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RESEARCH ARTICLE

Jacobsthal Graceful Labeling

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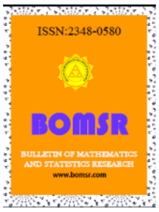

ABSTRACT

An injective function $f : V(G) \to \{0, 1, 1, 3, \ldots, J_q\}$ is said to be Jacobsthal graceful labeling of a graph $G(p;\ q)$ if the labeling of induced edge $f^*(uv) = |f(u) - f(v)|$ is a bijection onto $\{J_1, J_2, \ldots, J_q\}$ where J_q is the q^th^ Jacobsthal number in the Jacobsthal sequence. If G admits a Jacobsthal graceful labeling, then G is called a Jacobsthal graceful graph. In this paper some standard graphs are shown to be Jacobsthal graceful labeling.

AMS classification: 05C78

Keywords : Labeling, Graceful labeling, Jacobsthal sequence, Jacobsthal graceful labeling.

1 Introduction

Let $G = (p, q)$ be a simple, undirected, finite graph. An assignment of integers to the edges or vertices or both subject to certain conditions is a graph labeling. For basic terminologies we refer Harrary [2]. Gallian [1] has given a vast survey on graph labeling. Alex Rosa [3] introduced the concept of graceful labeling in 1966. A function f of G is called a graceful labeling if f is an injection from the vertex set of G to the set $\{0, 1, \ldots, m\}$ such that when each edge uv is assigned the label

$|f(u) - f(v)|$ and the resulting edge labels are distinct, then G is graceful. Acharya and Hedge introduced Fibonacci graceful labeling and further studied by [4, 5]. In this paper we introduced a new labeling called Jacobsthal graceful labeling and showed certain family of graphs which admits Jacobsthal graceful labeling.

2 Jacobsthal Graceful Labeling

Definition 2.1. *The Jacobsthal sequence is an additivesequence defined by* $J_n = J_{n-1} + 2J_{n=2}$ *with begining terms* $J_0 = 0$ and $J_1 = 1$ *and the numbers in the sequence are called Jacobsthal numbers. The Jacobsthal numbers are* 0, 1, 1, 3, 5, 11, 21, 43,85, . . .

Definition 2.2. *A graph $G(p, q)$ with injective function$f : V(G) \rightarrow \{0, 1, 1, 3, \dots, J_q\}$ is said to be Jacobsthal graceful labeling G if the induced edge labeling $|f(u) - f(v)|$is a bijection onto the set $\{J_1, J_2, \dots, J_q\}$where $J_q$is the q^{th} Jacobsthal number in the Jacobsthal sequence. A graph is referred to be a Jacobsthal graceful graph (G) if it admits a Jacobsthal graceful labelling.*

Definition 2.3. *The Comb is a graph obtained by Jogging a each vertex of a path P_n to a single pendant edge. It is denoted by* $P_n \odot K_1$

Definition 2.4. A Coconut Tree $CT_{(m,n)}$ is the graph obtained from the path P_m by appending n pendant edges at an end vertex of P_m:

Definition 2.5. The Bistar $B_{m,n}$ is the graph obtained from K_2 by joining m pendant edges to one end of K_2 and n pendant edges to the other end of K_2:

Definition 2.6. Subdivision of a graph G is a graph obtained from G by replacing certain edges of G with internally vertex - disjoint paths.

Definition 2.7. A rooted tree consisting of n branches where the jth branch is a path of length j is called Olive tree and is denoted by Tn:

Definition 2.8. The Jelly fish graph $J(m, n)$ is obtained from a 4-cycle $v_1; v_2; v_3; v_4$ by joining v_1 and v_3 with an edge and appending m pendant edges to v_2 and n pendant edges to v_4 . Jelly fish $J(m, n)$ is a graph with order of vertices $m + n + 4$ and sizes of edges is $m + n + 5$.

Theorem 2.9. The path P_n, $n > 3$ is Jacobsthal graceful labeling.

Proof. Let the vertices of P_n be $v_0, v_1, \dots, v_n$ and $e_1, e_2, \dots, e_n$ be the corresponding edges. Now,

$|V(P_n)| = n + 1$ $|E(P_n)| = n$.

