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About $(a^4 + b^4 + c^4 + d^4) = (a + b + c + d)^4$ (After Jacobi & Madden Results)

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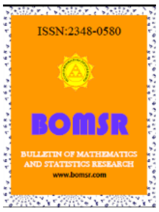

ABSTRACT

In this paper, we show the further research developments in the field for the equation:

$(a^4 + b^4 + c^4 + d^4) = (a + b + c + d)^4$

Introduction

In 1772 Euler[2] conjectured that $a^4 + {}^{b4} + c^4 + d^4 = e^4$ would have integer solutions, but he couldn't find a solution. Simcha Brudno [1] in 1964 found a solution $5400^4 + 1700^4 + (-2634)^4 + 955^4 = (5400 + 1700 - 2634 + 955)^4$..

Jacobi & Madden[3] in 2008 assumed that $a^4 + b^4 + c^4 + d^4 = (a + b + c + d)^4$..

They used identity $a^4 + b^4 + (a + b)^4 = 2(a^2 + ab + b^2)^2$, the equation was reduced to elliptic curve problem and they proved that the equation has infinitely many integer solutions.

One of solutions they gave is $1338058950^4 + (-89913570)^4 + 504106884^4 + (-404747255)^4 = (1338058950 - 89913570 + 504106884 - 404747255)^4$.

Jaroslaw Wroblewski[7] found many integer solutions of $a^4 + b^4 + c^4 + d^4 = e^4$.

Inspired by Jacobi and Madden's article, in 2008 this author found many integer solutions of $a^4+b^4+c^4+d^4 = (a+b+c+d)^4$ using their method.

Allan J. MacLeod[4] in 2017 found integer solutions of $a^4 + b^4 + c^4 + d^4 = (a + b + c + d)^4$.

In particular, he found large height solution using 2-isogenous curve.

We show, $a^4 + b^4 + c^4 + d^4 = (a + b + c + d)^4$ has infinitely any integer solutions using twodifferent methods.

One method is Jacobi and Madden method, another is quartic method.

In addition, we summarized many small solutions (a+b+c+d< 10^{18}). This contains all presently known small solutions.

2. Jacobi and Madden's method

Theorem 1. $a^4 + b^4 + c^4 + d^4 = (a + b + c + d)^4$ has infinitely many integer solutions

Proof.

Assume that, $a^4 + b^4 + c^4 + d^4 = (a + b + c + d)^4 = e^4$

We use identity $a^4 + b^4 + (a + b)^4 = 2(a^2 + ab + b^2)^2$.

So,

$a^4 + b^4 + (a + b)^4 + c^4 + d^4 + (c + d)^4 = a + b)^4 + (c + d)^4 + (a + b + c + d)^4$. $(a^2 + ab + b^2)^2 + (c^2 + cd + d^2)^2 = ((a + b)^2 + (a + b)(c + d) + (c + d)^2)^2$

We obtain

$$c^2 + cd + d^2 = m((a+b)^2 + (a+b)(c+d) + (c+d)^2 - a^2 - ab - b^2)$$

$$c^2 + cd + d^2 = \frac{(a+b)^2 + (a+b)(c+d) + (c+d)^2 + a^2 + ab + b^2}{m}$$

We use a known solution to determine the value of m.

$48150^4 + (-31764)^4 + 27385^4 + 7590^4 = 51361^4$ (Wroblewski)

Set $a_0 = 48150, b_0 = -31764, c_0 = 27385, d_0 = 7590$ and substitute to equation (1).

We get $m = 1807/475$.

To find the solution easy to obtain, we convert the variables.

