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RESEARCH ARTICLE

# APPROXIMATE CONTROLLABILITY OF SECOND ORDER FRACTIONAL NEUTRAL INTEGRO-DIFFERENTIAL INCLUSIONS WITH STATE-DEPENDENT DELAY IN BANACH SPACES

**A. Stephan Antony Raj[1], M. Muthuchelvam[2], A. Anuradha[3]**

[1]Associate professor, Department of Mathematics, Rathinam Group of Institutions, Kovilpalayam Campus, Coimbatore - 641 107, Tamil Nadu, India.
Email: stephanraj138@gmail.com

[2]Assistant professor, Department of Mathematics, Govt College of Technology, Coimbatore - 641 013, Tamil Nadu, India. Email: muthuthe1@gmail.com

[3]Assistant professor, PG & Research Department of Mathematics, Sri Ramakrishna college of arts & science, Coimbatore - 641 006, Tamil Nadu, India. Email: anuradha@srcas.ac.in
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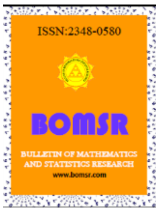


**ABSTRACT**

The primary objective of the present research is on the approximate controllability of mild solutions for fractional neutral integro-differential inclusions with state-dependent delay in Banach spaces. Applying the Dhage-derived fixed point theorem for multi-valued operators. With the assistance of the highly continuous -order fractional cosine family, we put up the existence result. The theoretical results are finally expressed through a concrete instance.

Keywords: Neutral fractional integro-differential inclusions, α-order cosine family, fixed point theorem.

2010 Mathematics Subject Classification: 34K09, 34K37, 34K40.

## 1 Introduction

Differential equations are used in many scientific fields to characterise physical processes mathematically. However, it has recently been demonstrated that due to material and inherited characteristics, the majority of these models may be better described by fractional order equations (or inclusions). As a result, fractional order differential systems have several applications (see [16, 17, 22, 31]).

Delay differential equations allow the incorporation of past activities into mathematical models, in contrast to ordinary differential equations. A delay differential equation with discrete delay is often presented in the type

$$\dot{x}(t) = f(t, x(t), x(t - \tau)) \tag{1.1}$$

with $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Depending on the complexity of the problem, the delay $\tau$ may be a constant value ($\tau \geq 0$), a function of the time ($\tau(t) \geq 0$), or a function of the solution $x$ itself ($\tau(x(t)) \geq 0$). Accordingly, equation (1.1) is called a differential equation with constant delay, time-dependent delay, or state-dependent delay, respectively. For more details refer [5, 16, 19]. When the right-hand side of the problem depends not only on the history of the solution $x$, but also on the history of the derivative $\dot{x}$, that is,

$$\dot{x}(t) = g(t, x(t), x(t - \tau), \dot{x}(t - \tau)),$$

we have a neutral delay differential equation or neutral functional differential equation [6, 14].

Fractional differential equations with state-dependent delays are a prevalent sort of problem, and as a result, study of this type of equation has attracted a lot of attention in recent years. We recommend the reader consult the handbook by Canada et al. [9] and the scientific papers for further information on differential equations with state-dependent delay and their applications, see [1, 2, 4, 6, 7, 14, 19].

Abstract linear second order differential equations are linked to the idea of the cosine function. We advise the reader to review Fattorini [12] and Travis and Webb [25] for the fundamental principles and applications of this approach.

The most active field of research is on the existence, controllability, and other qualitative and quantitative elements of fractional differential systems; in specific, see [3, 17, 23, 24, 28, 29].

Santos et al. investigated in [3] whether fractional integro-differential equations with unbounded delay had solutions in Banach spaces. The authors of [28, 29] showed the existence and roughly controllability of stochastic differential systems with indefinite latency and fractional order neutral differentials. By utilising the suitable fixed point theory, Sakthivel et al. [23] recognised the approximative controllability of the fractional dynamical system.

The authors recently used the appropriate fixed point theorem to explain the approximation controllability results for fractional neutral integro-differential systems with state-dependent delay in [26, 30]. The existence findings for fractional differential equations with nonlocal conditions of order $\alpha \in (1, 2)$ were examined by Shu and Wang et al. [24]. The controllability of nonlocal fractional differential equations of order $\alpha \in (1, 2]$ was later examined by Kexue et al. [17]. Recent research on optimum controllers for fractional stochastic functional differential equations of rank $\alpha \in (1, 2]$ in Hilbert spaces was conducted by Yan and Jia [27].

However, existence results for fractional neutral integro-differential inclusions with state-dependent delay in $B$ phase space adages have not yet been fully investigated.

Motivated by the above mentioned works [17, 24, 27], in this manuscript, we are concerned with the existence of mild solutions for fractional neutral differential inclusions with state-dependent delay of the form

$$ {}^{C}D_t^{\alpha}\,[y(t) + k(t, y_t)] \in A[y(t) + k(t, y_t)] + F(t, y_{\rho(t,y_t)}) + Bu(t), \quad t \in Q = [0, T], \tag{1.1}$$

$$ y(t) = \phi(t), \qquad t \in (-\infty, 0], \quad y'(0) = x_0 \in H, \tag{1.2}$$

where ${}^{C}D_t^{\alpha}$ is the Caputo's fractional derivatives of order $\alpha \in (1, 2]$. $A$ is the infinitesimal generator of a strongly continuous $\alpha$-order cosine family $\{C_\alpha(t)\}_{t\geq 0}$ on $H$, the state $y(\cdot)$ takes values in $H$, the control function $u(\cdot)$ is given in $L^2([0, T]; U)$, a Banach space of admissible control functions, with $U$ a Banach space, $B$ is a bounded linear from $U$ into $H$. The function $k : Q \times \mathcal{B} \to H$ is a continuous function and $F : Q \times \mathcal{B} \to P(H)$ is a multi-valued map. For any function $y$ described on $(-\infty, T]$ and any $t \in Q$, we represent by $y_t$ the element of $\mathcal{B}$ described by $y_t(\vartheta) = y(t + \vartheta)$, $\vartheta \in (-\infty, 0]$. Here $y_t$ represents the history of the state up to the current time $t$ and $\rho : Q \times \mathcal{B} \to (-\infty, \infty)$ is an apposite function. $\mathcal{B}$ is the theoretical phase space axioms characterized in Section 2.

Further, we additionally consider the subsequent fractional neutral integro-differential inclusions with state-dependent delay of the form

$$ {}^{C}D_t^{\alpha}[y(t) + k\left(t, y_t, \int_0^t K_1(t,s,y_s)\right)ds)] \in A[y(t) + k\left(t, y_t, \int_0^t K_1(t,s,y_s)\right)ds)] + F\left(y_p(t,y_t)\ \int_0^t K_2(t,s,y_{p(s,,y_s)})\right)ds) + Bu(t), t \in Q = [0,T], \tag{1.3}$$

$$ y(o) = \phi(t), \quad t \in (-\infty, 0], \quad y' = x_0 \in H, \tag{1.4}$$

where ${}^{C}D_t^{\alpha}$, $A$, $\mathcal{B}$, $\rho$, $B$, $u$, $\phi$ and $y_0$ are same as defined in (1.1) - (1.2). Further $K_i : Q \times Q \times \mathcal{B} \to H$, (for $i = 1, 2$) and $k_t : Q \times \mathcal{B} \times H \to H$ are continuous functions and $F : J \times \mathcal{B} \times H \to P(H)$ is a multi-valued map.

This is way the manuscript was put together. In the second section, we review a number of findings, definitions, and lemmas that will be applied later to support our key findings. The primary conclusions in the third portion are based on Dhage's fixed point theorem. An instance of the application is provided in the final section to illustrate how well our key findings worked.

## 2 Preliminaries

In this part, we remember some basic definitions, lemmas and notations which will be utilized all through this manuscript. Let $H$ be a Banach space. By $C(Q, H)$ we denote the Banach space of continuous functions from $J$ into $H$ with norm

$$\| y \| = sup\{|y| : t \in Q\}.$$

$B(H)$ denotes the Banach space of all bounded linear operators from H into H, with the norm

$$\|N\|_{B(H)} = \sup\{|N(y)| : |y| = 1\}.$$

$L^1(Q, H)$ denotes the Banach space of measurable functions $y : Q \to H$ which are Bochner integrable, normed by

$$\| y \|_{L^1} = \int_0^T [y(t)]\, dt$$

$L^\infty(Q, H)$ denotes the Banach space of measurable functions $y : Q \to H$ which are bounded equipped with the norm

$$\|y\|_{L^\infty} = \inf\{d > 0 : \|y(t)\| < d, \quad \text{a.e} \quad t \in Q\}.$$

We denote the notation P($H$) is the family of all subsets of $H$. Next, we denote the subsequent notations:

$P_{cl}(H) = \{x \in P(H) : x \text{ is closed}\}$, $P_{bd}(H) = \{x \in P(H) : x \text{ is bounded}\}$, $P_{cv}(H) = \{x \in P(H) : x \text{ is convex}\}$, $P_{cp}(H) = \{x \in P(H) : x \text{ is compact}\}$.

The definitions of multi-valued analysis like, convexity, bounded, upper semi-continuous, completely continuous and closed graph theorem are well-known results[10, 13, 15], for this reason, we omit here.

