Vol.10.Issue.4.2022 (Oct-Dec)
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RESEARCH ARTICLE

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

INTERNATIONAL STANDARD SERIAL NUMBER 2348-0580

POSITIVE INTEGER SOLUTIONS OF PELL EQUATIONS $x^2 - Cy^2 = \pm 1$ VIA GENERALIZED BI-PERIODIC FIBONACCI AND LUCAS SEQUENCES FOR THE CHOICES OF $C = m^2 \pm 4$

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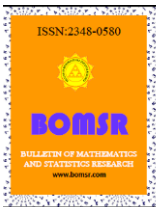
DOI:10.33329/bomsr.10.4.82



ABSTRACT

Let C be a non-perfect square positive-integer and $C = m^2 \pm 4$. The basic solution of the Pell equation is found in the present article $x^2 - Cy^2 = \pm 1$ by using Continued fraction expansion of $\sqrt{C}$. Also, in terms of Generalized Bi-Periodic Fibonacci & Lucas sequences, we obtain all positive-integer solutions of the Pell equation $x^2 - Cy^2 = \pm 1$.

Keywords: Continued fraction, Pell equations, Generalized Bi-Periodic Fibonacci and Lucas sequences.

2010 Mathematics Subject Classifications: 11A55, 11B39, 11D55, 11D09, 11J70.

1 Introduction

It is generally recognized that the Pell equation $x^2 - Cy^2 = 1$ always have positive-integer solutions, where C is a positive integer which is not a perfect square. When N is not equal to 1, there may be no positive-integer solution for $x^2 - Cy^2 = N$ Pell equation. The positive-integer solution for $x^2 - Cy^2 = -1$ equation depends on the period length of $\sqrt{C}$ continued fraction expansion. In [10], we gave all positive-integer solutions of the Pell equation $x^2 - Cy^2 = \pm 1$ in terms of Generalized Bi-Periodic Fibonacci and Lucas sequences for the choices $C = m^2 \pm 1, m^2 \pm 2, m^2 \pm m$. In the present article, when m is a positive integer as well as $C = m^2 \pm 4$, particularly if a solution is available, all positive integer solutions are provided in terms of Generalized Bi-Periodic Fibonacci and Lucas sequences by utilizing $\sqrt{C}$ continued fraction expansion.

2 Preliminaries

Some writers have generalized the sequences, Fibonacci & Lucas, by altering their initial conditions and recurring relations. Yayenie & Edson ([4]) generalize the Fibonacci sequence to the new set of sequences denoted as $\{p_n\}$ and is defined by

$$p_0 = 0, p_1 = 1,\ p_n = \begin{cases} ap_{n-1} + p_{n-2}, & if\ n\ is\ even \\ bp_{n-1} + p_{n-2}, & if\ n\ is\ odd \end{cases} \quad (n \geq 2).$$

Bilgici ([6]), on contrary, generalized the Lucas sequence by presenting a bi-periodic Lucas sequence denoted as $\{l_n\}$ and is expressed as:

$$l_0 = 2, l_1 = a,\ l_n = \begin{cases} bl_{n-1} + l_{n-2}, & if\ n\ is\ even \\ al_{n-1} + l_{n-2}, & if\ n\ is\ odd \end{cases} \quad (n \geq 2).$$

as well as several interesting associations between $\{p_n\}$ & $\{l_n\}$ have been proven.

We now consider a generalized bi-periodic Fibonacci sequence $\{f_n\}$ and Lucas sequence $\{q_n\}$ which are the generalization of $\{p_n\}$ and $\{l_n\}$, termed as:

$$f_0 = 0, f_1 = 1,\ f_n = \begin{cases} af_{n-1} + cf_{n-2}, & if\ n\ is\ even \\ bf_{n-1} + cf_{n-2}, & if\ n\ is\ odd \end{cases} \quad (n \geq 2),$$

and

$$q_0 = 2d, q_1 = ad,\ q_n = \begin{cases} bq_{n-1} + cq_{n-2}, & if\ n\ is\ even \\ aq_{n-1} + cq_{n-2}, & if\ n\ is\ odd \end{cases} \quad (n \geq 2),$$

where a, b, c, d are nonzero real numbers.

