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RESEARCH ARTICLE

Sum of Three Fourth Powers a Multiple of ‘n’ Fourth Powers
$A^4 + B^4 + C^4 = nD^4$

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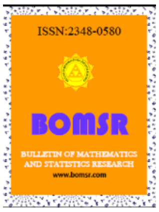

ABSTRACT
In this paper, we proved that there are infinitely many integer solutions of $A^4 + B^4 + C^4 = nD^4$.

1. Introduction

Let us mention the equation $A^4 + B^4 + C^4 = D^4$ first.

In 1986, Noam D. Elkies[1] found the first solution

$(a, b, c, d) = (2682440, 15365639, 18796760, 20615673)$.

Furthermore, he proved that there are infinitely many solutions.

After Elkies’s achievement, Roger Frye[3] found the smallest solution

$(a, b, c, d) = (95800, 217519, 414560, 422481)$ using exhaustive computer search.

The method of Elkies can be reduced to the problem of solving two quadratic diophantine equation.

Many solutions are known for $A^4 + B^4 + C^4 = D^4$, but little is known for $A^4 + B^4 + C^4 = nD^4$ with $n > 1$.

For $n = 2$, it is easy to prove that there are an infinite many integer of solutions.

As far as we know, little is known about the existence of a solution when $n > 2$.

In 2009 first author[4] searched the numeric solutions for n where $n < 100$.

In 2023 problem of $A^4 + B^4 + C^4 = 2D^4$ with $C \neq A + B$ was posted on the Mathoverflow website[2]. Inspired by this problem, we decided to take a closer look. We used two methods. One method is using an identity, another is elliptic curve method. When $C = A + B$, we used an identity to find some parametric solutions, when $C \neq A + B$, we used an elliptic curve to prove the infinity of solutions.

One of the results is that $A^4 + B^4 + C^4 = 2(3p^2 + q^2)^2D4$ with $C \neq A + B$ has infinitely many integer solutions.

In section 2, we searched the numeric solutions for n where $n < 1000$.

In section 3, we proved that $A^4 + B^4 + C^4 = 2(u^2 + uv + v^2)^4$ with $C = A + B$ has some parametric solutions.

In section 4, we proved that the problem of $A^4 + B^4 + C^4 = nD^4$ with $C \neq A + B$ can be reduced to solving simultaneous equations.

In section 4.1-4.5, we showed the solutions for $n = 2, 18, 242, 578, 2(3p^2 + q^2)^2$.

Finally, we stated the conjecture that $A^4 + B^4 + C^4 = 2m^2D^4$ with $C \neq B + C$ has infinitely many integer solutions where $m \equiv 1, 3, 7, 9 \ (mod\ 10)$.

2. Search for the numerical solutions with n < 1000

We consider the conditions for primitive solution of the $A^4 + B^4 + C^4 = nD^4$ by modular arithmetic consideration.

1. look mod 16

   $A^4 \equiv 0, 1 \ (mod\ 16)$ then ,the left side is in $(0, 1, 2, 3)\ mod\ 16$, hence $n \equiv 1, 2, 3 \ (mod\ 16)$

2. look mod 5

   $A^4 \equiv 0, 1$ mod 5 then ,the left side is in (0, 1, 2, 3) mod 5, hence $n \equiv 1, 2, 3$ (mod 5) By looking mod 16 and mod 5, we were able to reduce n as follows.

n = (2, 3, 17, 18, 33, 51, 66, 67, 82, 83, 97, 98, 113, 131, 146, 147, 161, 163, 177, 178,

193, 211, 226, 227, 241, 242, 257, 258, 273, 291, 306, 307, 321, 322, 323, 337, 338, 353,

371, 386, 387, 401, 402, 403, 417, 418, 433, 451, 466, 467, 481, 482, 483, 497, 498, 513,

531, 546, 547, 561, 562, 563, 577, 578, 593, 611, 626, 627, 641, 642, 643, 657, 658, 673,

691, 706, 707, 721, 722, 723, 737, 738, 753, 771, 786, 787, 801, 802, 803, 817, 818, 833,

851, 866, 867, 881, 882, 883, 897, 898, 913, 931, 946, 947, 961, 962, 963, 977, 978, 993)

We were able to show the infinity of solutions for $n = 2m^2$ where $m \equiv 0, 1$ (mod 3), but were unable to show the infinity of solutions for other cases.