$a = 2z+2w$

$b = 2z-2w$ $c = -x - y - z$ $d = x - y - z$

(1),(2) become to following equation.

$$-5979z^2 + 1807x^2 + 3521y^2 + 10842yz - 1900w^2 = 0$$

$$1425z^2 + 475x^2 - 5803y^2 + 2850yz + 7228w^2 = 0$$

$$5415000z^2 - 12158496y^2 + 13963496w^2 = 0$$

To find a parametric solution of (5),we use a known solution$[yo, wo, zo = \left(-21584, \frac{39957}{2}, \frac{8193}{2}\right)$

$$
\begin{aligned}
y &= \frac{-2(256310000k^2 + 575502144 + 97291875k)}{2103} \\
z &= \frac{616181875k^2 + 1383535524 + 14579387648k}{13319} \\
w &= 225625k^2 - 506604
\end{aligned}
$$

Substitute x,y,z of (6) to (3), then we get a quartic equation.

$$Y^2 = 651331010426062856347656 25k^4 + 97687206080225110285876875 00k^3$$

$$+2153580681346349883606603970 8k^2 + 2193406286939229330582442920 0k + 3283708116005390782967076 00 \quad (7)$$

$$Y = 72202299x$$

Transform (7) to minimal Weierstrass form (8).

$V^2 + UV = U^3 - U^2 + 69371210083521741309859 5U - 925623290959491513363159202180191099$ .....(8)

We get a point $P(U,V) = \left(\frac{4347462759616958 34}{11449}, \frac{-28671777510404760884167 7439}{1225043}\right)$

As this point on the curve (8) does not have integer coordinates, there are infinitely many rational points on the curve (8) by Nagell-Lutz theorem.

Point 2 , $P(U,V) = \left(\frac{527116173485232351969071249351364614855982417 1}{555979487782184246211680800056025}, \frac{-383967827631115132325778408795184936425282259780975975668812642463671}{1310956114852454158411010073358661938648674762012 5}\right)$

$$k = \frac{16865743515210194231445 0}{13242963993478038442743 41}$$

$$Y = \frac{-287049884891605933917141949569922910223374378499787229457540}{4858063582619274122389752417972653863736484721}$$

$$x = \frac{-529514637320800468395391901069771094306169301788945267 40}{647045488569061120361091124549777768111062399989 99}$$

Substitute k to (6),then we get $y = \frac{-191747809500336625620349410150763982441686328378297371 6}{340550257141611115979521644499883035847927578942 1}$

$$z = \frac{157901933389707068073273612296392638273750660431286779 04}{647045488569061120361091124549777768111062399989 99}$$

$$w = \frac{-244333611168920132754808738287503816584518385403498 4}{4858063582619274122389752417972653863736484721}$$

We get a new solution.

a = -9920529841425116466284680710894526792238279430539 4

b = 2862175605239546822899562799473665653544014839698 50

c = 2179020128093342104486291565186506687683938535650 40

d = -9566517417742495552443376900292304224030483522676 5

e= a+b+c+d = 3092491007416127725513048603541489239601077080027 31

We can obtain infinitely many integer solutions of $a^4 + b^4 + c^4 + d^4 = (a + b + c + d)^4$ by applying the group law.The proof is completed.

Small New Solutions

We searched for small new solutions without using elliptic curve as follows.

$$a^4 + b^4 + c^4 + d^4 = (a + b + c + d)^4 = e^4$$

$$
\begin{aligned}
(b + c + d)(a + c + d) &= pt^2\\
(a + b + d)(a + b + c) &= pr^2/2\\
ab + ac + ad + bc + bd + cd &= -prt \qquad (9)
\end{aligned}
$$

Change the variables by (10).

$$
\begin{aligned}
a &= -2A + B + C + D\\
b &= A - 2B + C + D\\
c &= A + B - 2C + D\\
d &= A + B + C - 2D \qquad (10)
\end{aligned}
$$

e=(a+b+c+d)

(9) becomes to (11).

$AB = pt^2$ $CD = pr^2/2$

$AB + AC + AD + BC + BD + CD - A^2 - B^2 - C^2 - D^2 = -3prt$ (11)

We use a known solution again.