Case (i) If n is odd,

Define $f : V(P_n) \rightarrow \{0, 1, 1, \dots, J_q\}$ by $f(v_0) = 0$, $f(v_1) = J_q$

and $f(v_i) = f(v_{i-1}) - J_{q-(i-1)}$ where $i = 2, 3, \dots, n$

Case(ii) If n is even

Define $f : V(P_n) \rightarrow \{0, 1, 1, \dots, J_q\}$ by

$f(v_0) = 0$, $f(v_1) = J_q$

$f(v_i) = f(v_{i-1}) - J_{q-(i-1)}$, where $i = 2, 3, \dots, n - 2$

$f(v_{n-1}) = J_3$, $f(v_n) = J_4 - 1$

In the above two cases, the edge labels induced are Jacobsthal numbers. Hence the path P_n is Jacobsthal graceful labeling for all $n > 3$

Example

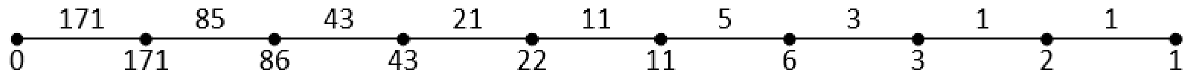

Figure 1: P_{10}

Theorem 2.10. *The Cycle C_n of even length is Jacobsthal graceful labeling*

Proof. Let C_n be the cycle of even length. Let $v_0, v_1, v_2, \ldots, v_{n-1}$ be the vertices and $e_0, e_1, \ldots e_{n-1}$ be the corresponding edges of the cycle C_n. Now, $|V(C_n)| = n$, $|E(C_n)| = n$.

We define $f : V(C_n) \to \{0, 1, 1, \ldots, J_q\}$ *by*

$f(v_0) = 0$, $f(v_1) = 1$, $f(v_i) = f(v_{i-1}) + J_i$ where $i = 2, 3, \ldots, n-1$

This shows that the induced edge labels are Jacobsthal numbers. Hence even cycles are Jacobsthal graceful labeling.

Example

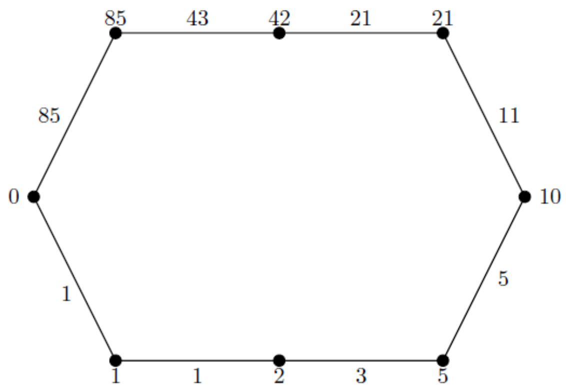

Figure 2: C_8

Remark 2.11. *Cycle of odd length is not Jacobsthal graceful*

Example 2.12. *The cycle C_5 is not a Jacobsthal graceful labeling.*

Proof. If C_5 is a Jacobsthal graceful graph,then $f : V(C_5) \to \{0, 1, 1, \ldots, J_q\}$ is an injective function such that the edge labels are Jacobsthal numbers $\{J_1, J_2, J_3, J_4, J_5\} = \{1, 1, 3, 5, 11\}$

Let {a, b, c, d} be the vertices of the cycle C_5.

By Figure 3, $f(a) = 0$, $f(b) = 1$, $f(e) = 11$, $f(d) = 6$. Let $f(c) = x$ if $x = 3$, then $f^*(bc) = 2$,

2 is not a Jacobsthal number and if $x = 2$, then $f^*(cd) = 4$,

4 is not a Jacobsthal number. Both cases leads to a contradiction that *f* is a Jacbsthal graceful labeling. Hence C_5 is not a Jacobsthal graceful labeling

Example

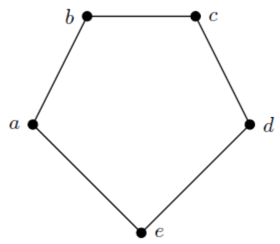

Figure 3: C_5

Theorem 2.13. *The combs $P_n \odot K_1$ are Jacobsthal graceful labeling*

Proof. Let $G = P_n \odot K_1$ be the comb graph. Let the vertices of the comb be $\{u_1, u_2, \ldots, u_{n+1}, v_1, v_2, \ldots, v_{n+1}\}$, where $u_1, u_2, \ldots, u_{n+1}$ are the vertices of P_n and $v_1, v_2, \ldots, v_{n+1}$ are the vertices attached to P_n by the edge $\{u_i v_i / i = 1, 2, 3, \ldots, n\}$ defined by,

