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SHORT COMMUNICATION



PARTITIONS AND EULER'S TOTIENT FUNCTION

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ABSTRACT

Merca-Schmidt obtained an interesting expression for $p(n)$ involving the values of the Euler's totient function. Here we show an alternative approach to write the number of partitions of n in terms of $\varphi(k)$.

Keywords: Partition function, Complete Bell polynomials, Euler's totient function.

1.- Introduction

Merca-Schmidt [1] deduced the following relation for the number of partitions of n in terms of the Euler's totient function [2]:

$$p(n) = \frac{1}{2} \sum_{k=3}^{n+3} \varphi(k) S_{n+3,k}^{(3)}, \quad n \geq 0, \quad (1)$$

where $S_{m,k}^{(r)}$ is the number of k 's in the partitions of m with the smallest part at least r .

In the next Section we use the connection between the divisor, partition, sum of divisors and Euler's totient functions to show an alternative procedure to (1).

2.- $p(n)$ in terms of $\varphi(k)$

We know [3] how to obtain $p(n)$ if we have the values of $\sigma(n)$, in fact:

$$n! p(n) = B_n(0! \sigma(1), 1! \sigma(2), 2! \sigma(3), \dots, (n-1)! \sigma(n)), \quad (2)$$

in terms of the complete Bell polynomials [4, 5]. On the other hand, the sum of divisors function is the Dirichlet convolution [2] of the divisor function and the Euler's totient function, that is:

$$\sigma(n) = \sum_{d|n} \varphi(n) d \left(\frac{n}{d}\right). \quad (3)$$

Hence the combination of (2) and (3) allows to deduce that:

$$\begin{aligned} p(1) &= \varphi(1), & 2! p(2) &= 3 \varphi(1) + \varphi(2), & 3! p(3) &= 11 \varphi(1) + 3 \varphi(2) + 2 \varphi(3), \\ 4! p(4) &= 59 \varphi(1) + 33 \varphi(2) + 8 \varphi(3) + 6 \varphi(4), \end{aligned} \quad (4)$$

$$5! p(5) = 339 \varphi(1) + 185 \varphi(2) + 80 \varphi(3) + 30 \varphi(4) + 24 \varphi(5),$$

$$6! p(6) = 2629 \varphi(1) + 1995 \varphi(2) + 880 \varphi(3) + 360 \varphi(4) + 144 \varphi(5) + 120 \varphi(6), \quad \text{etc.}$$

In (4) we observe the following structure:

$$1! p(1) = \mathbf{1} \cdot \varphi(1), \quad 2! p(2) = \mathbf{3} \cdot \varphi(1) + \dots, \quad 3! p(3) = \mathbf{11} \cdot \varphi(1) + \dots, \quad (5)$$

$$4! p(4) = \mathbf{59} \cdot \varphi(1) + \dots, \quad 5! p(5) = \mathbf{339} \cdot \varphi(1) + \dots, \quad 6! p(6) = \mathbf{2629} \cdot \varphi(1) + \dots,$$

where it appears the sequence A028342: 1, 3, 11, 59, 339, 2629, ... , which is in the known link <http://oeis.org>, in fact:

$$1, 3, 11, 59, 339, 2629, 20\ 677, 202\ 089, 2\ 066\ 201, 24\ 322\ 931, \dots \quad (6)$$

then we hope that $7! p(7) = \mathbf{20\ 677} \cdot \varphi(1) + \dots$, that is:

$$n! p(n) = a(n) \varphi(1) + \dots, \quad a(1) = 1, \quad a(2) = 3, \quad a(3) = 11, \quad a(4) = 59, \quad a(5) = 339, \dots \quad (7)$$

where:

$$a(n) = (n-1)! \sum_{r=1}^n d(r) \frac{a(n-r)}{(n-r)!} = B_n(0! d(1), 1! d(2), 2! d(3), \dots, (n-1)! d(n)), \quad a(0) = 1. \quad (8)$$

On the other hand, we know [2] that the Euler's totient function has connection with the Möbius function:

$$\varphi(n) = n \sum_{d|n} \frac{\mu(d)}{d}, \quad (9)$$

then the expressions (4) are equivalent to:

$$\begin{aligned} 1! p(1) &= \mu(1), & 2! p(2) &= 5 \mu(1) + \mu(2), & 3! p(3) &= 23 \mu(1) + 3 \mu(2) + 2 \mu(3), \\ 4! p(4) &= 173 \mu(1) + 45 \mu(2) + 8 \mu(3) + 6 \mu(4), \end{aligned} \quad (10)$$

$$5! p(5) = 1189 \mu(1) + 245 \mu(2) + 80 \mu(3) + 30 \mu(4) + 24 \mu(5),$$

$$6! p(6) = 12\ 139 \mu(1) + 3\ 075 \mu(2) + 1120 \mu(3) + 360 \mu(4) + 144 \mu(5) + 120 \mu(6), \dots,$$

thus the values of the Möbius function determine the number of partitions of n .

Finally, the results (4) lead to propose the following relation:

$$n! p(n) = a(n) \varphi(1) + \cdots + n \cdot (n-2)! \varphi(n-1) + (n-1)! \varphi(n). \quad (11)$$

References

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