiv 1 \text{ (mod 4)} \tag{9}$$

$$w = \frac{5k-3}{4} \quad \text{and} \quad v = \frac{k-3}{4} \qquad case\ k \equiv 3 \text{ (mod 4)} \tag{10}$$

In case $n = 10$ we have $k = 5$ and $w = 8, v = 3$, which leads to

$$x = 256p^{10} \quad \text{and} \quad y = 8q^{10} \tag{11}$$

and

$$a = 8p^3q,\quad b = 4pq^3,\quad c = 256p^{10}+8q^{10} \quad \text{and} \quad d = 256p^{10}-8q^{10}. \tag{12}$$

In case n = 14 we have k = 7 and w = 8, v = 1, which leads to

$$x = 256p^{14} \quad \text{and} \quad y = 2q^{14} \tag{13}$$

and

$$a = 4p^3q,\quad b = 2pq^3,\quad c = 256p^{14} + 2q^{14} \ \text{and}\ d = 256p^{14} - 2q^{14}\ . \tag{14}$$

In case $n = 2022$ we have $k = 1011$ and $w = 1263,\ v = 252$, which leads

to

$$x = 2^{1263}\,p^{2022} \quad \text{and} \quad y = 2^{252}q^{2022} \tag{15}$$

and

$$a = 4p^3q,\ \ b = 2pq^3,\ \ c = 2^{1263}\,p^{2022} + 2^{252}q^{2022},\ \ d = 2^{1263}\,p^{2022} - 2^{252}q^{2022}. \tag{16}$$

3. Solution to $a^6 + b^6 = c^4 - d^4$ with $a = 2b$

While examining numerical solutions to the equation (1) we have noticed quite a few solutions satisfying additional equation

$$a = 2b. \tag{17}$$

We have also observed that those numerical solutions satisfied condition GCD(a, b, c, d) = b. Therefore we assumed

$$a = 2x,\ b = x,\ \ c = sx \quad and \quad d = tx. \tag{18}$$

This leads to

$$(2x)^6 + x^6 = (sx)^4 - (tx)^4 \tag{19}$$

which simplifies to the form

$$\mathrm{s}^4 - \mathrm{t}^4 = 65x^2. \tag{20}$$

Taking $t = 2$ we get

$$\mathrm{s}^4 - 65x^2 = 16 \tag{21}$$

Equation (21) can be reduced to the quartic curve

$$v^4 = 65\mathrm{w}^4 - 1040, \tag{22}$$

which is bi-rationally equivalent to the elliptic curve

$$y^2 = x^3 + 16900x. \tag{23}$$

Elliptic curve (23) has infinitely many rational points (s, x), e.g. (274/7, 9312/(7)2)

$(59/29, 111/(29)^2)$

$(13794/3761, 22533056/(3761)^2)$

$(5839979/232571, 4230163499697/(232571)^2)$

For $(s, x) = (274/7, 9312/7^2)$ we get new numerical solution:

$(a, b, c, d) = (2x, x, sx, tx)$

$(a, b, c, d) = (18624, 9312, 2551488, 130368)$.

And $for\ (s, x) = (13794/3761, 22533056/3761^2)$

we get new numerical solution:

$(a, b, c, d) = (45066112, 22533056, 310820974464, 169493647232)$.

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