Here, we will utilize a common axioms for the phase space $\mathcal{B}$ which is identical to those presented by Hale and Kato. In particular, $\mathcal{B}$ will be a linear space of function mapping $(-\infty, 0]$ into $H$ endowed with a semi norm $\|\cdot\|_{\mathcal{B}}$ and fulfills the next conditions:

(i) $y \in (-\infty, T] \to H$ is continuous on $Q$ and $y_0 \in \mathcal{B}$, then $y_t \in \mathcal{B}$ and $y$ is continuous in $t \in Q$ and

$$\|y(t)\| \leq L\|y_t\|_{\mathcal{B}}$$

where $L \geq 0$, is a constant.

(ia) From the above condition is equivalent to $\|\varphi(0)\| \leq L\|\varphi\|_{\mathcal{B}}$, for all $\varphi \in \mathcal{B}$.

(ii) There exists a continuous function $c_1(t) \geq 0$ and a locally bounded function $c_2(t) \geq 0$ in $t \geq 0$ such that

$$\| y_t \|_{\mathcal{B}} \leq c_1(t) \sup_{s\in[0,t]} |y(s)| + c_2(t) \| y_0 \|_{\mathcal{B}},$$

for $t \in [0, T]$ and $y$ as in (i).

(iii) The space $\mathcal{B}$ is complete.

Designate $c_1^* = \sup\{c_1(t) : t \in Q\}$ and $c_2^* = \sup\{c_2(t) : t \in Q\}$.

Set

$$\mathbb{R}(\rho^-) = \{\rho(s, \phi) : (s, \phi) \in Q \times \mathcal{B}, \rho(s, \phi) \leq 0\}.$$

We generally assume that $\rho : Q \times \mathcal{B} \to (-\infty, 0]$ is continuous, we make the subsequent assumption:

(H0*) The function $t \to \phi$ is continuous from $\mathbb{R}(\rho^-)$ into $\mathcal{B}$ and we can find a continuous and bounded function $L^\phi : \mathbb{R}(\rho^-) \to (0, \infty)$ in a way that

$$\|\phi_t\|_B \leq L^\phi(t)\|\phi\|_B \text{ for every } t \in \mathbb{R}(\rho^-).$$

**Lemma 2.1.** *[4] If $y : (-\infty, T] \to H$ is a function to ensure that $y_0 = \phi$, then*

$$\| y_s \|_{\mathcal{B}} \leq (c_2^* + (L^\phi)\|\phi\|_{\mathcal{B}} \sup_{\theta\in[0,s]} |y(\theta)|, s \in \mathbb{R}(\rho^-) \cup Q$$

where $L^\phi = \sup_{t\in\mathbb{R}(\rho^-)} L^\phi(t)$

Let $I$ be the identity operator on $H$. If $A$ is a linear operator on $H$, then $R(\lambda, A) = (\lambda I - A)^{-1}$ means the resolvent operator of $A$. Next, we utilize the note for $\eta > 0$,

where $\Gamma(\eta)$ is the Gamma function. If $\eta = 0$, we fixed $k_0(t) = \delta(t)$, the delta distribution. Next, we recall some basic definitions and concepts of fractional integral and derivative of order $\alpha \in (1, 2]$ from [27], Definitions 2.1-2.7, Remark 2.1].

Before we define the mild solution for the system (1.1) - (1.2), first we consider the following linear problem,

$$k_\eta(t) = \frac{t^{\eta-1}}{\Gamma(\eta)}, \quad t > 0, \tag{2.1}$$

$$^{C}D_t^\alpha\, [y(t) + k(t)] = A[y(t) + k(t)] + F(t), \qquad t \in [0, T], \tag{2.2}$$

$$y(t) = \phi(t), \qquad t \in B. \quad y'(0) = x_0 \in H. \tag{2.3}$$

Assume that the Laplace transform of $y(t)$, $k(t)$ and $F(t)$, with respect to $t$ exists. Taking the Laplace transform to (2.2) - (2.3), by (2.3) of [27], we receive

$$\lambda^\alpha[\hat{y}(\lambda) + \hat{k}(\lambda)] - \lambda^{\alpha-1}[y(0) + k(0)] - \lambda^{\alpha-2}[y0 + \eta] = A[\hat{y}(\lambda) + \hat{k}(\lambda)] + \hat{F}(\lambda),$$

where $\hat{y}(\lambda), \hat{k}(\lambda)$ and $\hat{F}(\lambda)$ denote the Laplace transform of $y(t), k(t), F(t)$ and $d\, k(t)|t = 0 = \eta$, where η is independent of $y$. Then

$$y(\lambda) + k(\lambda) = \lambda^{\alpha-1}R(\lambda^\alpha, A)[\phi + k(0)] + \lambda^{\alpha-2}R(\lambda^\alpha, A)[x0 + \eta] + R(\lambda^\alpha, A)\hat{F}(\lambda)$$

By (2.5)-(2.7) of [27] and the properties of Laplace transforms,

$$y(t) = C_\alpha(t)[\phi + k(0)] + S_\alpha(t)[x0 + \eta] - k(t) + \int_0^t P_\alpha(t-s)F(s)ds. \tag{2.4}$$

Let

$$S_{F,y} = \{v \in L^1(Q, H) : v(t) \in F(t, y), \text{ a.e. } t \in Q\}$$

is nonempty.

Let $\Omega = \{y : (-\infty, T] \to H$ such that $y|_{(-\infty,0]} \in B, y|_Q \in C(Q, H)\}$.

Based on the above results, we define the mild solution for the given system (1.1)-(1.2).

**Definition 2.1**. *We say that a continuous function $y \in \Omega$ is a mild solution of problem (1.1)-(1.2) if $y(t) = \phi(t)$for all $t \leq 0$, the constraint of $y(\cdot)$to the interval $[0, T]$is continuous and there exists $v(\cdot) \in L^1(Q, H)$, such that $v(t) \in F(t, y_{\rho(t,yt)})$ a.e. $t \in [0, T]$ and y fulfills the consecutive integral equation*

$$y(t) = C_\alpha(t)\big[\phi(0) + k\big(0, \phi(0)\big)\big] + S_\alpha(t)[x0 + \eta] - k(t, y_t) + \int_0^t P_\alpha(t-s)\, v(s)ds$$
$$+ \int_0^t P_\alpha(t-s)\, Bu(s)ds, t \in Q. \tag{2.5}$$

In order to address the problem, it is convenient at this point to introduce two relevant operators and basic assumptions on these operations:

$$\Upsilon_0^T = \int_0^t P_\alpha(T-s)\,BB^*P_\alpha^*(T - s)ds : H \to H,$$

$$R(a,\Upsilon_0^T) = (aI + \Upsilon_0^T)^{-1} : H \to H.$$

It is straightforward that the operator $\Upsilon_0^T$ is a linear bounded operator.

To investigate the approximate controllability of system (1.1) − (1.2) and (1.3) − (1.4), we impose the following condition:

$(H0)$ $aR(a,\Upsilon_0^T) \to 0^+ + as\ a \to 0^+$ in the strong operator topology.

## 3 Main Results

We show below the controllability results for the systems (1.1) - (1.2) and (1.3) - (1.4) under Dhage's fixed point theorem. To establish the existence result for the system (1.1) - (1.2), we list the subsequent conditions:

$(\boldsymbol{H1})$ $A : D(A) \subset H \to H$ is the infinitesimal generator of a uniformly continuous cosine family

$\{C\alpha(t)\}_{t\geq 0}$. Let

$$M_c = \sup\{\|C_\alpha(t)\|;\ t \geq 0\} \quad \text{and} \quad M_s = \sup\{\|S_\alpha(t)\|;\ t \geq 0\}.$$

**(H2)** The multi-valued map $F : Q \times \mathcal{B} \to P_{cv}(H)$ is an $L^1$-Caratheodory function and there exists a function $\mu \in L^1(Q, \mathbb{R}^+)$ and a continuous non-decreasing function $\psi : \mathbb{R}^+ \to (0, \infty)$ in a way that

$$F(t, u) \leq \mu(t)\psi(\|u\|_B), \quad \text{for every} \quad (t, u) \in Q \times \mathcal{B},$$

**(H3)** (i) The function $k : Q \times \mathcal{B} \to H$ is continuous on $Q$ and there exist constants $L_k > 0$ and $\tilde{L}_k > 0$ such that

$$\|k(t, u)\| \leq L_k\|u\|_B + \tilde{L}_k, \quad \text{for each } u \in \mathcal{B}.$$

(ii) There exist a function $Lk*$ $\in L^1(Q,$ R+) in a way that

$$\|k(t, u) - k(t, v)\| \leq L\ \|u - v\|_B, \quad t \in Q,\ u, v \in \mathcal{B}.$$

**Theorem 3.1.** *Assume that the hypotheses (H0) and (H1) - (H3) holds. Then the problem (1.1) - (1.2) has at least one mild solution such that,*

$$\Lambda = L_k^* c_1^* M^* < 1\ and\ \int_0^T \mu(s)ds < \int_{w1}^\infty \frac{du}{\psi\,(u)} \tag{3.1}$$