Solutions for $n = 2m^2$ where $m \equiv 2$ (mod 3).

(n, A, B, C, D) = (242, 50, 27, 19, 13), (578, 527, 336, 257, 113)

Extended search for $n = 17 (n \neq 2m^2)$ where $A < 5000$.

(A, B, C, D) = (1066, 765, 758, 583), (2035, 884, 62, 1011), (2448, 1610, 533, 1259), (4751, 1750, 1224, 2353).

We show smallest numerical solutions by brute force.

Search range: n < 1000, (A, B, C, D) < 1000

Table 1: Smallest solutions

n	A	B	C	D
2	8	5	3	7
3	1	1	1	1
18	2	1	1	1
33	2	2	1	1
66	8	5	5	3
67	7	7	5	3
83	3	1	1	1
98	3	2	1	1
113	3	2	2	1
146	75	55	16	23
163	3	3	1	1
178	3	3	2	1
193	24	19	6	7
242	50	27	19	13
257	26	20	3	7
258	4	1	1	1
273	4	2	1	1
338	4	3	1	1
353	4	3	2	1
418	4	3	3	1
433	324	299	198	83
467	325	289	87	79
482	14	5	1	3
513	4	4	1	1
561	34	10	5	7
563	13	11	7	3
578	527	336	257	113
593	4	4	3	1
627	5	1	1	1
642	5	2	1	1
657	5	2	2	1
673	172	161	56	39
707	5	3	1	1
722	5	3	2	1
753	34	26	11	7
787	5	3	3	1
817	16	5	2	3
818	426	277	13	83
882	5	4	1	1
883	49	13	1	9
897	5	4	2	1
962	5	4	3	1
963	89	89	11	19
993	16	11	4	3

Table 2: n = 2 : A < 300

n	A	B	C	D
2	8	5	3	7
2	15	8	7	13
2	21	16	5	19
2	35	24	11	31
2	40	33	7	37
2	48	35	13	43
2	55	39	16	49
2	65	56	9	61
2	77	45	32	67
2	80	63	17	73
2	91	51	40	79
2	96	85	11	91
2	99	80	19	91
2	112	57	55	97
2	117	77	40	103
2	119	95	24	109
2	133	120	13	127
2	143	120	23	133
2	153	88	65	133
2	160	91	69	139
2	168	143	25	157
2	171	115	56	151
2	176	161	15	169
2	187	112	75	163
2	207	175	32	193
2	209	105	104	181
2	221	165	56	199
2	224	195	29	211
2	225	208	17	217
2	247	160	87	217
2	253	168	85	223
2	255	224	31	241
2	264	145	119	229
2	275	203	72	247
2	280	187	93	247
2	280	261	19	271
2	285	221	64	259
2	299	155	144	259

Table 3: n = 18 : A < 300

n	A	B	C	D
18	2	1	1	1
18	13	11	2	7
18	23	22	1	13
18	37	26	11	19
18	59	46	13	31
18	73	47	26	37
18	83	61	22	43
18	94	71	23	49
18	121	74	47	61
18	122	109	13	67
18	142	131	11	79
18	143	97	46	73
18	169	167	2	97
18	179	118	61	91
18	181	107	74	91
18	194	157	37	103
18	214	143	71	109
18	241	218	23	133
18	251	229	22	139
18	253	146	107	127
18	263	166	97	133
18	278	199	191	149
18	286	227	59	151
18	299	262	37	163

3. $A^4 + B^4 + C^4 = nD^4$ with $C = A + B$

Theorem 3.1 Suppose C = A + B and $n = 2(u^2 + uv + v^2)^2$.