Yayenie and Choo ([4] and [5]) gave Binet's formulas for $\{f_n\}$ & $\{q_n\}$ are represented as:

$$f_n(a,b,c) = \frac{a^{\zeta(n+1)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) \tag{1}$$

$$q_n(a,b,c,d) = \frac{d}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor} b^{\zeta(n)}} (\alpha^n + \beta^n) \tag{2}$$

where $\alpha = \frac{ab+\sqrt{a^2b^2+4abc}}{2}$ and $\beta = \frac{ab-\sqrt{a^2b^2+4abc}}{2}$, i.e., α and β are the roots of the equation $x^2 - abx - abc = 0$, and $\zeta(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function such that

$$\zeta(n) = \begin{cases} 0\ if\ n\ is\ even \\ 1\ if\ n\ is\ odd \end{cases}$$

We now provide the fundamental solution to an equation $x^2 - Cy^2 = \pm 1$ utilizing the length of a period of $\sqrt{C}$ continued fraction expansion.

Lemma 2.1: Suppose l be the period length of $\sqrt{C}$ continued fraction expansion. When l is even, then the fundamental solution for $x^2 - Cy^2 = 1$ equation is represented as:

$$x_1 + y_1\sqrt{C} = p_{l-1} + q_{l-1}\sqrt{C}$$

and $x^2 - Cy^2 = -1$ equation has no integer solution. In case of l is odd, then the fundamental solution for $x^2 - Cy^2 = 1$ equation is represented as:

$$x_1 + y_1\sqrt{C} = p_{2l-1} + q_{2l-1}\sqrt{C}$$

and fundamental solution for $x^2 - Cy^2 = -1$ equation is represented as:

$$x_1 + y_1\sqrt{C} = p_{l-1} + q_{l-1}\sqrt{C}$$

Cognition 2.1 Let $x_1 + y_1\sqrt{C}$ be the fundamental solution of $x^2 - Cy^2 = 1$ equation. Then all positive-integer solutions of $x^2 - Cy^2 = 1$ equation is represented as:

$$x_n + y_n\sqrt{C} = \left(x_1 + y_1\sqrt{C}\right)^n$$

with $n \geq 1$.

Cognition 2.2 Let $x_1 + y_1\sqrt{C}$ be the fundamental solution of $x^2 - Cy^2 = -1$. Then all positive-integer solutions for $x^2 - Cy^2 = -1$ are represented as:

$$x_n + y_n\sqrt{C} = \left(x_1 + y_1\sqrt{C}\right)^{2n-1}$$

with $n \geq 1$.

Cognition 2.3 Let $C = m^2 \pm 4$. Then $\sqrt{C}$ continued fraction expansion is given by

$$\sqrt{C} = \begin{cases} \left[m; \overline{\frac{m}{2}, 2m}\right] & \text{if C} = m^2 + 4 \ and\ m\ is\ even\ with\ m \geq 1 \\ \left[m; \overline{\frac{m-1}{2}, 1,1, \frac{m-1}{2}, 2m}\right] & \text{if C} = m^2 + 4\ and\ m\ is\ odd\ with\ m \geq 1 \\ \left[m-1;\ \overline{1, \frac{m-3}{2}, 2, \frac{m-3}{2}, 1,2(m-1)}\right] & \text{if C} = m^2 - 4\ and\ m\ is\ odd\ with\ m \geq 3 \\ \left[m-1;\ \overline{1, \frac{m-4}{2}, 1,2(m-1)}\right] & \text{if C} = m^2 - 4\ and\ m\ is\ even\ with\ m > 2, m \neq 4 \\ [3;\ \overline{2,6}] & if\ m = 4 \end{cases}$$