Where $M^* = \left[1 + \frac{M^2 M_B^2\, T}{\alpha}\right]$, $\|\mathcal{B}\| = M_{B,} \|\mathcal{B}^*\| = M_{B,}$ $and$

$$w_1 = c_n + \frac{c_1^*}{1 - L_k\, c_1^*}\{M_c\big(L_k\,\mathbb{L}\|\phi\|_{\mathcal{B}} + \tilde{L}_k\big) + M_{s,}\|x0 + \eta\| + L_k\, c_n^* + \tilde{L}_k\}$$

**Proof.** Now, we define the the multi-valued operator $\Upsilon : \Omega \rightarrow P(\Omega)$ described by $\Upsilon(e) = \{e \in \Omega\}$ with

$$et = \begin{cases} \emptyset(t), \ \ t \in (-\infty, o) \\ C_\alpha(t)[\emptyset(0) + k(0, \emptyset(0)] + S_\alpha(t)[y0 + \eta] - k(t, yt) \\ + \int_0^t P_\alpha(t-s)v(s)ds + \int_0^t P_\alpha(t-s)Bu(s)ds, \quad t \in Q, \end{cases} \quad (3.2)$$

$$where\ v \in S_{F,y\rho(s,ys)}\ and\ u(t) = B^* P_\alpha^* (T,t) R(a, \Upsilon_0^T) P(x(\ \cdot\ )), where$$

$$P(y(\cdot)) = y_T - C_\alpha(T)[\phi(0) + k(0, \phi(0))] - S_\alpha(T)[y0 + \eta] + k(T, y_T) - \int_0^t P_\alpha(t-s)v(s)ds.$$

$For\ \phi \in \mathcal{B}, we\ express\ the\ function\ x(\ \cdot\ ) : (-\infty, T] \rightarrow H\ by$

$$x(t) = \begin{cases} \emptyset(t), \ \ t \leq 0 \\ C_\alpha(t)[\emptyset(0) \ \ t \leq Q \end{cases}$$

$If\ y(\ \cdot\ )\ fulfills$ (2.5), we can able to decompose $y(t) = x(t) + z(t), t \in Q$, with infer that $y_t = x_t + z_t, for\ each\ t \in Q$ and the function $z(\ \cdot\ )$ fulfills

$$z(t) = C_\alpha(t)k\big(0, \phi(0)\big) + S_\alpha(t)[x_0 + \eta] - k(t, zt + xt) + \int_0^t P_\alpha(t-s)v(s)ds + \int_0^t P_\alpha(t-s)Bu(s)ds, \quad t \in Q, \quad (3.3)$$

$where\ v \in S_{F,z\rho(s,zs+xs)+x\rho(s,zs+xs)}$.

Let $Z_0 = \{z \in \Omega : z_0 = 0\}$. For any $z \in \mathbb{Z}_0$, we receive

$$\| z \|_{\mathbb{Z}_0} = \sup_{t \in J} \| z(t) \| + \| z_0 \|_{\mathcal{B}} = \sup_{t \in J} \| z(t) \|.$$

Therefore $(z_0 \ \|.\|_{z_0})$ is a Banach space. Now, we designate the operator $\Phi : z_0 \rightarrow P(z_0)\ by\ \Phi(z) = \{h \in z_0\}$ with

$$h(t) = C\alpha(t)k(0, \phi(0)) + S\alpha(t)[x_0 + \eta] - k(t, z_t + x_t) + \int_0^t P_\alpha(t-s)v(s)ds + \int_0^t P_\alpha(t-s)Bu(s)ds, \quad t \in Q,$$

where $v \in S_{F,z\rho(s,zs+xs)+y\rho(s,zs+xs)}$.

From this, we observe that the operator Υ having a fixed point is equivalent to Φ having one, so it turns to prove that Φ has a fixed point.

Now, we are in a position to utilize the Dhage's fixed point theorem [11]. To apply this, first, we split the multi-valued operator Φ as

$$\Phi_1(z) = \Big\{ h \in z_0 : h(t) = C_\alpha(t)k\big(0, \phi(0)\big) + S_\alpha(t)[x_0 + \eta] - k(t, zt + xt) \int_0^t P_\alpha(t-s)\ B^* P_\alpha^* (T-s) R(a, \Upsilon_0^T)[y_T - C_\alpha(T)k\big(0, \phi(0)\big) - S_\alpha(T)[x_0 + \eta] + k(T, z_T + x_T) \Big], t \in Q\}$$

$$\Phi_1(z) = \Big\{ h \in Z_0 : h(t) = \int_0^t P_\alpha(t-s)v(s)ds + \int_0^t P_\alpha(t-s)\, B^* P_\alpha^* (T-s) R(a, \Upsilon_0^T)(\times) \int_0^t P_\alpha(T-\tau)v(\tau)ds\, v(s) \in S_{F, z\rho(s, zs + xs) + y\rho(s, zs + xs)}, t \in Q \Big\}$$

Now, our aim is to show that the multi-valued operators Φ1 and Φ2 satisfy all the conditions of Dhage's fixed point theorem[11]. For better readability, we break the proof into sequence of steps.

Step 1. Φ1 is a contraction.

Let z, $z^*$∈ $Z_0$ and h ∈ Φ1(z)

$$\|\Phi_1(z) - \Phi_1(z^*)\| \le \|k(t, z_t + x_t) - k(t, z_t^* + x_t)\|_{\mathcal{B}} + \left\| \int_0^t P_\alpha(t-s)\, BB^* P_\alpha^* (T-s) R(a, \Upsilon_0^T) k(T, z_T + x_T) + k(T, z_T^* + x_T)] \right\|_{\mathcal{B}}$$

$$\le L_k^* \|z_t + z_t^*\|_{\mathcal{B}} + \int_0^t \frac{M^2 M_B^2}{a} L_k^* \|z_t + z_t^*\|_{\mathcal{B}} ds$$

$$\le L_k^* c_1(t) \sup_{t\in J} |z(t) - z^*(t)| + c_2(t)\|z_0 + z_0^*\| + \frac{M^2 M_B^2}{a} L_k^* \int_0^t c_1(t) \sup_{t\in J} |z(t) - z^*(t)| + c_2(t)\|z_0 + z_0^*\|\, ds$$

$$\le L_k^* c_1 \left[1 + \frac{M^2 M_B^2}{a}\right] \|z + z^*\|_{\mathcal{B}}$$

$$\le L_k^* c_1^* M^* \|z + z^*\|_{\mathcal{B}}$$

Since

$$\|(z_t + x_t)\|_{\mathcal{B}} \le \|(z_t)\|_{\mathcal{B}} + \|(x_t)\|_{\mathcal{B}}$$

$$\le c_1^* \sup_{t\in J} |z(t)| + c_2(t)\|z_0\|_{\mathcal{B}} + c_1^* \sup_{t\in J} |x(s)| + \|(x_0)\|_{\mathcal{B}}$$

$$\le c_1^* \sup_{t\in J} |z(t)| + c_1^* M_c \mathbb{L} \|\emptyset\|_{\mathcal{B}} + c_2^* \|\emptyset\|_{\mathcal{B}}$$

$$\le c_1^* r + [c_1^* M_c \mathbb{L} \|\emptyset\|_{\mathcal{B}} + c_2^* \|\emptyset\|_{\mathcal{B}}$$

$$\le c_1^* r + c_n^*, \qquad where\ c_n^* = [c_1^* M_c \mathbb{L} + c_2^*] \|\emptyset\|_{\mathcal{B}}$$

Then

$$\|\Phi_1(z) - \Phi_1(z^*)\| \le \Lambda \| z - z^* \|.$$

From (3.1), we see that Φ1 is a contraction.

Step 2. Φ2 has compact, convex valued and it is completely continuous. This will show in several steps.

Claim (i). $\Phi_2$ is convex for each z ∈ $Z_0$.

Indeed, if $h_1$, $h_2$ ∈ $\Phi_2$, then there exist $v_1$, $v_2$ ∈ $S_{F, z\rho(t,zt+xt)+x\rho(t,zt+xt)}$ , such that for t ∈ Q, we receive

$$h_i = \int_0^t P_\alpha(t-s)v(s)ds + \int_0^t P_\alpha(t-s)\, BB^* P_\alpha^* \,(T-s)R(a, \Upsilon_0^T\,)(\times)\int_0^t P_\alpha(T-\tau)vi(\tau)d\tau\, ds\; for\; i = 1,2,$$

t ∈ Q

Let d ∈ [0, 1]. Then for each t ∈ J, we get

$$[dh_1 + (1-d)h_2](t) = \int_0^t P_\alpha(t-s)[dv_1(s) + (1-d)v_2(s)]ds - \int_0^t P_\alpha(t-s)\, BB^* P_\alpha^*\,(T - s)R(a,\Upsilon_0^T\,)(\times)\int_0^t P_\alpha(T-\tau)[dv_1(\tau) + (1-d)v_2(\tau)]d\tau ds$$

Actually $S_{F_{z\rho(t,zt+xt)+x\rho(t,zt+xt)}}$ ,is convex (because F has convex values), we have $dh_1 + (1-d)h_2 \in$ Φ2.