Let

$A = (-v - u)b^2 - 2vab + ua^2$

$B = ub^2 + (2v + 2u)ab + va^2$

$C = -vb^2 + 2uab + (u + v)a^2$

$D = a^2 + ba + b^2$

a, b, u, v are arbitrary.

Then diophantine equation $A^4 + B^4 + C^4 = nD^4$ has infinitely many integer solutions.

Proof.

We use Proth's identity $A^4 + B^4 + (A + B)^4 = 2(A^2 + AB + B^2)^2$.

Thus we need to find the rational solution of $A^2 + AB + B^2 = mt^2$.

We know that $\mathbb{Z}[\omega]$ is a UFD where $\omega = \frac{1+\sqrt{-3}}{2}$

Now we can factorize $A^2 + AB + B^2 = (A - B\omega)(A - B\omega^2)$ in $\mathbb{Z}[\omega]$.

Let $m = u - v\omega$ and $t = a - b\omega$.

We observe the equation $A - B\omega = (u - v\omega)(a - b\omega)^2$, then we obtain

$A = (-v - u)b^2 - 2vab + ua^2$

$B = ub^2 + (2v + 2u)ab + va^2$

$C = -vb^2 + 2uab + (u + v)a^2$

$D = a^2 + ba + b^2$

Hence we obtain $A^4 + B^4 + (A + B)^4 = 2(uv + u^2 + v^2)^2(a^2 + ba + b^2)^4$

Example: $A^4 + B^4 + C^4 = 18D^4$

$(u, v) = (1, 1)$

$(A, B, C, D) = (-2b^2 - 2ab + a^2, b^2 + 4ab + a^2, -b^2 + 2a^2 + 2ab, a^2 + ab + b^2)$

Example: $A^4 + B^4 + C^4 = 98D^4$

$(u, v) = (2, 1)$

$(A, B, C, D) = (-3b^2 - 2ab + 2a^2, 2b^2 + 6ab + a^2, -b^2 + 3a^2 + 4ab, a^2 + ab + b^2)$

The proof is completed.

Remark: Several other parametric solutions are known as follows. Second author found below parametric solution.

$(A, B, C, D) = (pm^2 - 2qm + r, qm^2 - 2rm + p, rm^2 - 2pm + q, m^2 + m + 1)$

$A^4 + B^4 + C^4 = 2(p^2 + pq + q^2)^2D^4$ where $p + q + r = 0$.

Ramanujan found the parametric solution,

$(A, B, C, D) = (a^2b + b^2c + c^2a,\ ab^2 + bc^2 + ca^2,\ 3abc,\ ab + ac + bc)$

$A^4 + B^4 + C^4 = 2(ab + ac + bc)^2D^4$ where $a + b + c = 0$.

4. $A^4 + B^4 + C^4 = nD^4$ with $C \neq A + B$

Theorem 4.1

If the elliptic curve corresponding to the below simultaneous equations has a positive rank, then then $A^4 + B^4 + C^4 = nD^4$ has infinitely many integer solutions.

$c^2(2 + u^2) = -4(-2 + u^2)x^2 - 4um$

$y^2(2 + u^2) = -(6 + 3u^2 + 8u)x^2 - 2m + mu^2$

m is some integer. u is some rational number.

Proof.

Let $(a, b, c) = (\frac{A}{D}, b = \frac{B}{D}, c = \frac{C}{D})$ then we get

$$a^4 + b^4 + c^4 = n \tag{1}$$

Let $a = x + y$, $b = x - y$, $z = c^2$, $Y = y^2$ then

$$2x^4 + 12x^2Y + 2Y^2 + z^2 = n \tag{2}$$

Hence

$$y^2 = -3x^2 \pm \frac{\sqrt{32x^4 + 2n - 2z^2}}{2} \tag{3}$$

So we find the rational solutions of (4).

$$v^2 = 32x^4 + 2n - 2z^2 \tag{4}$$

(3), (4) can be parameterized by m and u using $(z, y) = (4x^2, -3x^2 + m)$ with $n = 2m^2$ as follows.

$$c^2(2 + u^2) = -4(-2 + u^2)x^2 - 4um \tag{5}$$

$$y^2(2 + u^2) = -(6 + 3u^2 + 8u)x^2 - 2m + mu^2 \tag{6}$$

m is some integer.

u is some rational number.

