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RESEARCH ARTICLE

# Equal Sums of sixth powers with (w+1) terms (where 'w' is an integer)

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**ABSTRACT**

Equation, $(x^6 + w * y^6) = (u^6 + w * v^6)$ has been studied by Ajai Choudhry (ref #1). In that paper he gave an example for w=12. In this paper the authors have done analysis & given results for many different values of the constant 'w' in the above equation. In the four sections of the paper the authors have demonstrated solutions by using algebraic & brute force method & also a combination of both methods.

Equation: $(x^6 + w * y^6) = (u^6 + w * v^6)$

[where, (x,y,u,v) are variables & 'w' is a constant]

Consider the equation, $(a^6 - b^6) = w(c^6 - d^6)$ (1)

**Section -1:**

$a = (c - d)(3c - 2d)$

$b = (c - d)(2c - 3d)$

$w = 5p(c - d)^4$

where, $p = (7c^2 - 11cd + 7d^2)$

After substituting in (1) & simplifying we get a five factor expression:

$0 = ((2c - d)(c - 2d)(c + d)(7c^2 - 11cd + 7d^2)(3c^2 - 5cd + 3d^2)$

For, (c-2d)=0 we get, c=2d & the below numerical solution:

For, c=4, d=2 we get:

(a,b,c,d)=(8,2,4,2) & w=4160

$(8^6 - 2^6) = 4160 * (4^6 - 2^6)$

**Method (B),**

$(a^6 - b^6) = w(c^6 - d^6)$ − − − −(1)

$p^3(a^6 - b^6) = (c^6 - d^6)$ − − − (2)

We take:

$(c, d, p) = [(a^2 + 3ab - b^2), (a^2 - 3ab - b^2), (2a^2 + ab)]$

And we get the below seven factors from eqn (2),

$0 = a(a - 6b)(a + b)(a - b)(a^2 - ab + b^2)(a^2 + ab + b^2) *$

$(8a^4 + 24a^3b + 15a^2b^2 + ab^3 + 6b^4)$ − − − (3)

Above has solution at, (a,b)=(6,1) & we get:

$(53^6 - 17^6) = (78)^3 * (6^6 - 1^6)$

$(a^6 - b^6) = w(c^6 - d^6)$

If we take, $w = (78)^3$ then we get:

$(a, b, c, d) = (53,17,6,1)$

**Method (C):**

$(a^6 - b^6) = w(c^6 - d^6)$ − − − −(1)

We take:

$(a, b, c, d) = [(8q - 3p), (3p), (5q), (5q - 3p)]$

After substituting in (1) we get:

& we get five factors expression:

$0 = pq(4p - 3q)(2916p^6 - 17253p^5q + 55998p^4q^2 - 94419p^3q^3 + 69048p^2q^4 + 3125pq^5 - 18750q^6)$

Above is satisfied at (p,q)=(3,4) & we get:

$3^3(23^6 - 9^6) = 4^3(20^6 - 11^6)$ − − − (3)eqn (3) can also be written as:

$3^3 * (23^6 - 9^6) = (40^6 - 22^6)$ − − − (4)

If we take, $w = 3^3 = 27$ then we have:

$$(a^6 - b^6) = w(c^6 - d^6)$$

(a,b,c,d)=(40,22,23,9) & w=27

$$(40^6 - 22^6) = 27 * (23^6 - 9^6)$$

**Section 2:**

$$(a^6 - b^6) = w(c^6 - d^6)$$

Table (1) see below:

| w | a | b | c | d |
|---|---|---|---|---|
| 12 | 64 | 10 | 43 | 29 |
| 27 | 40 | 22 | 23 | 9 |
| 65 | 4 | 1 | 2 | 1 |
| 73 | 47 | 15 | 23 | 9 |
| 97 | 11 | 8 | 5 | 2 |
| 183 | 5 | 4 | 2 | 1 |
| 199 | 164 | 163 | 47 | 44 |
| 215 | 22 | 3 | 9 | 4 |
| 247 | 5 | 2 | 2 | 1 |
| 248 | 5 | 1 | 2 | 1 |
| 296 | 8 | 6 | 3 | 1 |
| 297 | 10 | 1 | 4 | 3 |
| 360 | 8 | 2 | 3 | 1 |
| 473 | 17 | 16 | 5 | 2 |
| 494 | 134 | 88 | 47 | 1 |
| 608 | 58 | 46 | 19 | 5 |
| 712 | 132 | 46 | 45 | 31 |
| 730 | 9 | 1 | 3 | 1 |
| 768 | 128 | 20 | 43 | 29 |
| 793 | 9 | 4 | 3 | 2 |
| 824 | 61 | 25 | 20 | 11 |
| 872 | 34 | 4 | 11 | 1 |
| 900 | 433 | 367 | 129 | 25 |

**Section -3:**

$$X^6 + w * U^6 = y^6 + w * V^6$$

$let\ (X, Y, U, V) = (dn^2 + an + b, dn^2 - an + b, cn - 1, cn + 1)$

Hence,

$$w = \frac{\{X6 - Y6\}}{\{V6 - U6\}}$$

$$w = \frac{[a(dn^2 + b)(3b^2 + a^2n^2 + 6dn^2b + 3d^2n^4)(b^2 + 3a^2n^2 + 2dn^2b + d^2n^4)]}{m}$$

where, $m = c(c^2n^2 + 3)(3c^2n^2 + 1)$

For (a,b,c,d)=(8,3,1,1) we get:

$(x, y, u, v) = [(n^2 + 8n + 3), (n^2 - 8n + 3), (n - 1), (n + 1)]$ &

$w = 8(n^2 + 27)(n^4 + 198n^2 + 9)$

For other suitable parameters (a,b,c,d) we get the other formulas as shown in the table (2) below for different values of 'w'.

$$x^6 + w * u^6 = y^6 + w * v^6$$

$$where: (u, v) = [(n - 1), (n + 1)]$$

$(w, x, y)$ is *listed below*:

$8(n^2 + 27)(n^4 + 198n^2 + 9)$, $(n^2 + 8n + 3), (n^2 - 8n + 3)$

$18(n^2 + 1)(27n^4 + 58n^2 + 27)$, $(3n^2 + 2n + 3), (3n^2 - 2n + 3)$

$4(3n^2 + 5)(27n^2 + 25)(3n^2 + 25)$, $(3n^2 + 4n + 5), (3n^2 - 4n + 5)$

$8(27n^2 + 1)(9n^4 + 198n^2 + 1)$, $(3n^2 + 8n + 1), (3n^2 - 8n + 1)$

$72(n^2 + 27)(27n^4 + 226n^2 + 243)$, $(3n^2 + 8n + 9), (3n^2 - 8n + 9)$

$4(25n^2 + 27)(5n^2 + 3)(25n^2 + 3)$, $(5n^2 + 4n + 3), (5n^2 - 4n + 3)$

$72(27n^2 + 1)(243n^4 + 226n^2 + 27)$, $(9n^2 + 8n + 3), (9n^2 - 8n + 3)$

$8(9n^4 - 21n^2 + 64)(27n^6 + 18n^4 + 3n^2 + 64)$, $(3n^3 + n + 8), (3n^3 + n - 8)$

$4(3n^2 + 5)(9n^4 + 3n^2 + 16)(9n^4 + 27n^2 + 16)$, $3n^3 + 5n + 4, 3n^3 + 5n - 4$

$2(3n^2 + 7)(3n^2 + 4)(27n^6 + 126n^4 + 147n^2 + 4)$, $3n^3 + 7n + 2, 3n^3 + 7n - 2$

$18(3n^2 + 7)(3n^2 + 4)(9n^6 + 42n^4 + 49n^2 + 108)$, $3n^3 + 7n + 6, 3n^3 + 7n - 6$

$72(9n^4 + 51n^2 + 64)(3n^6 + 18n^4 + 27n^2 + 64)$, $3n^3 + 9n + 8, 3n^3 + 9n - 8$

*Note*: $(x$ & $y)$ *are of the* form, $(p + q)$ & $(p - q)$

**Section -4**

$$a^6 + w * b^6 = c^6 + w * d^6 \text{ ---(1)}$$

We take:

$$a = -s\alpha + t\beta,\ b = s - t,\ c = s\alpha + t\beta,\ d = -s - t$$

Referring to Choudhry's paper (ref #1) we take:

$(a0, a1, a2, a3, a4, a5, a6)$ = ( 1, 0, 0, 0, 0, 0, w ) & we get:

$$\emptyset_1\ (\alpha, \beta) = 6\alpha^5\beta + 6w$$

$$\emptyset2\ (\alpha, \beta) = 20(\alpha^3\beta^3) + 20w$$

$$\emptyset_3(\alpha, \beta) = 6\alpha\beta^5 + 6w$$

$$\emptyset_4\ (a, \beta) = 2\emptyset_1[(-\emptyset_2) \pm ((\emptyset_2)^2 - 4\emptyset_1\emptyset_3)^{\frac{1}{2}}]$$

Where:

$$s = [\emptyset_4(\alpha, \beta)]^{1/2}$$

$$t = 2(\emptyset_1\alpha, \beta)$$

We take,($\emptyset_2 - 4\emptyset_1\emptyset_3$) as a perfect square, to obtain a rational solution:

Table 3: Solutions of above equation (1):

$$a^6 + w * b^6 = c^6 + w * d^6$$

| w | a | b | c | d | α | β |
|---|---|---|---|---|---|---|
| | | | | | | |
| 12 | 64 | 29 | 10 | 43 | 3/4 | -37/7 |
| | | | | | | |
| 12 | 64 | 29 | 10 | 43 | 37/36 | -27/7 |
| | | | | | | |
| 27 | 40 | 9 | 22 | 23 | 9/7 | -31/16 |
| | | | | | | |
| 27 | 69 | 22 | 27 | 40 | 21/31 | -16/13 |

Note: Alpha & beta (α,β) values were arrived at by solving above simultaneous equations for [$\emptyset_1$ *to* $\emptyset_4$ ] by brute force. Thus we get the below numerical solutions:

$$64^6 + 12 * (29)^6 = 10^6 + 12 * (43)^6$$

$$40^6 + 27 * (9)^6 = 22^6 + 27 * (23)^6$$

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