Claim (ii). $\Phi_2$ maps bounded sets into bounded sets in $Z_0$.

In fact, it is sufficient to demonstrate that for any r > 0, there exists a positive constant l in ways that for every z ∈ $B_r = \{z \in Z_0 : \|z\|_{Z_0} \le r\}$, we sustain $\|\Phi_2(z)\|_{Z_0} \le l$. Then for every h ∈ $\Phi_2(z)$, there exists v ∈ $S_{F_{z\rho(t,zt+xt)+x\rho(t,zt+xt)}}$ such that,

$$\begin{aligned}
\|h(t)\| &\le \left\| \int_0^t P_\alpha(t-s)v(s)ds - \int_0^t P_\alpha(t-s)BB^*P_\alpha^*(T-s)R(a,\Upsilon_0^T) (\times)\int_0^T P_\alpha(T-\tau)v(\tau)d\tau ds\right\| \\
&\le \int_0^t \|P_\alpha(t-s)v(s)\|ds + \left\|\int_0^t P_\alpha(t-s)BB^*P_\alpha^*(T-s)R(a,\Upsilon_0^T) (\times)\int_0^T P_\alpha(T-\tau)v(\tau)d\tau ds\right\| \\
&\le M\int_0^t \mu(s)\psi\left(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathscr{B}}\right)ds + \frac{M^2M_B^2}{a} (\times)M\int_0^t\int_0^T \mu(\tau)\psi\left(\|z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}\|_{\mathscr{B}}\right)d\tau ds \\
&\le M\psi(c_1^*r + c_n)\int_0^t \mu(s)ds + M\frac{M^2M_B^2}{a}T\psi(c_1^*r+c_n)\int_0^t \mu(s)ds \\
&\le M\left[1 + \frac{M^2M_B^2}{a}T\right]\psi(c_1^*r+c_n)\|\mu\|_{L^1} \\
&\le MM^*\psi(c_1^*r+c_n)\|\mu\|_{L^1} \\
&\le l.
\end{aligned}$$

Since,

$$\begin{aligned}
&\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathscr{B}} \\
&\le \|z_{\rho(s,z_s+x_s)}\|_{\mathscr{B}} + \|x_{\rho(s,z_s+x_s)}\|_{\mathscr{B}} \\
&\le (c_2^* + L^\phi)\|z_0\| + c_1^*\sup_{s\in t}|z(s)| + c_1^*\sup_{s\in t}|x(s)| + (c_2^*+L^\phi)\|x_0\|_{\mathscr{B}}
\end{aligned}$$

$$\le c_1^* \sup_{s\in t} |z(s)| + c_1^*|C_\alpha(s)\phi(0)| + (c_2^* + L^\phi)\|C_\alpha(0)\phi\|_{\mathscr{B}}$$
$$\le c_1^* \sup_{s\in t} |z(s)| + c_1^* M_c \mathbb{L}\|\phi\|_{\mathscr{B}} + (c_2^* + L^\phi)\|\phi\|_{\mathscr{B}}$$
$$\le c_1^* r + [c_1^* M_c \mathbb{L} + c_2^* + L^\phi]\|\phi\|_{\mathscr{B}}$$
$$\le c_1^* r + c_n, \text{ where } c_n = [c_1^* M_c \mathbb{L} + c_2^* + L^\phi]\|\phi\|_{\mathscr{B}}.$$

Thus $\Phi_2(Br)$ is bounded.

Claim (iii). $\Phi_2$ maps bounded sets into equi-continuous sets of $Z_0$.

Let $h \in \Phi_2(z)$ for $z \in Z_o$ and let $\tau_1, \tau_2 \in [0, T]$, with $\tau_1 < \tau_2$, we have

Let $h \in \Phi_2(z)$ for $z \in \mathcal{Z}_0$ and let $\tau_1, \tau_2 \in [0,T]$, with $\tau_1 < \tau_2$, we have

$$\begin{aligned}\|h(\tau_2) - h(\tau_1)\| \le & \left\| \int_0^{\tau_2} P_\alpha(\tau_2 - s)v(s)ds - \int_0^{\tau_1} P_\alpha(\tau_1 - s)v(s)ds \right\| \\ & + \left\| \int_0^{\tau_2} P_\alpha(\tau_2 - s)BB^* P_\alpha^*(T-s)R(a, \Upsilon_0^T) \int_0^T P_\alpha(T-\tau)v(\tau)d\tau ds \right. \\ & \left. - \int_0^{\tau_1} P_\alpha(\tau_1 - s)BB^* P_\alpha^*(T-s)R(a, \Upsilon_0^T) \int_0^T P_\alpha(T-\tau)v(\tau)d\tau ds \right\| \\ & \le I_1 + I_2,\end{aligned}$$

Where

$$\begin{aligned} I_1 \le & \left\| \int_0^{\tau_1-\epsilon} \Big[P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s)\Big] v(s)ds \right. \\ & + \int_{\tau_1-\epsilon}^{\tau_1} \Big[P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s)\Big] v(s)ds \\ & \left. + \int_{\tau_1}^{\tau_2} P_\alpha(\tau_2 - s)v(s)ds \right\| \\ \le & \left\| \int_0^{\tau_1-\epsilon} \Big[P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s)\Big] v(s)ds \right\| \\ & + \left\| \int_{\tau_1-\epsilon}^{\tau_1} \Big[P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s)\Big] v(s)ds \right\| \\ & + \left\| \int_{\tau_1}^{\tau_2} P_\alpha(\tau_2 - s)v(s)ds \right\| \\ \le & \psi(c_1^* r + c_n)\Big\{ \int_0^{\tau_1-\epsilon} \|P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s)\|\mu(s)ds \\ & + \int_{\tau_1-\epsilon}^{\tau_1} \|P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s)\|\mu(s)ds \\ & + M \int_{\tau_1}^{\tau_2} \mu(s)ds \Big\}. \end{aligned}$$

And

$$\begin{aligned} I_2 \le & \left\| \int_0^{\tau_1-\epsilon} \Big[P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s)\Big] BB^* P_\alpha^*(T-s)R(a, \Upsilon_0^T) \int_0^T P_\alpha(T-\tau)v(\tau)d\tau ds \right\| \\ & + \left\| \int_{\tau_1-\epsilon}^{\tau_1} \Big[P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s)\Big] BB^* P_\alpha^*(T-s)R(a, \Upsilon_0^T) \int_0^T P_\alpha(T-\tau)v(\tau)d\tau ds \right\| \\ & + \left\| \int_{\tau_1}^{\tau_2} P_\alpha(\tau_2 - s) BB^* P_\alpha^*(T-s)R(a, \Upsilon_0^T) \int_0^T P_\alpha(T-\tau)v(\tau)d\tau ds \right\| \end{aligned}$$

$$\leq \left\| \int_0^{\tau_1-\epsilon} \left[P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s)\right] \frac{MM_B^2}{a} TMv(s)ds \right\|$$
$$+ \left\| \int_{\tau_1-\epsilon}^{\tau_1} \left[P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s)\right] \frac{MM_B^2}{a} TMv(s)ds \right\|$$
$$+ \left\| \int_{\tau_1}^{\tau_2} P_\alpha(\tau_2 - s) \frac{MM_B^2}{a} TMv(s)ds \right\|$$
$$\leq \frac{M^2M_B^2}{a} T\psi(c_1^* r + c_n) \Big\{ \int_0^{\tau_1-\epsilon} \|P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s)\|\mu(s)ds$$
$$+ \int_{\tau_1-\epsilon}^{\tau_1} \|P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s)\|\mu(s)ds$$
$$+ M \int_{\tau_1}^{\tau_2} \mu(s)ds \Big\}.$$

From $I_1$ and $I_2$, we get

$$\|h(\tau_2) - h(\tau_1)\|$$
$$\leq M^*\psi(c_1^* r + c_n) \Big\{ \int_0^{\tau_1-\epsilon} \|P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s)\|\mu(s)ds$$
$$+ \int_{\tau_1-\epsilon}^{\tau_1} \|P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s)\|\mu(s)ds + M \int_{\tau_1}^{\tau_2} \mu(s)ds \Big\}.$$

Clearly the right-hand side of the above inequality tends to zero as $\tau_2 \to \tau_1$.

Then, Φ2(Br) is equi-continuous.