$a_0 = 48150$, $b_0 = -31764$, $c_0 = 27385$,

$d_0 = 7590$

Set,

$\mathbf{A_0} = b_0 + c_0 + d_0 = 3211 = (13)^2 \cdot (19)$

$\mathbf{B_0 = a_0 + c_0 + d_0 = 83125 = (5)^4 \cdot (7) \cdot (19)}$

$\mathbf{C_0 = a_0 + b_0 + d_0 = 23976 = (2) \cdot (3) \cdot (37)}$

$\mathbf{D_0 = a_0 + b_0 + c_0 = 43771 = (7) \cdot (13)^2 \cdot (37)}$

$$
\begin{aligned}
A &= p_1 t_1 t_2^2\\
C &= p_1 r_1 r_3^2/2\\
D &= p_2 r_1 r_2^2 \qquad (12)
\end{aligned}
$$

We obtain the equation similar to their[3] lemma4, that is

If $(p_1, p_2, r_1, r_2, r_3, t_1, t_2, t_3)$ is a solution of (13) then (14) is a solution of $a^4 + b^4 + c^4 + d^4 = (a + b + c + d)^4$.

$$\mathbf{4p_1t_1^2t_2^2p_2t_3^2 + 2p_1^2t_1t_2^2r_1r_3^2 + 4p_1t_1t_2^2p_2r_1r_2^2 + 2p_2t_1t_3^2p_1r_1r_3^2 + 4p_2^2t_1t_3^2r_1r_2^2 + 2p_1r_1^2r_3^2p_2r_2^2 - 4p_1^2t_1^2t_2^4 - 4p_2^2t_1^2t_3^4}$$

$$-p^2r_1^2r_3^4 - 4p_2^2r_1^2r_2^4 + 12p_1p_2r_1r_2r_3t_1t_2t_3 = 0 \qquad (13)$$

$$
\begin{aligned}
\mathbf{a} &= \mathbf{-4p_1t_1t_2^2 + 2p_2t_1t_3^2 + p_1r_1r_3^2 + 2p_2r_1r_2^2}\\
\mathbf{b} &= \mathbf{2p_1t_1t_2^2 - 4p_2t_1t_3^2 + p_1r_1r_3^2 + 2p_2r_1r_2^2}\\
\mathbf{c} &= \mathbf{2p_1t_1t_2^2 + 2p_2t_1t_3^2 - 2p_1r_1r_3^2 + 2p_2r_1r_2^2}\\
\mathbf{d} &= \mathbf{2p_1t_1t_2^2 + 2p_2t_1t_3^2 + p_1r_1r_3^2 - 4p_2r_1r_2^2} \qquad (14)
\end{aligned}
$$

Define $L4(p_1, p_2, r_1, r_2, r_3, t_1, t_2, t_3)$ with left hand side of (13).

It is understood that $L4(1,7,37,13,36,19,13,25) = 0$ by the values of A_0, B_0, C_0 and D_0.

We consider $L4(p_1, 7, 37, 13, 36, 19, 13, 25) = 2032688644p_1^2 - 22781495408p_1 + 20748806764 = 0.$

Since above solutions are 1 and $\frac{741028813}{72596023}$, then we get $L4(\frac{741028813}{72596023}, 7, 37, 13, 36, 19, 13, 25) = 0.$

Hence we get a new solution by using (14).

a = 1058103081810

b = 535945811334

c = -1140105961325

d = 944080652640

e= a+b+c+d =1398023584459

Next, we consider $L4(1,7,r_1,13,36,19,13,25) = 4211236r_1^2 - 875076464r_1 + 26612647084 = 0.$

Since above solutions are 37 and $\frac{9463957}{55411}$, then we get L4(1,7, $\frac{9463957}{55411}$, 13,36,19,13,25)=0.

In the same way, we get a new solution by using (14).

a = 378573600

b = 145514934

c = 65167315

d = -201317790

a+b+c+d = 387938059

This solution is smaller than found by Jacobi and Madden [3].