Thus we must find the rational solutions of simultaneous equations (5), (6).

In general, intersections of two quadratic equations are reduced to elliptic curve problem as follows.

First, substitute parametric solution x of (5) to (6) then we obtain a quartic equation (7),

$$V^2 = a_4U^4 + a_3U^3 + a_2U^2 + a_1U + a_0 \tag{7}$$

If the quartic equation (7) has a rational point, (7) can be transformed to an elliptic curve $E(Q)$.

If the corresponding elliptic curve $E(Q)$ has positive rank, then we can say

$A^4 + B^4 + C^4 = nD^4$ has infinitely many integer solutions.

The proof is completed.

4.1 m = 1 : $A^4 + B^4 + C^4 = 2D^4$

First we get a parametric solution of (8) using giving some rational number u.

$y^2(2 + u^2) = -(6 + 3u^2 + 8u)x^2 - 2 + u^2$ (8)

$c^2(2 + u^2) = -4(-2 + u^2)x^2 - 4u$ (9)

Take $u=\frac{-26}{9}$ the we get a parametric solution (*x, y*) of (8) for arbitrary rational number *s*.

$$x = \frac{1}{83}\,\frac{5447S^2 - 4173 - 53632s}{419S^2 - 321}$$

$$y = \frac{-2}{83}\,\frac{13408S^2 - 10272 - 4173s}{419S^2 + 321}$$

Substituting this value of x into (9), we have

$$\frac{838}{81}c^2 = \frac{13408}{558009}\,\frac{79880255s^2 - 304733485s^2 + 46883655 - 68637504s}{(419S^2 + 321)^2}$$

Hence we must find the rational solutions (*s, t*) of (10)

$Q : t^2 = 79880255s^4 + 89592256s^3 - 304733485s^2 - 68637504s + 46883655$ (10)

Quartic equation (10) is birationally equivalent to the elliptic curve using a known solution (s, t) = $(\frac{-1353}{419}, \frac{-22010355}{419})$ as follows.

$E : V^2 = U^3 - U^2 - 1097465452U + 3288951361780$

Since elliptic curve E has rank 1, we get infinitely many rational solutions using group law. Finally, we get a solution using Q(s, t) = $(\frac{-1353}{419}, \frac{-22010355}{419})$ as follows.

$$(a, b, c) = \left(\frac{-32}{1973}, \frac{2321}{1973}, \frac{1065}{1973}\right)$$

$$(A, B, C, D) = (32, 2321, 1065, 1973)$$

Large solution is obtained by using 2Q(s,t)(i.e.,doubling Q(s,t)).

A = 8331871536210073175631303374584

B = 4014369822293252845298556668977

C = 8028813922964684804294250901215

D = 8243128117136361914922521992201

Using 3Q(s,t):

A = 260201608895851514794364864822335510259200856988915349382616045151032504574658222134696

B = 35005523891745643921143648760580918653118928668410183533857015945186796301863835892687

C = 5380570727475226049666861210420510782062613064764655492156937060669469177198689737775

D = 218820526348816859964945789812126758717268326220649213329491535068239235702562298192769

4.2 m = 3 : $A^4 + B^4 + C^4 = 18D^4$

As in the case m=1,

$$y^2(2 + u^2) = -(6 + 3u^2 + 8u)x^2 - 6 + 3u^2$$

$$c^2(2 + u^2) = -4(-2 + u^2)x^2 - 12u$$

$$x = \frac{6}{19}\,\frac{2166m^2 - 4658 - 9709\text{m}}{1083m^2 + 2329}$$

$$y = \frac{-1}{361}\,\frac{553413m^2 - 1190119 + 1062024\text{m}}{1083m^2 + 2329}$$

$$\frac{2166}{25}c^2 = \frac{-24}{9025}\,\frac{9464041341\text{m}^4 + 708945694890m^2 - 521368173648m^3 + 43768200629 + 1121206349424\text{m}}{(1083m^2 + 2329)^2}$$