Corollary 2.1 Let $C = m^2 \pm 4$. The basic solution of $x^2 - Cy^2 = 1$ equation is represented as:

$$x_1 + y_1\sqrt{C} = \begin{cases} \frac{m^6 + 6m^4 + 9m^2 + 2}{2} + \frac{m^5 + 4m^3 + 3m}{2}\sqrt{C} & if\ C = m^2 + 4\ and\ \ m\ is\ odd \\ \frac{m^2 + 2}{2} + \frac{m}{2}\sqrt{C} & if\ C = m^2 + 4\ and\ \ m\ is\ even \\ \frac{m^3 - 3m}{2} + \frac{m^2 - 1}{2}\sqrt{C} & if\ C = m^2 - 4\ and\ \ m\ is\ odd \\ \frac{m^2 - 2}{2} + \frac{m}{2}\sqrt{C} & if\ C = m^2 - 4\ and\ \ m\ is\ even \end{cases}$$

Corollary 2.2 Let $m > 0$ and $C = m^2 \pm 4$. The basic solution of $x^2 - Cy^2 = -1$ is

$$x_1 + y_1\sqrt{C} = \begin{cases} \frac{m^3+3m}{2} + \frac{m^2+1}{2}\sqrt{C} & if\ C = m^2 + 4\ and\ m\ is\ odd \\ nosolution & if\ C = m^2 + 4\ and\ m\ is\ even \\ \quad nosolution & if\ C = m^2 - 4\ and\ m > 3 \end{cases}$$

3 Main Theorems

Theorem 3.1 Let $m > 1$ and $C = m^2 + 4$. Then all positive-integer solutions of the equation $x^2 - Cy^2 = 1$ are given by

$$(x_n, y_n) = \begin{cases} \begin{cases} \frac{1}{2} m^{\lfloor 3n \rfloor}\left(q_{6n}\left(m, 1, \frac{1}{m}, 1\right), \frac{1}{m} f_{6n}\left(m, 1, \frac{1}{m}\right)\right) \\ \quad or \\ \frac{1}{2} m^{\lfloor 3n \rfloor}\left(q_{6n}\left(1, m, \frac{1}{m}, 1\right), f_{6n}\left(1, m, \frac{1}{m}\right)\right) \end{cases} & if\ m\ is\ odd \\ \begin{cases} \frac{1}{2} m^{\lfloor n \rfloor}\left(q_{2n}\left(m, 1, \frac{1}{m}, 1\right), \frac{1}{m} f_{2n}\left(m, 1, \frac{1}{m}\right)\right) \\ \quad or \\ \frac{1}{2} m^{\lfloor n \rfloor}\left(q_{2n}\left(1, m, \frac{1}{m}, 1\right), f_{2n}\left(1, m, \frac{1}{m}\right)\right) \end{cases} & if\ m\ is\ even \end{cases}$$

with $n \geq 1$.

Proof

Case I

Let m is odd.

By Corollary 2.1, Cognition 2.1, and Cognition 2.3, all positive integer solutions of the equation $x^2 - Cy^2 = 1$ are given by

$$x_n + y_n\sqrt{C} = \left(\frac{m^6 + 6m^4 + 9m^2 + 2}{2} + \frac{m^5 + 4m^3 + 3m}{2}\sqrt{C}\right)^n$$

with $n \geq 1$. Let $\alpha_1 = \frac{(m^3+3m)^2+2}{2} + \frac{m\left((m^2+2)^2-1\right)}{2}\sqrt{C}$ and $\beta_1 = \frac{(m^3+3m)^2+2}{2} - \frac{m\left((m^2+2)^2-1\right)}{2}\sqrt{C}$.