$|V(P_n \odot K_1)| = 2n + 2,$

$|E(P_n \odot K_1)| = 2n + 1.$

Case(i) G is even

Define $f : V \rightarrow \{0, 1, 1, \ldots, J_q\}$ by

$f(u_1) = J_{n+1}$

$f(u_i) = f(u_{i-1}) + Jn+(i-1)$, where $i = 2, 3, 4, \ldots,$

$n\, f(u_{n+1}) = 0$

$$and\ f(v_i) = \begin{cases} J_{n+(i)} - J_{n-(n-1)};\ for\ i \text{ is odd} \\ J_{n+i} - J_{n-(n-1)} + 1;\ for\ i \text{ is even} \end{cases}$$

where, $i = 1, 2, \ldots, n.$

$f(v_{n+1}) = 1.$

Case(ii) *G is odd*

Define $f : V \rightarrow \{0, 1, 1, \ldots, J_q\}$ by $f(u_1) = J_{n+1} + 1$

$f(u_i) = f(u_{i-1}) + Jn+(i-1)$, where $i = 2, 3, \ldots, n.$

$f(u_{n+1}) = 0$

and $f(v_1) = J_n$

$$f(v_i) = \begin{cases} f(v_{i-1}) + J_{n-1-(i-2)} + J_{n+(i-1)} + 1;\ \text{for i is even} \\ f(v_{i-1}) + J_{n-1-(i-2)} + J_{n+(i-1)} - 1;\ \text{for i is odd} \end{cases}$$

where $i = 2, 3, \ldots, n\ f(v_{n+1}) = 1.$

we get the induced edge labels are Jacobsthal numbers. Hence the combs are Jacobsthal graceful .

Example

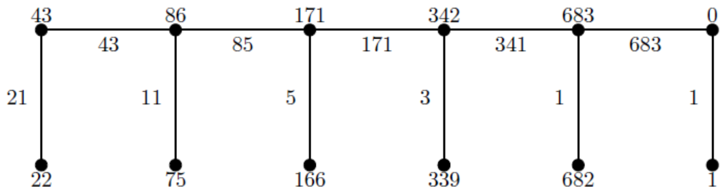

Figure 4: Comb $P_6 \odot K_1$

Theorem 2.14. *The coconut tree CT (m, n) is Jacobsthal graceful for all n;m > 2*

Proof. Let $CT(m, n)$ be the coconut tree. Let $\{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$ be the vertex set. By the definition of coconut tree $u_1, u_2, \ldots, u_m$ be the vertices of the path P_m and $v_1, v_2, \ldots, v_n$ be the pendant vertices joined to the end vertex u_m of the path P_m and the edge set is $\{u_iu_j, i = 1, 2, \ldots, m - 1, j = 2, 3, \ldots, m\}$ and $\{u_mv_i, i = 1, 2, \ldots, n\}$;

Now, $|V(CT(m, n)| = m + n$,

$|E(CT(m, n)| = (m - 1) + n$

Define the function{ f : V ! f0; 1; 1; : : : jq} by

$$f(u_i) = \begin{cases} j_{m+n-1} + \sum_{i=1}^{m-3} j_{m+n-1} - i - J_1 \text{ for } m > 3 \\ j_{m+n-1} \text{ for } m = 3 \end{cases}$$

$f(u_2) = f(u_1) + 1$

$f(u_i) = f(u_{i-1}) + J_{n+i-1}, \quad i = 3, 4, 5, \ldots, m - 1, f(u_m) = 0$

and $f(v_i) = J_{i+1}, \quad i = 1, 2, \ldots, n$

Hence the induced edge labels are Jacobsthal numbers. Therefore the coconut tree is Jacobsthal graceful .

Example

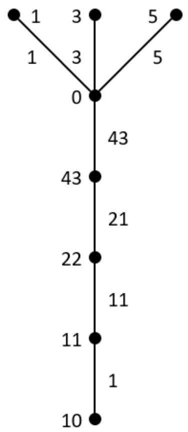

Figure 5: *CT* (5, 3)

Theorem 2.15. *The bistar $B_{m,n}$ is Jacobsthal graceful for all $m, n \geq 2$*

Proof. Let $\{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n, u, v\}$ be the vertex set and $\{uu_i, i = 1, 2, \ldots, m\}$, $\{vv_i, i = 1, 2, \ldots, m\}$, $\{uv\}$ are the edge sets of $B_{m,n}$.