$Q : t^2 = -9464041341s^4 + 521368173648s^3 - 708945694890s^2 - 1121206349424s - 43768200629$

$E : V^2 = U^3 - 2773832216493U - 1323991146937260508$

Since elliptic curve E has rank 2, we get infinitely many rational solutions using group law. Finally, we get a solution using Q(s, t) = ($\frac{-17}{19}$, −111146) as follows.

(A, B, C, D) = (34622, 15299, 3269, 16967)

Using Q(s, t) = ($\frac{1071}{209}, \frac{757827394}{121}$):

(A, B, C, D) = (31873, 32842, 21577, 19133)

Using Q(s, t) = ($\frac{587481508311}{82605686471}, \frac{-20659275939769587303517383 8}{18902214507881308681}$):

(A, B, C, D) = (22561627039795221418, 28653872511286931281, 16294035416689673639, 15366818880060336173)

4.3 m = 11 : $A^4 + B^4 + C^4 = 242D^4$

As in the case m=1,

$$y^2(2 + u^2) = -(6 + 3u^2 + 8u)x^2 - 22 + 11u^2$$

$$c^2(2 + u^2) = -4(-2 + u^2)x^2 - 44u$$

$$x = \frac{1}{26}\,\frac{20861m^2 - 43263 - 139678\text{m}}{907m^2 + 1881}$$

$$y = \frac{-11}{26}\,\frac{6349m^2 - 13167 + 7866\text{m}}{907m^2 + 1881}$$

$$\frac{1814}{25}c^2 = \frac{1814}{4225}\,\frac{296976289m^4 - 13791614738m^2 + 5506386116m^3 + 1277276121 - 11419528428\text{m}}{(907m^2 + 1881)^2}$$

$$Q : t^2 = 296976289s^4 + 5506386116s^3 - 13791614738s^2 - 11419528428s1277276121$$

$E : V^2 = U^3 - U^2 - 17479103758200U - 19826700713763173100$

Since elliptic curve E has rank 3, we get infinitely many rational solutions using group law. Finally, we get a solution using $Q(s, t)$ = (9, 2178072) as follows.

(A, B, C, D) = (485, 549, 358, 161)

Using Q(s, t) = ($\frac{77}{23}, \frac{121080388}{529}$):

(A, B, C, D) = (50, 27, 19, 13)

Using Q(s, t) = ($\frac{11891}{560}, \frac{3238310635417}{313600}$):

(A, B, C, D) = (2259203131, 3338268210, 1741963763, 900950669)

Using Q(s, t) = ($\frac{-1813173743}{22311936375}, \frac{228741707791414520504297 08}{4978225048020481406 25}$):

(A, B, C, D) = (2318435855965414337406, 3549982154602363308025, 1780922670440785740457, 950795530208383453411)

4.4 m = 17 : $A^4 + B^4 + C^4 = 578D^4$

As in the case m=1,

$$y^2(2 + u^2) = -(6 + 3u^2 + 8u)x^2 - 34 + 17u^2$$

$$c^2(2 + u^2) = -4(-2 + u^2)x^2 - 68u$$

$$x = \frac{1}{18}\ \frac{44443m^2 - 43263 - 139678\text{m}}{907m^2 + 1881}$$

$$y = \frac{-1}{6}\ \frac{4535m^2 - 9405 + 61446\text{m}}{907m^2 + 1881}$$

$$\frac{1814}{25}c^2 = \frac{-1814}{2025}\ \frac{1341740519m^4 - 9217051614m^2 - 2285259060m^3 + 5770740591 + 4739329980\text{m}}{(907m^2 + 1881)^2}$$

$Q : t^2 = -1341740519m^4 + 2285259060m^3 + 9217051614m^2 - 4739329980m - 5770740591$

$E : V^2 + UV = U^3 - 2609225715971U - 1143507824283861960$

Since elliptic curve E has rank 3, we get infinitely many rational solutions using group law. Finally, we get a solution using Q(s, t) = ($\frac{-3}{4}, \frac{-635931}{16}$) as follows.