Then,

$$x_n + y_n\sqrt{C} = {\alpha_1}^n \text{ and } x_n - y_n\sqrt{C} = {\beta_1}^n$$

Thus, it follows that

$$x_n = \frac{{\alpha_1}^n + {\beta_1}^n}{2} \text{ and } y_n = \frac{{\alpha_1}^n - {\beta_1}^n}{2\sqrt{C}}$$

Let

$$\alpha = \frac{ab + \sqrt{a^2b^2 + 4abc}}{2} \text{ and } \beta = \frac{ab - \sqrt{a^2b^2 + 4abc}}{2}$$

Sub Case (i)

Take $a = m, b = 1, c = \frac{1}{m}$, we get

$$\alpha = \frac{m + \sqrt{m^2 + 4}}{2} \text{ and } \beta = \frac{m - \sqrt{m^2 + 4}}{2}$$

Thus, $\alpha^6 = \alpha_1$ and $\beta^6 = \beta_1$

Therefore, we get,

$$x_n = \frac{(\alpha^6)^n + (\beta^6)^n}{2} = 2^{-1}(\alpha^{6n} + \beta^{6n}) = 2^{-1}m^{\lfloor 3n \rfloor} q_{6n}\left(m, 1, \frac{1}{m}, 1\right) \quad \text{by (2)}$$

and

$$y_n = \frac{(\alpha^6)^n - (\beta^6)^n}{2\sqrt{m^2+4}} = 2^{-1}\frac{\alpha^{6n} - \beta^{6n}}{\alpha - \beta} = 2^{-1}\frac{m^{\lfloor 3n \rfloor}}{m} f_{6n}\left(m, 1, \frac{1}{m}\right) \quad \text{by (1)}$$

Thus,

$$(x_n, y_n) = \left(\frac{1}{2}m^{\lfloor n \rfloor} q_{6n}\left(m, 1, \frac{1}{m}, 1\right), \frac{1}{2m}m^{\lfloor 3n \rfloor} f_{6n}\left(m, 1, \frac{1}{m}\right)\right)$$

Sub Case (ii)

Take, $a = 1, b = m, c = \frac{1}{m}$, we get

$$\alpha = \frac{m + \sqrt{m^2+4}}{2} \text{ and } \beta = \frac{m - \sqrt{m^2+4}}{2}$$

Thus, $\alpha^6 = \alpha_1$ and $\beta^6 = \beta_1$

Therefore, we get,

$$x_n = \frac{(\alpha^6)^n + (\beta^6)^n}{2} = 2^{-1}(\alpha^{6n} + \beta^{6n}) = 2^{-1}m^{\lfloor 3n \rfloor} q_{6n}\left(1, m, \frac{1}{m}, 1\right) \quad \text{by (2)}$$

and

$$y_n = \frac{(\alpha^6)^n - (\beta^6)^n}{2\sqrt{m^2+1}} = 2^{-1}\frac{\alpha^{6n} - \beta^{6n}}{\alpha - \beta} = 2^{-1}m^{\lfloor 3n \rfloor} f_{6n}\left(1, m, \frac{1}{m}\right) \quad \text{by (1)}$$

Thus,

$$(x_n, y_n) = \left(\frac{1}{2}m^{\lfloor 3n \rfloor} q_{6n}\left(1, m, \frac{1}{m}, 1\right), \frac{1}{2}m^{\lfloor 3n \rfloor} f_{6n}\left(1, m, \frac{1}{m}\right)\right)$$

Case II

Let m is even.

By Corollary 2.1, Cognition 2.1, and Cognition 2.3, all positive integer solutions of the equation $x^2 - Cy^2 = 1$ are given by

$$x_n + y_n\sqrt{C} = \left(\frac{m^2+2}{2} + \frac{m}{2}\sqrt{C}\right)^n$$

with $n \geq 1$. Let $\alpha_1 = \frac{m^2+2}{2} + \frac{m}{2}\sqrt{C}$ and $\beta_1 = \frac{m^2+2}{2} - \frac{m}{2}\sqrt{C}$.