Now, $|V(B_{m,n})| = m + n + 2$

$|E(B_{m,n})| = m + n + 1$

Define the function $f : V(B_{m,n}) \rightarrow \{0, 1, 1, \ldots, J_q\}$ by

$f(u) = 0$

$f(u_i) = J_{n+i+1}, \quad i = 1, 2, \ldots, m$

and $f(v) = 1$

$f(v_i) = J_1 + J_{i+1}, \quad i = 1, 2, \ldots, n.$

Now the induced edge labels are Jacobsthal numbers. Therefore the bistar $B_{m,n}$ is Jacobsthal graceful for all $m, n \geq 2$.

Example

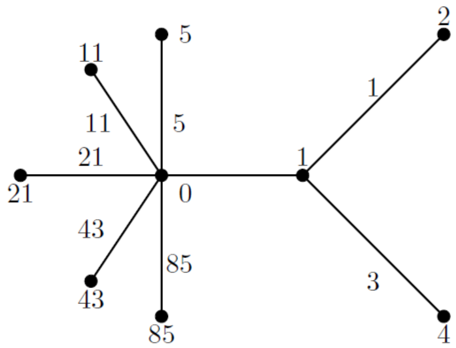

Figure 6: B(5; 2)

Theorem 2.16. The subdivision of bistar $S(B_{m,n})$ is Jacobsthal graceful for all m, n ≥ 2.

Proof. Let $S(B_{m,n})$ be the subdivision of bistar . Let $\{u_1, u_2, \ldots u_m, u'_1, u'_2, \ldots u'_m, v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n, u, v\}$ be the vertex set and $\{u_i u'_i, i = 1, 2, \ldots, m\}$, $\{u'_i u, i = 1, 2, \ldots, m\}$, $\{uv\}$, $\{vv'_i, i = 1, 2, \ldots, n\}$, $\{v'_i v_i, i = 1, 2, \ldots, n\}$ are the edge set of $B_{m,n}$.

Now, $|V(S(B_{m,n})| = 2(m + n) + 2$

and $|E(S(B_{m,n})| = 2(m + n) + 1$

Define the function $f : V(S(B_{m,n})) \rightarrow \{0, 1, 1, \ldots, J_q\}$ by $f(u) = 0$,

$f(v) = J_{2(m+n)+1}$

$f(u_1) = 2$

$f(u_i) = J_{2m-(i-2)} + J_{i+1}, \quad i = 2, 3, \ldots, m$

$f(u'_i) = J_{i+1}, \quad i = 1, 2, \ldots, m$

$f(v'_i) = J_{2(m+n)+1} - J_{2m+i}, \quad i = 1, 2, \ldots, n$

$f(v_i) = J_{2(m+n)+1} - J_{2(m+n)-(i-1)} - J_{2m+i}, \quad i = 1, 2, \ldots, n.$

Now, we get the induced edge labels are Jacobsthal numbers. Hence the subdivision of bistar is Jacobsthal graceful for all *m, n* ≥ 2

Example

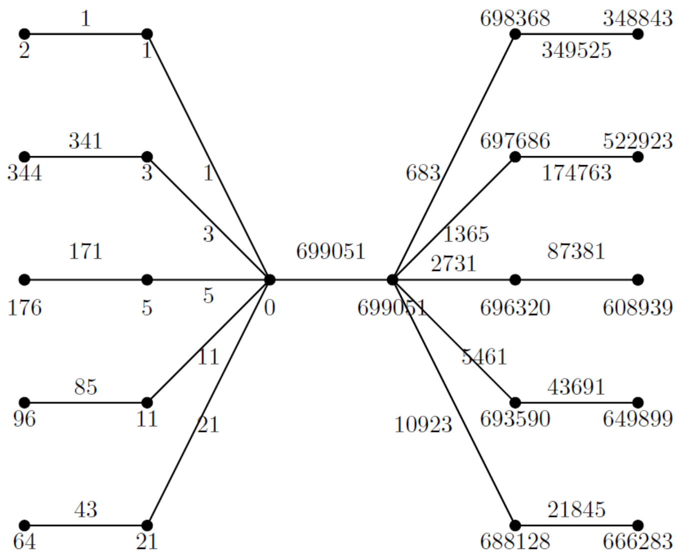

Figure 7: S(B(5; 5))