Claim (iv). $\Phi_2(Br)$ is relatively compact for every $t \in J$, we sustain

$$\Phi_2(Br)(t) = \{h(t) : h \in \Phi_2(Br)\}.$$

Allow $0 \leq t \leq T$ be fixed and let $\epsilon$ be a real number fulfilling $0 < \epsilon < t$. For $\delta > 0$, we specify

$$h_{\epsilon,\delta}(t) = \int_0^{t-\epsilon} P_\alpha(t - s)v(s)ds - \int_0^{t-\epsilon} P_\alpha(t - s)BB^* P_\alpha^*(T - s)R(a, \Upsilon_0^T)$$
$$(\times) \int_0^T P_\alpha(T - \tau)v(\tau)d\tau ds,$$

where $v(s) \in S_{F,z\rho(s,z_s+x_s)+y\rho(s,z_s+x_s)}$. Since $C\alpha(t)$ is a compact operator,

$$h_{\epsilon,\delta} = \{h_{\epsilon,\delta}(t) : h \in \Phi 2(Br)\}$$

is relatively compact. Furthermore,

$$\|h(t) - h_{\epsilon,\delta}(t)\| \leq \left\| \int_0^t P_\alpha(t - s)v(s)ds - \int_0^{t-\epsilon} P_\alpha(t - s)v(s)ds \right\|$$
$$+ \Big\| \int_0^t P_\alpha(t - s)BB^* P_\alpha^*(T - s)R(a, \Upsilon_0^T) \int_0^T P_\alpha(T - \tau)v(\tau)d\tau ds$$
$$- \int_0^{t-\epsilon} P_\alpha(t - s)BB^* P_\alpha^*(T - s)R(a, \Upsilon_0^T) \int_0^T P_\alpha(T - \tau)v(\tau)d\tau ds \Big\|$$
$$\leq \left\| \int_{t-\epsilon}^t P_\alpha(t - s)v(s)ds \right\| + \Big\| \int_{t-\epsilon}^t P_\alpha(t - s)BB^* P_\alpha^*(T - s)R(a, \Upsilon_0^T)$$
$$(\times) \int_0^T P_\alpha(T - \tau)v(\tau)d\tau ds \Big\|$$

$$\leq \int_{t-\epsilon}^{t} \|P_\alpha(t-s)\|\|v(s)\|ds + \frac{\ddot{M}^2 M_B^2}{a} T \int_{t-\epsilon}^{t} \|P_\alpha(t-s)\|\|v(s)\|ds$$
$$\leq M\left[1 + \frac{M^2 M_B^2}{a} T\right] \int_{t-\epsilon}^{t} \mu(s)\psi\left(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_B\right) ds$$
$$\leq M M^* \psi(c_1^* r + c_n) \int_{t-\epsilon}^{t} \mu(s) ds.$$

Therefore, $(\Phi_2(B_r))(t)$ is relatively compact.

As a consequence of claim (ii) - (iv) together with the Arzela-Ascoli's theorem we can conclude

that $\Phi_2$ is completely continuous.

Claim (v). $\Phi_2$ has a closed graph.

Let $z_n \to z_*$, $h_n \in \Phi_2(z_n)$ and $h_n \to h_*$.We will prove that $h_* \in \Phi_2(z_*)$. Indeed, $h_n \in \Phi_2(z_n)$ means that there exists $v_n \in S_{F,z_{\rho(n,z_n+x_n)}+x_{\rho(n,z_n+x_n)}}$ such that

$$h_n(t) = \int_0^t P_\alpha(t-s)v_n(s)ds - \int_0^t P_\alpha(t-s)BB^* P_\alpha^*(T-s)R(a,\Upsilon_0^T)$$
$$(\times)\int_0^T P_\alpha(T-\tau)v_n(\tau)d\tau ds, \quad t \in Q.$$

We must prove that there exists $v_* \in S_{F,z_{\rho(*,z_*+x_*)}+x_{\rho(*,z_*+x_*)}}$ such that

$$h_*(t) = \int_0^t P_\alpha(t-s)v_*(s)ds - \int_0^t P_\alpha(t-s)BB^* P_\alpha^*(T-s)R(a,\Upsilon_0^T)$$
$$(\times)\int_0^T P_\alpha(T-\tau)v_*(\tau)d\tau ds,\ t \in Q.$$

Consider the linear and continuous operator $\Upsilon : L^1(Q,H) \to C(Q,H)$, defined by

$$\mathcal{Y}(v)(t) = \int_0^t P_\alpha(t-s)v(s)ds - \int_0^t P_\alpha(t-s)BB^* P_\alpha^*(T-s)R(a,\Upsilon_0^T)$$
$$(\times)\int_0^T P_\alpha(T-\tau)v(\tau)d\tau ds,$$

From[18], it follows that $\Upsilon \circ S_F$ is a closed graph operator and from the definition of $h_n(t) \in \Upsilon S_{F,z_{\rho(n,z_n+x_n)}+x_{\rho(n,z_n+x_n)}}$.

As $z_n \to z_*$ and $h_n \to h_*$, there is a $v_* \in S_{F,z_{\rho(*,z_*+x_*)}+x_{\rho(*,z_*+x_*)}}$ such that

$$h_*(t) = \int_0^t P_\alpha(t-s)v(s)ds - \int_0^t P_\alpha(t-s)BB^* P_\alpha^*(T-s)R(a,\Upsilon_0^T)$$
$$(\times)\int_0^T P_\alpha(T-\tau)v(\tau)d\tau ds.$$

Therefore, the multi-valued operator $\Phi_2$ is a completely continuous multivalued map, upper semi-continuous with convex, closed and compact values.

Claim (vi). A priori bounds.

$\Gamma = \{z \in Z_0 : z \in \lambda\Phi_1(z) + \lambda\Phi_2(z)$, for some $0 < \lambda < 1\}$ is bounded

Let $z \in \Gamma$ be any element, then there exists $v \in S_{F,z_{\rho(s,z_s+x_s)}+y_{\rho(s,z_s+x_s)}}$, such that

$$\|z(t)\| \leq \Big\| C_\alpha(t)g(0,\phi(0)) + S_\alpha(t)[x_0+\eta] - k(t, z_t + x_t) + \int_0^t P_\alpha(t-s)F(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds + \int_0^t P_\alpha(t-s)Bu(s)ds \Big\|$$

$$\leq M_c(L_k\|\phi(0)\|_{\mathscr{B}} + \bar{L}_k) + M_s\|x_0+\eta\| + L_k\|z_t+x_t\|_{\mathscr{B}} + \bar{L}_k + M\int_0^t \mu(s)\psi\left(\|z_{\rho(s,z_s+x_s)} + y_{\rho(s,z_s+x_s)}\|_{\mathscr{B}}\right)ds + \frac{MM_B^2}{a}\int_0^t P_\alpha(t-s)\Big[\|z_T+x_T\| + M_c(L_k\|\phi(0)\|_{\mathscr{B}} + \bar{L}_k) + M_s\|x_0+\eta\| + L_k\|z_t+x_t\|_{\mathscr{B}} + \bar{L}_k + M\int_0^T \mu(\tau)\psi\left(\|z_{\rho(\tau,z_\tau+x_\tau)} + y_{\rho(\tau,z_\tau+x_\tau)}\|_{\mathscr{B}}\right)d\tau\Big]ds$$

$$\leq \frac{MM_B^2}{a}\int_0^t P_\alpha(t-s)\|z_T+x_T\|ds + M^*\Big(M_c(L_k\mathbb{L}\|\phi\|_{\mathscr{B}} + \bar{L}_k) + M_s\|x_0+\eta\| + L_k(c_1^*\sup_{t\in Q}|z(t)| + c_n^*) + \bar{L}_k + M\int_0^t \mu(s)\psi(c_1^*\sup_{s\in[0,t]}|z(s)| + c_n)ds\Big)$$

$$\leq \frac{1}{1-M^*L_kc_1^*}\Big\{\frac{M^2M_B^2T}{a}\|z_T+x_T\| + M^*\Big(M_c(L_k\mathbb{L}\|\phi\|_B + \bar{L}_k) + M_s\|x_0+\eta\| + L_kc_n^* + \bar{L}_k + M\int_0^t \mu(s)\psi(c_1^*\|z(s)\| + c_n)ds\Big)\Big\},$$

Then

$$c_n + c_1^*\|z(t)\| \leq c_n + \frac{c_1^*}{1-M^*L_kc_1^*}\Big\{\frac{M^2M_B^2T}{a}\|z_T+x_T\| + M^*\Big(M_c(L_k\mathbb{L}\|\phi\|_B + \bar{L}_k) + M_s\|x_0+\eta\| + L_kc_n^* + \bar{L}_k + M\int_0^t \mu(s)\psi(c_1^*\|z(s)\| + c_n)ds\Big)\Big\}$$
$$\leq \omega_1 + \omega_2\int_0^t \mu(s)\psi(c_n + c_1^*\|z(s)\|)ds,$$

Where

$$\omega_1 = c_n + \frac{c_1^*}{1-M^*L_kc_1^*}\Big\{\frac{M^2M_B^2T}{a}\|z_T+x_T\| + M^*\Big(M_c(L_k\mathbb{L}\|\phi\|_{\mathscr{B}} + \bar{L}_k) + M_s\|x_0+\eta\| + L_kc_n^* + \bar{L}_k\Big)\Big\}$$

and

$$\omega_2 = \frac{Mc_1^*}{1-M^*L_kc_1^*}M^*.$$

Denote

$$m(t) = \sup\{c_n + c_1^*\|z(s)\| : 0 \leq s \leq t\},\ t\in J.$$

From the above mentioned inequality, we receive

$$m(t) \leq \omega_1 + \omega_2\int_0^t \mu(s)\psi(m(s))ds.$$

Let us take the right hand side of the above inequality as $\nu(t)$. Thus, we get

$$m(t) \leq \nu(t), \text{ for every } t \in Q,$$

with

$$\nu(0) = \omega_1$$

and

$$\nu'(t) = \omega_2\mu(t)\psi(m(t)), \text{ a.e. } t \in Q.$$

Utilizing the non-decreasing character of $\psi$, we obtain

$$\nu'(t) \leq \omega_2\mu(t)\psi(\nu(t)), \text{ a.e. } t \in Q.$$

Integrating from 0 to t we get

$$\int_0^t \frac{\nu'(s)}{\psi(\nu(s))}ds \leq \omega_2 \int_0^t \mu(s)ds.$$

By change the variable, we get

$$\int_{\nu(0)}^{\nu(t)} \frac{du}{\psi(u)} \leq \omega_2 \int_0^t \mu(s)ds.$$

In view of (3.1), this ensures that for every $t \in Q$, we have

$$\int_{\nu(0)}^{\nu(t)} \frac{du}{\psi(u)} \leq \omega_2 \int_0^t \mu(s)ds \leq \omega_2 \int_0^T \mu(s)ds < \int_{\omega_1}^{\infty} \frac{du}{\psi(u)}.$$

Therefore, for every $t \in Q$, there exists a constant $\Lambda_1$ such that $\nu(t) \leq \Lambda_1$ and hence $m(t) \leq \Lambda_1$.