Then,

$$x_n + y_n\sqrt{C} = {\alpha_1}^n \text{ and } x_n - y_n\sqrt{C} = {\beta_1}^n$$

Thus, it follows that

$$x_n = \frac{{\alpha_1}^n + {\beta_1}^n}{2} \text{ and } y_n = \frac{{\alpha_1}^n - {\beta_1}^n}{2\sqrt{C}}$$

Let

$$\alpha = \frac{ab + \sqrt{a^2b^2 + 4abc}}{2} \text{ and } \beta = \frac{ab - \sqrt{a^2b^2 + 4abc}}{2}$$

Sub Case (iii)

Take $a = m, b = 1, c = \frac{1}{m}$, we get

$$\alpha = \frac{m + \sqrt{m^2 + 4}}{2} \text{ and } \beta = \frac{m - \sqrt{m^2 + 4}}{2}$$

Thus, $\alpha^2 = \alpha_1$ and $\beta^2 = \beta_1$

Therefore, we get,

$$x_n = \frac{(\alpha^2)^n + (\beta^2)^n}{2} = 2^{-1}(\alpha^{2n} + \beta^{2n}) = 2^{-1}m^{\lfloor n \rfloor}q_{2n}\left(m, 1, \frac{1}{m}, 1\right) \quad \text{by (2)}$$

and

$$y_n = \frac{(\alpha^2)^n - (\beta^2)^n}{2\sqrt{m^2 + 4}} = 2^{-1}\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} = 2^{-1}\frac{m^{\lfloor n \rfloor}}{m}f_{2n}\left(m, 1, \frac{1}{m}\right) \quad \text{by (1)}$$

Thus,

$$(x_n, y_n) = \left(\frac{1}{2}m^{\lfloor n \rfloor}q_{2n}\left(m, 1, \frac{1}{m}, 1\right), \frac{1}{2m}m^{\lfloor n \rfloor}f_{2n}\left(m, 1, \frac{1}{m}\right)\right)$$

Sub Case (iv)

Take, $a = 1, b = m, c = \frac{1}{m}$, we get

$$\alpha = \frac{m + \sqrt{m^2 + 4}}{2} \text{ and } \beta = \frac{m - \sqrt{m^2 + 4}}{2}$$

Thus, $\alpha^2 = \alpha_1$ and $\beta^2 = \beta_1$

Therefore, we get,

$$x_n = \frac{(\alpha^2)^n + (\beta^2)^n}{2} = 2^{-1}(\alpha^{2n} + \beta^{2n}) = 2^{-1}m^{\lfloor n \rfloor}q_{2n}\left(1, m, \frac{1}{m}, 1\right) \quad \text{by (2)}$$

and

$$y_n = \frac{(\alpha^2)^n - (\beta^2)^n}{2\sqrt{m^2 + 4}} = 2^{-1}\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} = 2^{-1}\frac{m^{\lfloor n \rfloor}}{m}f_{2n}\left(1, m, \frac{1}{m}\right) \quad \text{by (1)}$$

Thus,

$$(x_n, y_n) = \left(\frac{1}{2}m^{\lfloor n \rfloor}q_{2n}\left(1, m, \frac{1}{m}, 1\right), \frac{1}{2}m^{\lfloor n \rfloor}f_{2n}\left(1, m, \frac{1}{m}\right)\right)$$

From Cases (I) and (II) we get the required solution.

Theorem 3.2 Let $m > 1$ and $C = m^2 + 4$. Then all positive solutions of the equation $x^2 - Cy^2 = -1$ are given by

(i) If m is odd, then

$$(x_n, y_n) = \begin{cases} \frac{1}{2} m^{\left\lfloor \frac{6n-3}{2} \right\rfloor} \left(q_{6n-3}\left(m, 1, \frac{1}{m}, 1\right), f_{6n-3}\left(m, 1, \frac{1}{m}, 1\right) \right) \\ or \\ \frac{1}{2} m^{\left\lfloor \frac{6n-3}{2} \right\rfloor} \left(q_{6n-3}\left(1, m, \frac{1}{m}, 1\right), f_{6n-3}\left(1, m, \frac{1}{m}, 1\right) \right) \end{cases}$$

with $n \geq 1$.