Theorem 2.17. *Olive tree is Jacobsthal graceful for all n* ≥ 2

Proof. Let T_n be an olive tree. By the definition of Olive tree ,the vertices of thepaths be joined by the single vertex u_0 . Let the vertices are $\{u_0, u_{11}, u_{12}, \ldots\ldots, u_{1n}, u_{21}, u_{22}, \ldots, u_{2(n-1)}, \ldots, u_{n1}\}$ and the edges are denoted as in figure.Now, $|V(G)| = (n^2 + n + 2)/2$

$|E(G)| = (n(n + 1))/2$

Define the function $f : V \to \{0, 1, 1, \ldots, J_q\}$ by $f(u_0) =$

$0, f(u_{i1}) = J_{q-(i-1)}, \quad 1 \le i \le n$

$f(u_{i2}) = f(u_{i1}) - J_{q-[n+(i-1)]}, \quad 1 \le i \le n-1$

$f(u_{i3}) = f(u_{i2}) - J_{q-[(n+(i-1))+(n-1)]}, \quad 1 \le i \le n-2$

$f(u_{i4}) = f(u_{i3}) - J_{q-[(n+(i-1))+(n-1)+(n-2)]}, \quad 1 \le i \le n-3$

$f(u_{i5}) = f(u_{i4}) - J_{q-[(n+(i-1))+(n-1)+(n-2)+(n-3)]}, \quad 1 \le i \le n-4$

and so on. continuing like this we get the edge labels are Jacobsthal numbers. Hence the olive tree is Jacobsthal graceful labeling for all *n* ≥ 2

Example

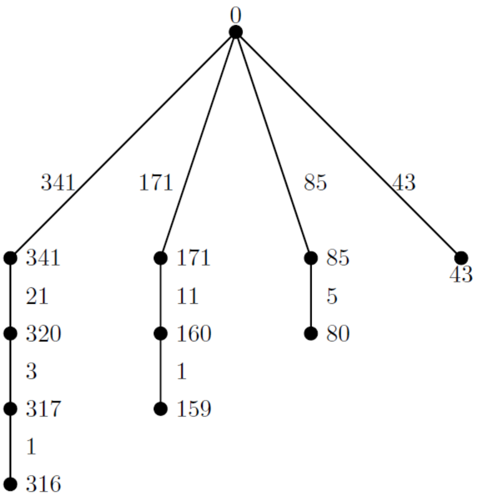

Figure 8: T_4

Theorem 2.18. The Jelly fish $J_{(m,n)} - e_1$ where $e_1 = x_1x_3$ is Jacobsthal graceful.

Proof. Let $G = J_{(m,n)} - e_1$, where $e_1 = x_1x_3$ Let $\{x_1, x_2, x_3, x_4, u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$ be the vertices of G. By definition x_1, x_2, x_3, x_4 are the vertices of the cycle C_4. $u_1, u_2, \ldots, u_m$ be the pendant vertices are attached to the vertex x_1 and the pendant vetices $v_1, v_2, \ldots, v_n$ are attached to the vertex x_3 . The edge set is defined as $\{x_1u_i, 1 \le i \le m\}, \{x_3v_i, 1 \le i \le n\}, \{x_2x_1\}, \{x_1x_4\}, \{x_2x_3\}, \{x_3x_4\}$.

Define $f : V \longrightarrow \{0, 1, 1, \ldots, J_q\}$ by $f(u_i) = J_{q-(i-1)}, \quad i = 1, 2, \ldots, m$

$f(x_1) = J_0$

$f(x_i) = f(x_{i-1}) + J_{i-1}, \ i = 2, 3, 4$

$f(v_i) = J_{i+4} + 2, \ i = 1, 2, \ldots, n$

Above conditions show that the induced edge labels are Jacobsthal numbers. Therefore the Jelly fish $J(m, n) - e_1$ is Jacobsthal graceful labeling.

Example

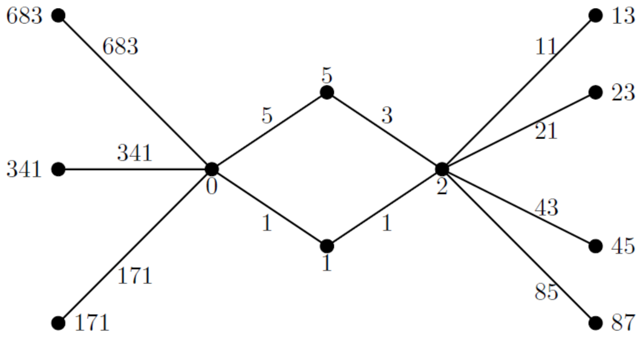

Figure 9: $J(3, 4) - e_1$

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