(A, B, C, D) = (3689, 6768, 2617, 1417)

Using Q(s, t) = ($\frac{161139}{124283}, \frac{1066567912382016}{15446264089}$):

(A, B, C, D) = (747341, 472107, 360368, 159967)

Using Q(s, t) = ($\frac{410343}{371870}, \frac{5505015688713}{152466700}$):

(A, B, C, D) = (115838763997, 55137925928, 32193074203, 23956381071)

4.5 $m = 3p^2 + q^2 : A^4 + B^4 + C^4 = 2m^2D^4$ with $C \neq A + B$

Theorem 4.2

Let $m = 3p^2 + q^2$ with p, q are arbitrary integers,

then $A^4 + B^4 + C^4 = 2m^2D^4$ has infinitely many integer solutions.

Proof.

$$c^2(2 + u^2) = -4(-2 + u^2)x^2 - 4um$$

$$y^2(2 + u^2) = -(6 + 3u^2 + 8u)x^2 - 2m + mu^2$$

Substitute (m, u) = ($3p^2 + q^2, \frac{-3p2+q2}{2p2}$) to first equation,

then we obtain a solution $(x, c) = (p, 2p)$.

Hence we have a parametric solution as follows.

$$x = \frac{p(8n^2p^4 + n^2m2 + 32p^4 - 4m^2 - 32p4n - 4nm^2)}{8n^2p^4 + n^2m^2 - 32p^4 + 4m^2}$$

$$c = \frac{-2p(8n^2p^4 + n^2m2 + 32p^4 - 4m^2 - 32p4n - 4nm^2)}{8n^2p^4 + n^2m^2 - 32p^4 + 4m^2}$$

Substitute parametric solution *x* to second equation then we obtain a quartic equation,

$Q : V^2 = (70q^6p^4 + 204q^4p^6 + 289q^2p^8 + 12q^8p^2 + q^10)n^4$

$+ (24q^8p^2 + 160q^6p^4 + 528q^4p^6 + 416q^2p^8 + 408p^10)n^3$

$+ (8q^10 + 96q^8p^2 + 432q^6p^4 + 96q^4p^6 - 376q^2p^8 - 768p^10)n^2$

$+ (-640q^6p^4 - 576q^4p^6 - 640q^2p^8 - 96p^{10} - 96q^8p^2)n$

$+ 16q^2(q^4 + 6q^2p^2 + p^4)^2$

The quartic equation has a rational point $Q(n, V) = (0, 4q(q^4 + 6q^2p^2 + p^4))$,

then it can be transformed to an elliptic curve E.

The coefficients of elliptic curve are so tedious, so we omit it. The point of elliptic curve E corresponding to $Q(n, V)$ is

$$P(X,Y) = \left(\frac{-8(q^{12} + 12p^2q^{10} + 36p^4q^8 - 12p^6q^6 - 91p^8q^4 - 120p^{10}q^2 - 18p^{12})}{q^2},\right.$$

$$\left.\frac{-128(3q^{16} + 38p^2q^{14} + 162p^4q^{12} + 270p^6q^{10} + 120p^8q^8 - 110p^{10}q^6 - 258p^{12}q^4 - 198p^{14}q^2 - 27p^{16})p^2}{q^3}\right)$$

By Nagell-Lutz theorem we know P (X, Y) has infinite order. Hence this point P (X, Y) has infinite order,

the quartic equation has infinitely many parametric solutions.