(ii) If m is even, then there is no solution

Proof

Case I

Let m is odd.

By Corollary 2.1, Cognition 2.2, and Cognition 2.3, all positive integer solutions of the equation $x^2 - Cy^2 = -1$ are given by

$$x_n + y_n\sqrt{C} = \left(\frac{m^3 + 3m}{2} + \frac{m^2 + 1}{2} \sqrt{C} \right)^{2n-1}$$

with $n \geq 1$. Let $\alpha_1 = \frac{m^3+3m}{2} + \frac{m^2+1}{2}\sqrt{C}$ and $\beta_1 = \frac{m^3+3m}{2} - \frac{m^2+1}{2}\sqrt{C}$.

Then,

$$x_n + y_n\sqrt{C} = {\alpha_1}^{2n-1} \text{ and } x_n - y_n\sqrt{C} = {\beta_1}^{2n-1}$$

Thus, it follows that

$$x_n = \frac{{\alpha_1}^{2n-1} + {\beta_1}^{2n-1}}{2} \text{ and } y_n = \frac{{\alpha_1}^{2n-1} - {\beta_1}^{2n-1}}{2\sqrt{C}}$$

Let

$$\alpha = \frac{ab + \sqrt{a^2b^2 + 4abc}}{2} \text{ and } \beta = \frac{ab - \sqrt{a^2b^2 + 4abc}}{2}$$

Sub Case (i)

Take $a = m, b = 1, c = \frac{1}{m}$, we get

$$\alpha = \frac{m + \sqrt{m^2 + 4}}{2} \text{ and } \beta = \frac{m - \sqrt{m^2 + 4}}{2}$$

Thus, $\alpha^3 = \alpha_1$ and $\beta^3 = \beta_1$

Therefore, we get,

$$x_n = \frac{(\alpha^3)^{2n-1} + (\beta^3)^{2n-1}}{2} = 2^{-1}(\alpha^{6n-3} + \beta^{6n-3}) = 2^{-1} m^{\left\lfloor \frac{6n-3}{2} \right\rfloor} q_{6n-3}\left(m, 1, \frac{1}{m}, 1\right) \quad \text{by (2)}$$

and

$$y_n = \frac{(\alpha^3)^{2n-1} - (\beta^3)^{2n-1}}{2\sqrt{m^2 + 4}} = 2^{-1} \frac{\alpha^{6n-3} - \beta^{6n-3}}{\alpha - \beta} = 2^{-1} m^{\left\lfloor \frac{6n-3}{2} \right\rfloor} f_{6n-3}\left(m, 1, \frac{1}{m}\right) \qquad \text{by (1)}$$

Thus,

$$(x_n, y_n) = \left(\frac{1}{2} m^{\lfloor n \rfloor} m^{\left\lfloor \frac{6n-3}{2} \right\rfloor} q_{6n-3}\left(1, m, \frac{1}{m}, 1\right), \frac{1}{2} m^{\left\lfloor \frac{6n-3}{2} \right\rfloor} f_{6n-3}\left(m, 1, \frac{1}{m}\right) \right)$$

Sub Case (ii)

Take, $a = 1, b = m, c = \frac{1}{m}$, we get

$$\alpha = \frac{m + \sqrt{m^2 + 4}}{2} \text{ and } \beta = \frac{m - \sqrt{m^2 + 4}}{2}$$

