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SHORT COMMUNICATION

PARTITIONS AND EULER'S TOTIENT FUNCTION

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ABSTRACT

Merca-Schmidt obtained an interesting expression for $p(n)$ involving the values of the Euler's totient function. Here we show an alternative approach to write the number of partitions of n in terms of $\varphi(k)$.

Keywords: Partition function, Complete Bell polynomials, Euler's totient function.

1.- Introduction

Merca-Schmidt [1] deduced the following relation for the number of partitions of n in terms of the Euler's totient function [2]:

$$p(n) = \frac{1}{2}\sum_{k=3}^{n+3} \varphi(k)\, S_{n+3,k}^{(3)}, \quad n \geq 0, \tag{1}$$

where $S_{m,k}^{(r)}$ is the number of k's in the partitions of m with the smallest part at least r.

In the next Section we use the connection between the divisor, partition, sum of divisorsand Euler's totient functions to show an alternative procedure to (1).

2.- $p(n)$ in terms of $\varphi(k)$

We know [3] how to obtain $p(n)$if we have the values of $\sigma(n)$, in fact:

$$n!\ p(n) = B_n\big(0!\ \sigma(1), 1!\ \sigma(2), 2!\ \sigma(3), \dots, (n-1)!\ \sigma(n)\big), \tag{2}$$

in terms of the complete Bell polynomials [4, 5]. On the other hand, the sum of divisors function is the Dirichlet convolution [2] of the divisor function and the Euler's totient function, that is:

$$\sigma(n) = \sum_{d/n} \varphi(n)\, d\left(\frac{n}{d}\right). \tag{3}$$

Hence the combination of (2) and (3) allows to deduce that:

$$p(1) = \varphi(1), \quad 2!\ p(2) = 3\,\varphi(1) + \varphi(2), \quad 3!\ p(3) = 11\,\varphi(1) + 3\,\varphi(2) + 2\,\varphi(3),$$

$$4!\ p(4) = 59\,\varphi(1) + 33\,\varphi(2) + 8\,\varphi(3) + 6\,\varphi(4), \tag{4}$$

$$5!\ p(5) = 339\,\varphi(1) + 185\,\varphi(2) + 80\,\varphi(3) + 30\,\varphi(4) + 24\,\varphi(5),$$

$$6!\, p(6) = 2629\,\varphi(1) + 1995\,\varphi(2) + 880\,\varphi(3) + 360\,\varphi(4) + 144\,\varphi(5) + 120\,\varphi(6), \qquad etc.$$

In (4) we observe the following structure:

$$1!\, p(1) = \mathbf{1} \cdot \varphi(1), \quad 2!\, p(2) = \mathbf{3} \cdot \varphi(1) + \cdots, \quad 3!\, p(3) = \mathbf{11} \cdot \varphi(1) + \cdots, \tag{5}$$

$$4!\, p(4) = \mathbf{59} \cdot \varphi(1) + \cdots, \quad 5!\, p(5) = \mathbf{339} \cdot \varphi(1) + \cdots, \quad 6!\, p(6) = \mathbf{2629} \cdot \varphi(1) + \cdots,$$

where it appears the sequence A028342: 1, 3, 11, 59, 339, 2629, … ,which is in the known link http://oeis.org , in fact:

1, 3, 11, 59, 339, 2629, 20 677, 202 089, 2 066 201, 24 322 931, …. (6)

then we hope that $7!\, p(7) = \mathbf{20\,677} \cdot \varphi(1) + \cdots$, that is:

$$n!\ p(n) = a(n)\, \varphi(1) + \cdots, \quad a(1) = 1,\ a(2) = 3,\ a(3) = 11,\ a(4) = 59,\ a(5) = 339, \dots \tag{7}$$

where:

$$a(n) = (n-1)!\ \sum_{r=1}^{n} d(r) \frac{a(n-r)}{(n-r)!} = B_n\big(0!\, d(1), 1!\, d(2), 2!\, d(3), \dots, (n-1)!\, d(n)\big), \quad a(0) = 1. \tag{8}$$

On the other hand, we know [2] that the Euler's totient function has connection with the Möbius function:

$$\varphi(n) = n \sum_{d/n} \frac{\mu(d)}{d}, \tag{9}$$

then the expressions (4) are equivalent to:

$$1!\, p(1) = \mu(1), \quad 2!\, p(2) = 5\,\mu(1) + \mu(2), \quad 3!\, p(3) = 23\,\mu(1) + 3\,\mu(2) + 2\,\mu(3),$$

$$4!\, p(4) = 173\,\mu(1) + 45\,\mu(2) + 8\,\mu(3) + 6\,\mu(4), \tag{10}$$

$$5!\, p(5) = 1189\,\mu(1) + 245\,\mu(2) + 80\,\mu(3) + 30\,\mu(4) + 24\,\mu(5),$$

$$6!\, p(6) = 12\,139\,\mu(1) + 3\,075\,\mu(2) + 1120\,\mu(3) + 360\,\mu(4) + 144\,\mu(5) + 120\,\mu(6), \dots,$$

thus the values of the Möbius function determine the number of partitions of n .

Finally, the results (4) lead to propose the following relation:

$$n!\ p(n) = a(n)\ \varphi(1) + \cdots + n \cdot (n-2)!\ \varphi(n-1) + (n-1)!\ \varphi(n). \tag{11}$$

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