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Short communication

Apostol-Robbins theorem applied to several arithmetic functions

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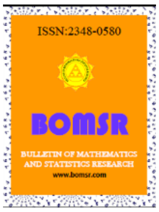

ABSTRACT

The study of numbers has fascinated humans for more than two millennia and it continues to do so forever. Over the period of time, several mathematicians have contributed enormously to the growth of number theory as we know today. Arithmetic functions play a significant role in understanding of numbers. Divisors function, Sum of divisors function, Euler's totient function, Liouville function, Mobius function are examples of multiplicative arithmetic functions in the sense that the product of the image is image of the product for two co-prime numbers. In this short paper, we have applied Apostol-Robbins theorem to several multiplicative arithmetic functions.

Keywords: Möbius function, Euler's totient function, Sum of divisors function, Liouville function.

1. Introduction

We have the following result for any multiplicative arithmetic function [1, 2]:

$$\sum_{d|n}(-1)^d f\left(\frac{n}{d}\right)=\left[\sum_{j=0}^{k-1} f(2^j)-f(2^k)\right]\sum_{d|m} f(d) \quad (1)$$

where $n = 2^k m$, $k \geq 0$, m is odd.

We shall apply (1) to several arithmetic functions of interest in number theory.

2. Applications of (1)

a) If $f(n) = I(n) = n$ [3], then (1) implies the property [2, 4]:

$$\sum_{d|n}(-1)^{d-1}\frac{n}{d}=\sum_{d|m} d=\sigma_0(n)=\sigma(m) \quad (2)$$

where $\sigma_O(n)$ is the sum of all odd divisors of n, and $\sigma(m)$ is the sum of all divisors of m [5].

b) If $f(n) = \varphi(n)$ is the Euler's totient function (n) [1, 3], then (1) gives the following Liouville's result [2, 3, 6]:

$$\sum_{d|n}(-1)^d\varphi\left(\frac{n}{d}\right)=\begin{cases}-m, & k=0\\ 0, & k\geq 1\end{cases} \quad (3)$$

hence:

$$\sum_{even\ d|n}\varphi\left(\frac{n}{d}\right)=\sum_{odd\ d|n}\varphi\left(\frac{n}{d}\right),\quad k\geq 1 \quad (4)$$

c) For the sum of divisors function and the divisor function [3], using (1), we obtain the expressions:

$$\sum_{d|n}(-1)^d\sigma\left(\frac{n}{d}\right)=-(k+1)\sum_{d|m}\sigma(d)\ \ and\ \ \sum_{d|n}(-1)^d\,\mathrm{d}\left(\frac{n}{d}\right)=\frac{1}{2}(k+1)(k-2)\sum_{d|m}\mathrm{d}(d) \quad (5)$$

d) If $f(n) = \lambda(n) =$ Liouville's function (n) [3, 7], then from (1) we obtain

$$\sum_{d|n}(-1)^d\lambda\left(\frac{n}{d}\right)=\begin{cases}\dfrac{1-3(-1)^k}{2}, & if\ m\ is\ a\ square\\ 0, & otherwise\end{cases} \quad (6)$$

Thus:

$$\sum_{even\ d|n}\lambda\left(\frac{n}{d}\right)=\sum_{odd\ d|n}\lambda\left(\frac{n}{d}\right),\quad m\neq square \quad (7)$$

e) If $f(n) = d^*(n) =$ Number of unitary divisors of n [3, 6], hence (1) gives the relation:

$$\sum_{d|n}(-1)^d\,\mathrm{d}^*\left(\frac{n}{d}\right)=\begin{cases}-d(m^2), & k=0\\ (2k-3)d(m^2), & k\geq 1\end{cases} \quad (8)$$

f) If $f(n) = \mu(n) =$ Möbius function (n) [1, 3, 8-10], then (1) implies the property:

$$\sum_{d|n}(-1)^d\mu\left(\frac{n}{d}\right)=\begin{cases}-e_0(m), & k=0\\ 2e_0(k)e_0(m), & k\geq 1\end{cases} \quad (9)$$

g) If $f(n) = \mu^*(n) =$ Unitary analogue of the Möbius function [3, 6], thus from (1) we obtain

$$\sum_{d|n}(-1)^d \mu*\left(\frac{n}{d}\right) = \begin{cases} -\sum_{d|m}\mu*(d) & , \quad k=0 \\ (3-k)\sum_{d|m}\mu*(d), & k \geq 1 \end{cases} \qquad (10)$$

3. **Conclusion**

Upon describing Apostol – Robbins theorem in (1), we have used it for seven multiplicative functions and obtained nice results provided through equations (2), (4), (5), (7), (8), (9) and (10). These results provide ways for much more deeper understanding of the arithmetic functions upon the application of Apostol – Robbins Theorem.

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