Thus, $\alpha^3 = \alpha_1$ and $\beta^3 = \beta_1$

Therefore, we get,

$$x_n = \frac{(\alpha^3)^{2n-1} + (\beta^3)^{2n-1}}{2} = 2^{-1}(\alpha^{6n-3} + \beta^{6n-3}) = 2^{-1} m^{\left\lfloor \frac{6n-3}{2} \right\rfloor} q_{6n-3}\left(1, m, \frac{1}{m}, 1\right) \quad \text{by (2)}$$

and

$$y_n = \frac{(\alpha^3)^{2n-1} - (\beta^3)^{2n-1}}{2\sqrt{m^2 + 4}} = 2^{-1} \frac{\alpha^{6n-3} - \beta^{6n-3}}{\alpha - \beta} = 2^{-1} m^{\left\lfloor \frac{6n-3}{2} \right\rfloor} f_{6n-3}\left(1, m, \frac{1}{m}\right) \quad \text{by (1)}$$

Thus,

$$(x_n, y_n) = \left(\frac{1}{2} m^{\lfloor 3n \rfloor} m^{\left\lfloor \frac{6n-3}{2} \right\rfloor} q_{6n-3}\left(1, m, \frac{1}{m}, 1\right), \frac{1}{2} m^{\left\lfloor \frac{6n-3}{2} \right\rfloor} f_{6n-3}\left(1, m, \frac{1}{m}\right)\right)$$

Case II

Let m is even.

Since by Cognition 2.3, the period length of continued fraction expansion of $\sqrt{C}$ is always even. Thus, by Lemma 2.1, it follows that there is no positive integer solution of the equation

$x^2 - Cy^2 = -1$.

From Cases (I) and (II) we get the required solution.

Now, we consider the other cases of C without giving their proof since they can be proved as similar to that of Theorem 3.1 and Theorem 3.2 was proved.

Theorem 3.3 Let $m > 0$ and $C = m^2 - 4$. Then all positive solutions of the equation

$x^2 - Cy^2 = 1$ are given by

(i) If m is even, then

$$(x_n, y_n) = \begin{cases} \frac{1}{2} m^{\lfloor n \rfloor}\left(q_{2n}\left(m, 1, \frac{-1}{m}, 1\right), f_{2n}\left(m, 1, \frac{-1}{m}\right)\right) \\ \quad or \\ \frac{1}{2} m^{\lfloor n \rfloor}\left(q_{2n}\left(1, m, \frac{-1}{m}, 1\right), f_{2n}\left(1, m, \frac{-1}{m}\right)\right) \end{cases}$$

(ii) If m is odd, then

$$(x_n, y_n) = \begin{cases} \frac{1}{2} m^{\left\lfloor \frac{3n}{2} \right\rfloor}\left(q_{3n}\left(m, 1, \frac{-1}{m}, 1\right), \frac{1}{m^{\zeta(3n+1)}} f_{3n}\left(m, 1, \frac{-1}{m}\right)\right) \\ \quad or \\ \frac{1}{2} m^{\left\lfloor \frac{3n}{2} \right\rfloor}\left(m^{\zeta(3n)} q_{2n}\left(1, m, \frac{-1}{m}, 1\right), f_{3n}\left(1, m, \frac{-1}{m}\right)\right) \end{cases}$$

with $n \geq 1$.

Theorem 3.8 Let $C = m^2 - 4$ then the equation $x^2 - Cy^2 = -1$ has no solution in positive integers.

Proof

Since by Cognition 2.3, the period length of continued fraction expansion of $\sqrt{C}$ is always even. Thus by Lemma 2.1, it follows that there is no positive integer solution of the equation

$x^2 - Cy^2 = -1.$

4 Conclusion

In this paper, we investigate the Pell equation $x^2 - Cy^2 = \pm 1, C = m^2 \pm 4$ and we are seeking positive integer solutions in x and y. We get all positive integer solutions of the Pell equations $x^2 - Cy^2 = \pm 1$ in terms of Generalized Bi-Periodic Fibonacci and Lucas sequences when $C = m^2 \pm 4$.

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