Since $\| z \|_{z_0} \leq m(t)$, we have $\| z \|_{z_0} \leq \Lambda_1$.

This shows that the set $\Gamma$ is bounded. As a consequence of Dhage's fixed point theorem[11], we realize that $\Phi_1 + \Phi_2$ has a fixed point $z$ defined on the interval $(-\infty, T]$ which is the mild solution of the system (1.1)-(1.2). The proof is now completed. Our next existence results for the problem (1.3)-(1.4) is based on Dhage's fixed point theorem. Before we present and prove the existence result for the problem, first we define the mild solution of (1.3)-(1.4).

**Definition 3.1.** We say that a continuous function $x \in \Omega$ is a mild solution of problem (1.3)-(1.4) if $y(0) = \phi \in \mathcal{B}$, $y'(0) = x_0 \in \mathcal{H}$, we have

$$\begin{aligned} y(t) = &C_\alpha(t)[\phi(0) + k(0,\phi(0),0)] + S_\alpha(t)[x_0 + \eta] - k\left(t, y_t, \int_0^t k_1(t,s,y_s)\,ds\right) \\ &+ \int_0^t P_\alpha(t-s)F\left(s, y_{\rho(s,y_s)}, \int_0^s k_2(s,\tau,y_{\rho(\tau,y_\tau)})d\tau\right)ds \\ &+ \int_0^t P_\alpha(t-s)Bu(s)ds, \quad t \in Q, \end{aligned} \tag{3.4}$$

Where

$u(t) = B^* P_\alpha^*(T,t)R(a,\Upsilon_0^T)P(x(\cdot))$, *where*

$$\begin{aligned} \overline{P}(y(\cdot)) = &y_T - C_\alpha(T)[\phi(0) + k(0,\phi(0),0)] - S_\alpha(T)[x_0 + \eta] + k\left(T, y_T, \int_0^t k_1(T,s,y_s)\,ds\right) \\ &- \int_0^T P_\alpha(T-s)F\left(s, y_{\rho(s,y_s)}, \int_0^s k_2(s,\tau,y_{\rho(\tau,y_\tau)})d\tau\right)ds. \end{aligned}$$

Next, to prove the existence result for the problem (1.3)-(1.4), we list the following additional hypotheses:

($H_2$*) The multivalued map $F : Q \times \mathcal{B} \times \mathcal{H} \to P_{cl,cv,bd}(\mathcal{H})$ is an $L_1$-Caratheodory function.

($H_3$*) t ∈ J, the function $F(t, \cdot, \cdot) : \mathcal{B} \times \mathcal{H} \to P_{cl,cv,bd}(\mathcal{H})$ is upper semicontinuous and for each $(u, v) \in \mathcal{B} \times \mathcal{H}$, the function $F(\cdot, u, v) : Q \to \mathcal{H}$ is strongly measurable. Also, for each fixed $u \in \mathcal{B}$ and the set

$$S_{F,y^*} = \{v^* \in L^1(Q,H) : v^*(t) \in F\left(t, y_{\rho(t,y_t)}, \int_0^t k_2(t,s,y_{\rho(s,y_s)})ds\right) \text{ for a.e. } t \in Q\}$$

is nonempty.

(H4*) (i) There exist a continuous function $\mu_1 \in L_1(Q,R_+)$ and a continuous non decreasing

function $\psi_1 : R_+ \to (0,\infty)$ in a way that

$\| F(t,u,v) \|_H \leq \mu_1(t)\psi_1(\| u \|_B + \| v \|_H)$, for a.e. $t \in Q$, $u \in \mathcal{B}$, $v \in H$,

(ii) We can find $L_f \in L_1(Q,R_+)$ to ensure that

$\| F(t,u_1,v_1) - F(t,u_2,v_2) \|_H \leq L_f [\| u_1 - v_1 \|_B + \| u_2 - v_2 \|_H]$, for a.e. $t \in Q$,

$u_1, v_1 \in \mathcal{B}$, $u_2, v_2 \in H$.

(iii) There is a function $m \in L^1(Q,R_+)$ and a non decreasing function $\Omega_1 : R^+ \to (0,\infty)$ to ensure that

$\| k_2(t,s,u) \|_H \leq m(s)\Omega_1(\| u \|_B)$, for every $(t, s, u) \in Q \times Q \times \mathcal{B}$.

(iv) There is a constant $C_1 > 0$, in a way that

$$\left\| \int_0^t [K_2(t,s,u_1) - K_2(t,s,u_2)]ds \right\|_H \leq C_1 \|u_1 - u_2\|_{\mathscr{B}}, \text{ for } (t,s) \in Q \times Q, \ u_1, u_2 \in \mathscr{B}.$$

($H_5$*) (i) The function $k : Q \times \mathcal{B} \times \mathcal{H} \to \mathcal{H}$ is continuous on Q and there exist a positive constant $M_k > 0$ such that for each $u_1, v_1 \in \mathcal{B}$, $u_2, v_2 \in \mathcal{H}$.

$\|k(t, u_1, v_1) - k(t, u_2, v_2)\| \leq M_k(\|u_1 - v_1\|_B + \|u_2 - v_2\|_H)$.

(ii) There exist positive constants $L_{k_1} > 0$ and $\tilde{L}_{k_1} > 0$, such that

**(iii)**$\|k(t, u, v)\| \leq L_{k_1}(\|u\|_B + \|v\|_H) + \tilde{L}_{k_1}$, $t \in J$, $u \in B$, $v \in H$.

($H_6$*) The function $K_1 : Q \times Q \times H \to H$ are continuous maps and there exists a positive constant $L_{K_1} > 0$ such that

$$\left\| \int_0^t [K_1(t,s,z_1) - K_1(t,s,z_2)]ds \right\|_H \leq L_{K_1} \|z_1 - z_2\|_{\mathscr{B}}, \text{ for each } z_1, z_2 \in \mathscr{B}$$

$$\left\| \int_0^t [K_1(t,s,z_1) - K_1(t,s,z_2)]ds \right\|_H \leq L_{K_1} \|z_1 - z_2\|_{\mathscr{B}}, \text{ for each } z_1, z_2 \in \mathscr{B}$$

and

$$\left\| \int_0^t K_1(t,s,z)ds \right\|_H \leq L_{K_1}(\|z\|_{\mathscr{B}} + 1), \quad \text{for } z \in \mathscr{B}.$$

**Theorem 3.2.** Assume that the hypotheses ($H_0$), ($H_1$) and ($H_2$*) - ($H_6$*) holds. Then the problem (1.3) - (1.4) has at least one mild solution such that

$$\Lambda^* = M_k c_1^* M^* [1 + (L_{K_1} + 1)] < 1 \quad \text{and} \int_0^T \nu(s)ds < \int_{\omega_1^*}^{\infty} \frac{ds}{\psi_1(s) + \Omega_1(s)}, \qquad (3.5)$$

$$\text{where } \omega_1^* = c_n + \frac{c_1^*}{1 - L_{k_1} c_1^* (1 + L_{K_1})} \left\{ M_c (L_{k_1} \mathbb{L} \|\phi\|_{\mathscr{B}} + \tilde{L}_{k_1}) + M_s \|x_0 + \eta\| + L_{k_1} c_n^* (1 + L_{K_1}) \right\}.$$

**Proof.** Consider the multivalued operator $\bar{\Upsilon} : \Omega \to P(\Omega)$ defined by $\Upsilon(h) = \{h \in \Omega\}$ with

$$h(t) = \begin{cases} \phi(t), & t \in (-\infty, 0] \\ C_\alpha(t)[\phi(0) + k(0, \phi(0), 0)] + S_\alpha(t)[x_0 + \eta] \\ -k\left(t, y_t, \int_0^t K_1(t,s,y_s)ds\right) + \int_0^t P_\alpha(t-s)v^*(s)ds + \int_0^t P_\alpha(t-s)Bu(s)ds, & t \in J, \end{cases}$$

where $v_*(s) \in S_{F,y^*}$.