3.Quartic method

According to Piezas [5], if simultaneous equation (17),(18) has a rational solution for some rational number m, equation (15) has a rational solution.

$$a^4 + b^4 + c^4 + d^4 = (a+b+c+d)^4$$
$$a = p - 2q + r, b = p - 2q - r, c = q + s, d = q - s$$
$$(m^2 - 7)p^2 + 24pq - 24q^2 = (m^2 + 1)r^2$$
$$8mp^2 - 24mpq - 3(m^2 - 8m + 1)q^2 = (m^2 + 1)s^2$$
$$m = \frac{3q^2 + s^2}{p^2 - r^2}$$

Above equations are numbered as, (15),(16),(17),(18) & (19) resp.

First, we search one solution (p,q,r) using Sage's qfsolve and we obtain the parametric solution (p,q,r) of (17) for given m.

Substituting the parametric solution (p,q) to (18), we obtain the quartic equation.

$V^2 = c_4U^4 + c_3U^3 + c_2U^2 + c_1U + c_0$ (20)

We can find the rational solution of quartic equation using Sage's ratpoints (Michael Stoll). Before using ratpoints, we should check local solubility using Is locally solvable (X) in Magma.

Since the Hasse Principle no longer holds for genus one curve in general, we don't know quartic has rational points or not really.

After checking ELS, we search the rational points using ratpoints for given height.

We searched the rational points within the range of height(m) < 10000 and height(U) < 10^6. After we find the rational solution, quartic equation can be transformed to an elliptic curve and we can obtain infinitely many integer solutions of (15).

Theorem 2. $a^4 + b^4 + c^4 + d^4 = (a + b + c + d)^4$ has infinitely many solutions form $m = \frac{211}{150}$ and $m = \frac{2851}{1626}$

Proof.

In the case of $m = \frac{211}{150}$

Taking $m = \frac{211}{150}$ and a known solution (p,q) = (825,311), we can obtain a parametric solution of (17).

$$p = \frac{75(2458031 + 737231k^2 - 670210k)}{112979 + 67021k^2}$$

$$q = \frac{126281119 + 20843531k^2 - 50265750k}{112979 + 67021k^2}$$

$$r = \frac{-75(-564895 + 335105k^2 - 1215262k)}{112979 + 67021k^2}$$

Hence equation (18) becomes to

$$V^2 = 9471547265521339197853197k^4 + 22062657079770078500959500k^3 + 756411501601022852455206k^2 + 5173869366750430780465500k - 11494681792596210024563403 \quad (21)$$

Using ratpoints (Sage,Stoll) we found a rational point $Q(k,V) = (\frac{48623}{40973}, \frac{-12031903478184804180000}{1678786729})$.

Then (21) can be transformed to an elliptic curve (22).

$$Y^2 + XY = X^3 - X^2 + 1524020321143902735X - 752432065809125643921039075 \quad (22)$$

Using mwrank (Sage) we know an elliptic curve has rank ≥ 1.

Hence we can obtain infinitely many rational solutions of simultaneous equation (17),(18).

We obtain a small new solution using $Q(\frac{48623}{40973}, \frac{-12031903478184804180000}{1678786729})$

a = 1229559

b = -1022230

c = 1984340

d = -107110

e= a+b+c+d = 2084559

In the case of $m = \frac{2851}{1626}$

Taking $m = \frac{2851}{1626}$ and a known solution $(p,q) = \left(-\frac{1873}{385}, -\frac{332677}{313005}\right)$, we can obtain a parametric solution of (17).

Substituting the parameterized p and q into (18), then we obtain

$$V^2 = 413311767680253945386291181k^4 - 276219739811225906909842860k^3 - 981803770684311390957676434k^2 - 3957328185693703225615980k - 495567955372658516362812531 \quad (23)$$

Then (23) can be transformed to an elliptic curve below.

$$E : Y^2 + XY = X^3 - X^2 + 971928215887051428490754955X - 55329591449597546272216211300713415224779$$

Though we couldn't obtain the rank, one point

$$P(X,Y) = \frac{456381414199597952657263}{79727041}, \frac{3083172587090396226431500762632704024}{7118827499089}$$

was found.