For instance, using 2P (X, Y) we obtain a point 2Q(n, V) of quartic equation,

$$2Q(n,V) = \left(\frac{4q^2}{3p^2}, \frac{4q(4q^8+24p^2q^6+41p^4q^4+30p^6q^2-27p^8)}{9p^4}\right).$$

Finally, we obtain a parametric solution of $A^4 + B^4 + C^4 = 2m^2D^4$ as follows.

$A = -9p^9 + 27qp^8 + 150q^2p^7 - 30q^3p^6 + 131q^4p^5 - 41q^5p^4 + 36q^6p^3 - 24q^7p^2 + 4q^8p - 4q^9$

$B = -9p^9 - 27qp^8 + 150q^2p^7 + 30q^3p^6 + 131q^4p^5 + 41q^5p^4 + 36q^6p^3 + 24q^7p^2 + 4q^8p + 4q^9$

$C = 2(9p^8 + 66q^2p^6 + 13q^4p^4 - 12q^6p^2 - 4q^8)p$

$D = 9p^8 + 54q^2p^6 + 77q^4p^4 + 24q^6p^2 + 4q^8$

For instance, let $(p, q) = (4, 7)$ then we obtain $m = 97$, $(A, B, C, D) = (9968351, 123938809, 15880664, 10582731)$.

If we use a point $3Q(n, V)$ then we obtain other new solution.

Thus the quartic equation has infinitely many parametric solutions,

$A^4 + B^4 + C^4 = 2m^2D^4$ has infinitely many integer solutions.

The proof is completed.

Finally we state conjecture.

As mentioned in the 4.1 and 4.2, we proved $A^4 + B^4 + C^4 = 2m^2D^4$ with $C \neq A + B$ has infinitely many integer solutions for $m \equiv 0, 1 \pmod 3$.

In the same way, in the 4.3 and 4.4, we proved $A^4 + B^4 + C^4 = 2m^2D^4$ with $C \neq A + B$ has infinitely many integer solutions for some $m \equiv 2 \pmod 3$.

In the 4.5 we proved $A^4 + B^4 + C^4 = 2m^2D^4$ with $C \neq A + B$ has infinitely many integer solutions for any $m = 3p^2 + q^2$.

As mentioned in the in 2., in order to $A^4 + B^4 + C^4 = nD^4$ with $C \neq A + B$ has a integer solution,

$n \equiv 1,2,3 \pmod 5$,

Hence $2m^2 \equiv 1, 2, 3 \pmod{16}$ and $2m^2 \equiv 1, 2, 3 \pmod 5 \Rightarrow m \equiv 1, 3, 7, 9 \pmod{10}$.

Table 4: Solutions for $m \equiv 1, 3, 7, 9 \pmod{10}$ $m < 100$

m	n	A	B	C	D
3	18	278	199	191	149
7	98	491	254	137	159
11	242	50	27	19	13
13	338	2228	1943	1237	591
17	578	527	336	257	113
19	722	63	46	25	13
21	882	272	149	25	51
23	1058	62	29	9	11
27	1458	326	109	91	53
31	1922	80858	185415	9109	28253
37	2738	2357361	570629	691532	326767
39	3042	866	665	539	129
43	3698	9309329	26667961	2363574	3432457
47	4418	839990066	10754417721	25232397949	3120163151
53	5618	453	217	28	53
57	6498	164	149	83	21
61	7442	28592719	13425635	9332556	3123599
63	7938	491	254	137	53
67	8978	221884469	170489659	46798074	24573059
71	10082	10	3	1	1
73	10658	21857987	9003797	3243816	2166827
79	12482	6565031	17528299	25834670	2566683
89	15842	2.78196E+11	1741159879	4.15156E+11	38743789163
91	16562	1715795117	501121955	527813898	151857193
93	17298	71538991	42289931	21776324	6432517
97	18818	123938809	9968351	15880664	10582731

Conjecture

Suppose $m \equiv 1, 3, 7, 9 \pmod{10}$ and m is squarefree.

Then $A^4 + B^4 + C^4 = 2m^2D^4$ with $C \neq A + B$ has infinitely many integer solutions.

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