Next, we split the multi-valued operator $\Phi_*$ as

$$\Phi_1^*(z) = \Big\{h \in \mathcal{Z}_0 : h(t) = C_\alpha(t)k(0,\phi(0),0) + S_\alpha(t)[x_0+\eta] - k\left(t, y_t, \int_0^t K_1(t,s,y_s)ds\right)$$
$$+ \int_0^t P_\alpha(t-s)BB^*P_\alpha^*(T-s)R(a,\Upsilon_0^T)\Big(y_T - C_\alpha(T)k(0,\phi(0),0) - S_\alpha(T)[x_0+\eta]$$
$$+ k\left(T, y_T, \int_0^T K_1(T,\tau,y_\tau)d\tau\right)\Big)ds, \ t \in Q\Big\}.$$

$$\Phi_2^*(z) = \Big\{h \in \mathcal{Z}_0 : h(t) = \int_0^t P_\alpha(t-s)v^*(s)ds - \int_0^t P_\alpha(t-s)BB^*P_\alpha^*(T-s)R(a,\Upsilon_0^T)$$
$$(\times)\int_0^T P_\alpha(T-\tau)v^*(\tau)d\tau ds, \ v^*(s) \in S_{F,x^*}, \ t \in Q\Big\}.$$

The proof of this theorem is very similar to Theorem 3.1. With necessary modifications, we can prove the steps 1 and 2 (Claims (i)-(v)) clearly, so we omit these steps. Now, we prove a priori bounds only.

**Claim (vi).** A priori bounds.

$\Gamma = \{z \in Z_0 : z \in \lambda\Phi_{*1}(z) + \lambda\Phi_{*2}(z)$, for some $0 < \lambda < 1\}$ is bounded.

Let $z \in \Gamma$ be any element, then there exists $v_* \in S_{F,x^*}$, such that

$$\|z(t)\| \le \Big\| C_\alpha(t)k(0,\phi(0),0) + S_\alpha(t)[x_0+\eta] - k\left(t, z_t + x_t, \int_0^t K_1(t,s,z_s+x_s)ds\right)$$
$$+ \int_0^t P_\alpha(t-s)F\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}, \int_0^s K_2(s,\tau,z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)})d\tau\right)ds$$
$$+ \int_0^t P_\alpha(t-s)Bu(s)ds\Big\|$$
$$\le \|C_\alpha(t)\|\|k(0,\phi(0),0)\| + \|S_\alpha(t)[x_0+\eta]\| + \left\|k\left(t, z_t+x_t, \int_0^t K_1(t,s,z_s+x_s)ds\right)\right\|$$
$$+ \left\|\int_0^t P_\alpha(t-s)F\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}, \int_0^s K_2(s,\tau,z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)})d\tau\right)ds\right\|$$
$$+ \left\|\int_0^t P_\alpha(t-s)Bu(s)ds\right\|$$
$$\le M_c(L_{k_1}\mathbb{L}\|\phi\|_{\mathscr{B}} + \bar{L}_{k_1}) + M_s\|x_0+\eta\| + L_{k_1}\left[\|z_t+x_t\|_{\mathscr{B}} + \left\|\int_0^t K_1(t,s,z_s+x_s)ds\right\|_H\right] + \bar{L}_{k_1}$$
$$+ M\int_0^t \mu_1(s)\psi_1\left[\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathscr{B}} + \int_0^s \|K_2(s,\tau,z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)})\|_H d\tau\right]ds$$
$$+ \frac{MM_B^2}{a}\int_0^t \|P_\alpha(t-s)\|\Big\{\|z_T + x_T\| + M_c(L_{k_1}\mathbb{L}\|\phi\|_{\mathscr{B}} + \bar{L}_{k_1}) + M_s\|x_0+\eta\|$$
$$+ L_{k_1}\left[\|z_T + x_T\|_{\mathscr{B}} + \left\|\int_0^T K_1(T,\tau,z_\tau+x_\tau)d\tau\right\|_H\right] + \bar{L}_{k_1}$$

$$+MT\mu_1(s)\psi_1\left[\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathscr{B}}+\int_0^s\|K_2(s,\tau,z_{\rho(\tau,z_\tau+x_\tau)}+x_{\rho(\tau,z_\tau+x_\tau)})\|_H d\tau\right]\Big\}ds$$

$$\le \frac{MM_B^2}{a}\int_0^t\|P_\alpha(t-s)\|\|z_T+x_T\|ds+\left[1+\frac{M^2M_B^2}{a}T\right]$$

$$(\times)\Big[M_c(L_{k_1}\mathbb{L}\|\phi\|_{\mathscr{B}}+\bar{L}_{k_1})+M_s\|x_0+\eta\|+L_{k_1}c_1^*\sup_{t\in Q}\|z(t)\|+L_{k_1}c_n^*+L_{k_1}L_{K_1}c_1^*\sup_{t\in Q}\|z(t)\|$$

$$+L_{k_1}L_{K_1}c_n^*+M\int_0^t\mu_1(s)\psi_1\Big(c_1^*\|z(s)\|+c_n+\int_0^s m(\tau)\Omega_1(c_1^*\|z(\tau)\|+c_n)]\Big)ds\Big],$$

Then

$$c_n+c_1^*\|z(t)\|\le c_n+\frac{c_1^*}{1-M^*L_{k_1}c_1^*(1+L_{K_1})}\Big\{\frac{M^2M_B^2}{a}T\|z_T+x_T\|$$

$$+M^*\Big(M_c(L_{k_1}\mathbb{L}\|\phi\|_B+\bar{L}_{k_1})+M_s\|x_0+\eta\|+L_{k_1}c_n^*(1+L_{K_1})\Big)\Big\}$$

$$+\frac{c_1^*M}{1-M^*L_{k_1}c_1^*(1+L_{K_1})}\left[1+\frac{M^2M_B^2}{a}T\right]\int_0^t\mu_1(s)\psi_1\Big\{c_1^*\|z(s)\|+c_n$$

$$+\int_0^s m(\tau)\Omega_1(c_1^*\|z(\tau)\|+c_n)d\tau\Big\}ds,$$

Where

$$\omega_1^*=c_n+\frac{c_1^*}{1-M^*L_{k_1}c_1^*(1+L_{K_1})}\Big\{M_c(L_{k_1}\mathbb{L}\|\phi\|_{\mathscr{B}}+\bar{L}_{k_1})+M_s\|x_0+\eta\|+L_{k_1}c_n^*(1+L_{K_1})\Big\}$$

and

$$\omega_2^*=\frac{c_1^*M}{1-M^*L_{k_1}c_1^*(1+L_{K_1})}.$$

Denote

$$\beta(t)=\sup\{c_n^*+D_1^*\|z(s)\|:0\le s\le t\},\ t\in Q.$$

From the above inequality, we receive

$$\beta(t)\le\omega_1^*+\omega_2^*\int_0^t\mu_1(s)\psi_1\Big\{\beta(s)+\int_0^s m(\tau)\Omega_1(\beta(\tau))d\tau\Big\}ds.$$

Let us take the right hand side of the above inequality as W(t). Thus, we get

$$\beta(t)\le W(t),\ \text{for every } t\in Q,$$

with

$$W(0)=\omega_1^*$$

and

$$W'(t)=\omega_2^*\mu_1(t)\psi_1\Big\{W(t)+\int_0^t m(s)\Omega_1(W(s))ds\Big\},\ \text{a.e. } t\in Q.$$

Utilizing the non-decreasing character of $\psi_1$, we obtain

$$W'(t)\le\omega_2^*\mu_1(t)\psi_1(W(t))+m(t)\Omega_1(W(t)),\ \text{a.e. } t\in Q.$$

We characterize the function $\gamma(t)=\max\{\omega_2^*\mu_1(t),m(t)\}$, $t\in Q$, we suggests that

$$\frac{W'(t)}{\psi_1(W(t))+\Omega_1(W(t))}\le\gamma(t).$$

Integrating from 0 to t we get

$$\int_0^t \frac{W'(s)ds}{\psi_1(W(s)) + \Omega_1(W(s))} \leq \int_0^t \gamma(s)ds.$$

In view of (3.5), this implies that for each $t \in Q$, we sustain

$$\int_{W(0)}^{W(t)} \frac{ds}{\psi_1(s) + \Omega_1(s)} \leq \int_0^t \gamma(s)ds \leq \int_0^T \gamma(s)ds < \int_{\omega_1^*}^{\infty} \frac{ds}{\psi_1(s) + \Omega_1(s)}.$$

Therefore, for each $t \in Q$, there exists a constant $\Lambda_1$ in a way that$W(t) \leq \Lambda_2$ and hence $\beta(t) \leq \Lambda_2$. Since $\| z \|_{Z_0} \leq \beta(t)$, we have $\| z \|_{Z_0} \leq \Lambda_2$.

From this, we observe that the set Γ is bounded. As a consequence of Dhage's fixed point theorem[11], we realize that $\Phi_{*1} + \Phi_{*2}$ has a fixed point z defined on the interval $(-\infty, T]$ which is the mild solution of the problem (1.3)-(1.4).

**Definition 3.2.** The control system (1.1) − (1.4) are said to be approximately controllable on Q if $\overline{R(T)} = H$, where $R(T) = \{x(T; u) : u_2L(J,U)\}$ is a mild solution of the system (1.1) − (1.4).