According to the Nagell-Luts theorem, since the point P is a point of infinite order then we can obtain infinitely many rational points on E.

Hence we can obtain infinitely many rational solutions of simultaneous equation (17) , (18). We obtain a small new solution using $Q(k,V) = (\frac{-69781}{48917}, \frac{120905808289176428743368}{2392872889})$.

a = 2434795

b = -1945570

c = -1483582

d = -1858600

e = a+b+c+d = -2852957

The proof is completed.

4. MacLeod's results

Allan MacLeod found many solutions by solving simultaneous equation as follows :

$$(-c^2 - cd - d^2)t + (a + b)^2 + (a + b)(c + d) + (c + d)^2 + a^2 + ab + b^2 = 0$$

$$(b + c + d)(a + d + c)t - c^2 - cd - d^2 = 0$$

Table 1: Solutions of $a^4 + b^4 + c^4 + d^4 = (a + b + c + d)$

t	a	b	c	d
193/18	27385	48150	7590	-31764
511/450	-2634	955	5400	1770
619/450	27385	-31764	48150	7590
1651/126	955	5400	1770	-2634
1141/666	7590	27385	48150	-31764
2041/150	-1229559	-1984340	1022230	107110
1423/1098	955	1770	5400	-2634
31/6	53902630	2542025	35847220	-34122866
157/150	-841263	792940	44410	-3852350
181/150	-460945405	189854902	732896170	303742360

He also found the solutions by solving simultaneous equation using Tito Piezas's [5] method.

$$(t^2 - 7)p^2 + 24pq - 24q^2 = (t^2 + 1)r^2$$

$$8tp^2 - 24tpq - 3(t^2 - 8t + 1)q^2 = (t^2 + 1)s^2$$

Table 2: Solutions of $a^4 + b^4 + c^4 + d^4 = (a + b + c + d)^4$

t	a	b	c	d
499/474	3868630767650	895775733285	21271390911326	-4745425061560
511/150	-6714317914	994485789915	-698106854980	864417463190
3163/1350	-16515508578	10824551825	-15627586290	1711841340
18913/438	123140611690	446604426005	-96985017746	-25263498320
1213/438	106185491830	80795489585	146163232960	-149806955726
1963/150	662971279500154	309770790508565	85290604949260	-371936154165950
1651/126	115711769730	58931380645	10424211666	-64829623500

a = −11590249845869269057824863556535439476779628603513075
b = 12097338013880728917779953989473028810920897155225060
c = 3561881391291690403489592769705028154469958565069524
d = 11315459134997579304238981942203181424806814023773640
e = (a+b+c+d)

5. Solution family

According to Piezas [5], if simultaneous equation {(17),(18)} has a rational solution for some rational number m, equation (15) has a rational solution.

From equation (15),(16),(19), m is given below.

$$m = \frac{c^2 + cd + d^2}{(c + b + d)(a + c + d)}$$

We used the equation (24) to obtain m from a known solution a,b,c,d.

$$3597130^4 + 1493309^4 + 561760^4 + (-1953890)^4 = 3698309^4$$

$$m = \frac{211}{150}, \ \frac{361}{61}, \ \frac{2041}{150}, \ \frac{2191}{1891}$$

give above solution

$$841263^4 + (-792940)^4 + (-44410)^4 + 3852350^4 = 3856263^4$$

$$m = \frac{157}{150}, \ \frac{307}{7}, \ \frac{8467}{150}, \ \frac{8617}{8317}$$

give above solution. Jeremey Rouse[6] pointed out that $m1 = \frac{m+1}{m-1}$ is solvable if m is solvable.