Theorem 3.3. Suppose that the assumptions $(H_0) - (H_8)$ holds. Assume additionally that there exists $N_i \in L^1(Q, [0,\infty))$, i = 1, 2, such that $\sup_{y\in H} \| F(t,y) \| \leq N_1(t)$ and $\sup_{y \in H} \| F(t,y,x) \| \leq N_2(t)$ for a.e. $t \in Q$, then the nonlinear fractional differential inclusion (1.1) − (1.4) are approximate controllable on Q.

**Proof.** Let $\hat{y}^a(\cdot)$ and $\bar{y}^a(\cdot)$ be fixed point of Φ in B. By Theorem 3.1 and Theorem 3.2, any fixed point of Φ are mild solution of (1.1) − (1.4) under the control

$$\hat{u}^a(t) = B^* P_\alpha(T,t) R(a, \Upsilon_0^T) P(\hat{y}^a)$$

and

$$\bar{u}^a(t) = B^* P_\alpha(T,t) R(a, \Upsilon_0^T) P(\bar{y}^a)$$

and satisfies the following inequalities

(i) $\hat{y}^a(t) = y_T + aR(a, \Upsilon_0^T)P(\hat{y}^a)$;

(ii) $\bar{y}^a(t) = y_T + aR(a, \Upsilon_0^T)P(\bar{y}^a)$.

Define

$$w = x_T - C_\alpha(T)[\phi(0) + k(0,\phi(0))] - S_\alpha(T)[y_0 + \eta] + k(T, y_T) - \int_0^T P_\alpha(T-s)v(s)ds$$

and

$$\overline{w} = y_T - C_\alpha(T)[\phi(0) + k(0,\phi(0),0)] - S_\alpha(T)[x_0 + \eta] + k\left(T, y_T, \int_0^T K_1(T,s,y_s)\, ds\right) - \int_0^T P_\alpha(T-s) F\left(s, y_{\rho(s,y_s)}, \int_0^s K_2(s,\tau, y_{\rho(\tau,y_\tau)}) d\tau\right) ds.$$

By using infinite-dimensional version of the Ascoli-Arzela theorem, one can show that an operator $F(\cdot) \to R \cdot \int_0^{\cdot} s(\cdot, s)F(s)ds : L_1(Q,H) \to C$ is compact. Therefore, we obtain that $\| P(\hat{y}_a) - w \| \to 0$ and $\| P(\bar{y}_a) - w \| \to 0$ as $a \to 0_+$, respectively. Moreover, from (1.1) − (1.4) we get

$$\|\hat{y}^a(T) - y_T\| \leq \|aR(a, \Upsilon_0^T)(w)\| + \|aR(a, \Upsilon_0^T)\|\|P(\hat{y}^a) - w\|$$
$$\leq \|aR(a, \Upsilon_0^T)(w)\| + \|P(\hat{y}^a) - w\|.$$

It follows from $(H_0)$, we have that

$$\|\hat{y}^a(T) - y_T\| \to 0 \quad \text{as} \quad a \to 0^+ \quad \text{and} \quad \|\bar{y}^a(T) - y_T\| \to 0 \quad \text{as} \quad a \to 0^+.$$

This proves the approximate controllability of differential inclusion (1.1) − (1.4).

## 4 Application

In this section an illustration is provided for the existence results to the following neutral fractional integro-differential inclusion with state-dependent delay of the structure

$$\begin{aligned} D_t^\alpha\Bigg[ & w(t,x) + \int_0^t e_1(t,x,s-t) g\Big(t, y_t, \int_0^t k_1(t,s,y_s)\,ds\Big)(w(s,x))ds \\ & + \int_0^t \int_{-\infty}^s E_1(s-\tau) g\Big(t, y_t, \int_0^t k_1(t,s,y_s)\,ds\Big)(w(\tau,x))d\tau ds\Bigg] \\ \in \frac{\partial^2}{\partial x^2}\Bigg[ & w(t,x) + \int_0^t e_1(t,x,s-t) g\Big(t, y_t, \int_0^t k_1(t,s,y_s)\,ds\Big)(w(s,x))ds \\ & + \int_0^t \int_{-\infty}^s E_1(s-\tau) g\Big(t, y_t, \int_0^t k_1(t,s,y_s)\,ds\Big)(w(\tau,x))d\tau ds\Bigg] \\ & + \int_{-\infty}^t e_2(t,x,s-t) K_1(w(s-\rho_1(t)\rho_2(\|w(t)\|),x))ds \\ & + \int_0^t \int_{-\infty}^s f_1(s-\tau) K_2(w(s-\rho_1(t)\rho_2(\|w(t)\|),x))d\tau ds + u(t,x), \quad 0 \leq t \leq T, \end{aligned} \tag{4.1}$$

$$w(t,0) = 0 = w(t,\mathcal{P}), \quad t \geq 0, \tag{4.2}$$

$$w(0,x) = \phi(t,x), \ x \in (-\infty, 0], \ z \in [0,\mathcal{P}], \tag{4.3}$$

$$\frac{d}{dt} w(0,x) = y_0(x), \quad x \in [0,\mathcal{P}], \tag{4.4}$$

where $D_t^\alpha$ denotes the Caputo's fractional derivatives of order $\alpha \in (1, 2]$, $B = 1$. We consider $H = L_2[0,P]$ having the norm $\| . \|_{2L}$.

Define the operators $A : D(A) \subseteq H \to H$ by $A\omega = \omega''$ with the norm

$D(A) = \{\omega \in H : \omega, \omega'$ are absolutely continuous, $\omega'' \in H, \omega(0) = \omega(P) = 0\}$.

Then

$$A\omega = \sum_{n=1}^{\infty} n^2 \langle \omega, \omega_n \rangle \omega_n, \quad \omega \in D(A),$$

where $\omega_n(x) = \sqrt{\frac{2}{p}} \sin(nx)$, n = 1, 2, . . . is the orthogonal set of eigenvectors of A. it is well known that A is the infinitesimal generator of a strongly continuous cosine family C(t), and

$$C(t)\omega = \sum_{n=1}^{\infty} \cos nt \langle \omega, \omega_n \rangle \omega_n, \quad \omega \in H, \quad t \in \mathbb{R}.$$

For $\alpha = 2$, the associated sine family is given by

$$S(t)\omega = \sum_{n=1}^{\infty} \frac{1}{n} \sin nt \langle \omega, \omega_n \rangle \omega_n, \quad \omega \in H, \quad t \in \mathbb{R}.$$

It is clear that $\| C(t) \| \leq 1$ for all $t \in \mathbb{R}$. Thus, C(t) is uniformly bounded on R. It follows that A is the infinitesimal generator of a strongly continuous exponentially bounded fractional cosine family $C_\alpha(t)$ such that $C_\alpha(t) = I$, and

$$C_\alpha(t) = \int_0^\infty \varphi_{t,\frac{\alpha}{2}} C(s) ds, \quad t > 0,$$

where $\varphi_{t,\frac{\alpha}{2}} = t^{\frac{-\alpha}{2}} \phi_{\frac{\alpha}{2}}(st^{\frac{-\alpha}{2}})$, and

$$\phi_\delta(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\delta n + 1 - \gamma)}, \quad 0 < \delta < 1.$$

Then, there is a constant $M_C \geq 1$ such that $\|C_\alpha(t)\| \leq M_C$ for all $t \geq 0$.

For $(t, y) \in J \times \mathcal{B}$, where $w(t)(x) = w(t, x), (t, x) \in (-\infty, 0] \times [0, \mathcal{P}]$. Set

$$\rho(t, w) = \rho_1(t)\rho_2(\|w(t)\|) \quad \text{and} \quad u(t) = u(t, x), u : J \times [0, \mathcal{P}] \to [0, \mathcal{P}].$$

Define the function $g : J \times \mathcal{B} \times H \to H, F : J \times \mathcal{B} \times H \to \mathcal{P}(H)$ by

$$g\Big(t, y, \int_0^t a_1(t, s, y) ds\Big)(x) = \int_0^t e_1(t, x, s - t) g_1(w(s, x)) ds + \int_0^t \int_{-\infty}^s E_1(s - \tau) g_2(w(\tau, x)) d\tau ds$$

$$F\Big(t, y, \int_0^t a_2(t, s, y) ds\Big)(x) = \int_{-\infty}^t e_2(t, x, s - t) K_1(w(s - \rho_1(t)\rho_2(\|w(t)\|), x)) ds$$
$$+ \int_0^t \int_{-\infty}^s f_1(s - \tau) K_2(w(s - \rho_1(t)\rho_2(\|w(t)\|), x)) d\tau ds$$

$$Bu(t)(x) = u(t, x).$$

Furthermore, by applying the given conditions we can modify the system (4.1)-(4.4) into the abstract form of equation (1.3)-(1.4) and all the conditions of Theorem 3.2 are fulfilled.

## 5 Conclusion

We conclude that the overall focus of this work is on the approximate controllability of mild solutions for fractional neutral integro-differential inclusions with state-dependent delay in Banach spaces. Applying the Dhage-derived fixed point theorem for multi-valued operators. With the concourse of the highly continuous -order fractional cosine family, we put up the existence result. Ultimately, a scenario has been offered that illustrates the conclusions of the theory.

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