For example, if $m = \frac{211}{150}$,then $m1 = \frac{m+1}{m-1} - \frac{361}{61}$.

Using $m = \frac{211}{150}$ and $m1 = \frac{361}{61}$, we obtain same solution as follows.

$(-107110)^4 + 1984340^4 + 1229559^4 + (-1022230)^4 = 2084559^4$

When $m = \frac{211}{150}$,simultaneous equations {(17),(18)} is transformed to elliptic curve

$Y^2 + XY = X^3 - X^2 + 1524020321143902735X - 752432065809125643921039075$

Similarly, when m = $\frac{361}{61}$ we obtain

$Y^2 + XY = X^3 - X^2 + 1524020321143902735X - 752432065809125643921039075$

In this way, when $m = \frac{211}{150}$ *and* $m = \frac{361}{61}$ we obtain same elliptic curve

We summarized the small solutions found until now.

Table3: Small solutions

m	a	b	c	d	e	Finder
2521/325	5400	-2634	1770	955	5491	Brudno[1]
1807/475	48150	-31764	27385	7590	51361	JW
211/150	1229559	-1022230	1984340	-107110	2084559	ST
2851/1626	2434795	-1945570	-1483582	-1858600	-2852957	ST
211/150	561760	1493309	3597130	-1953890	3698309	ST
157/150	841263	-792940	-44410	3852350	3856263	JR
31/6	39913670	-23859495	15187700	10116014	41357889	JR
31/6	53902630	2542025	35847220	-34122866	58169009	JR
1069/169	378573600	145514934	65167315	-201317790	387938059	ST
3523/2623	336869940	-178944510	-210240721	-396470430	-448785721	ST
331/31	732896170	303742360	189854902	-460945405	765548027	ST
1159/259	753684930	294589950	558360120	-701876813	904758187	ST
961/61	1338058950	-89913570	504106884	-404747255	1347505009	JM
1159/259	500764020	1768211850	1297734853	-1510410870	2056299853	ST
961/61	3095408880	1655829870	-157072326	-1406590625	3187575799	JM
2977/2502	719130355	-2889516060	4672341330	2405612802	4907568427	ST
1807/475	4625798910	46140636	3744195265	-3172936050	5243198761	JM
3163/1350	-16515508578	10824551825	-15627586290	1711841340	-19606701703	AM
157/150	-7784423350	-2943361793	10692790190	-49328431840	-49363426793	ST
373/150	50627178820	1357751663	55867457830	-41572821650	66279566663	ST
1651/126	115711769730	-64829623500	10424211666	58931380645	120237738541	JM
1213/438	106185491830	80795489585	146163232960	-149806955726	183337258649	AM
18913/438	123140611690	446604426005	-96985017746	-25263498320	447496521629	AM
157/150	-35835675310	168853510327	-134075405440	-644955984250	-646013554673	ST
511/150	-6714317914	994485789915	-698106854980	864417463190	1154082080211	AM
1807/475	1058103081810	535945811334	-1140105961325	944080652640	1398023584459	ST
217/25	-259448373800	-1526478290216	889698809680	-687020381505	-1583248235841	ST
331/31	-150723250810	1751113229630	802797814305	-626137906588	1777049886537	ST
499/474	3868630767650	895775733285	21271390911326	-4745425061560	21290372350701	AM
31/6	-29175553438600	-5059968816155	21271610809130	-19158210038746	-32122121484371	ST
31/6	-34998027446475	-32309023830920	42457132181770	-25125873749306	-49975792844931	ST
217/25	237321095011880	-558974521862416	-22424373335225	-222795507072280	-566873307258041	ST
331/31	530920858665230	377970149282480	35966749745415	-360346958398438	584510799294687	ST
1963/150	662971279500154	309770790508565	85290604949260	-371936154165950	686096520792029	AM
211/150	1619214280915810	-3065097088334400	2940554547668260	2243306697670930	3737978437920590	ST

where e=(a+b+c+d)

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