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RESEARCH ARTICLE

Kumaraswamy-Poisson Mixtures and its Generalizations

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ABSTRACT

In this paper, direct techniques are applied in constructing the probability density functions of Kumaraswamy-Poisson mixture distributions based on minimum and maximum order statistics from Kumaraswamy distribution and Zero-Trucated Poisson distribution. The resulting mixtures are then generalized by exponentiating part of their cumulative density functions involving the cumulative density function and survival function of the Kumaraswamy distribution. For all the mixtures, probability density functions under different parameter values are plated, and the survival and hazard functions derived. Some statistical properties of the mixtures such as the rth moments, L-moments and quantile functions are explicitly expressed. Cumulative density functions of the mixtures are easily invertible to obtain simple quantile functions that can be applied in statistical modelling. Maximum likelihood estimation of parameters is discussed.

Key words: mixture distributions, order statistics, zero-inflated, cumulative density function, moments, maximum likelihood estimation

1. Introduction

When modeling real data, simple models derived from single distributions may fit the data poorly due to data heterogeneity. Technological advances and constant creation of more robust statistical softwares have made mixture distributions to become more popular among statisticians, data scientists and researchers at large. The complex computations of properties of mixture distributions such as MLEs, moments and quantiles are now possible by running just a few codes. It is common to see research papers where several mixture models are applied on the same data sets, comparisons of estimates made across the models and the best performing model selected according to model selection criteria such as AIC, BIC, Pearson X squared among other criteria. Mixture distributions can be finite, infinite or discrete constituting of a mixture of a single distribution with various parameters or more than one distribution. They can comprise of a mixture of either discrete or continuous distributions, or a combination of the two. Some examples of mixture distributions include Double-Gaussian, COM-Poisson, Binomial Poisson, Beta-binomial, Zero-Inflated Negative Binomial, Gamma-Poisson, Kurnaraswamy-Binornial,Gaussian-Cauchy and Beta-Weibull-Exponential distribution. In count data, for example, COM-Poisson and hyper Poisson models are commonly used to model under dispersed and over dispersed counts in place of the classical Poisson model which has the assumption of equidispersion (Kaltkawi et al.,2018). In bioinformatics, beta-mixture models are fitted to solve a variety of problems arising due to diverse correlation coefficients of genes which can be assumed to be coming from several underlying distributions (Yuan Ji et al.,2005).

2. Literature review

Mixture distributions date back to late 19th century when Karl Pearson mixed two univariate normal densities to explain observed high skewness in a data set on crabs collected by W.R.Weldon. This gave rise to a five parameter finite mixture distribution, popularly known as Double Gaussian distribution. By mid 20th century, finite mixture distributions had become increasingly popular and found wide application in the field of medicine. Poondi Kumaraswamy (1980) introduced a generalized pdf for random processes bounded at both the lower and upper limits. This was later named after him as Kumaraswamy distribution. He initially used the distribution in modelling hydrological processes such as daily rainfall and daily stream flow processes, therein its application confined for almost three decades. With advent of the computer era, computation of properties of the mixtures was much easier thus their increasing popularity among researchers Everitt et al (1981). Adamidis and Loukas (1989) introduced Exponential-Geometric mixture distribution derived from minimum order exponential statistic and zero truncated geometric distribution. Jones (2009) claimed that the distribution resembled the Beta distribution but was simpler to use since its pdf, cdf, and quantiles have a closed form. He studied more properties of Kumaraswamy distribution, paving way for further study and application of the distribution. The use of order statistics in mixing distributions became more common thereafter Mecha et al (2021). The class of Generalization of the Kumaraswamy (Kw-G) distribution and its extensions such as the Kumaraswamy-Normal, Kumaraswamy-Weibull, and Kumaraswamy-Gamma among others, was first introduced by Cordeiro and de Castro (2011). The quantile function of the distribution was used to come up with its generalization. Some mathematical and statistical properties of the generalized distribution followed. Finally, the application of two Kumaraswamy mixture distributions, Gamma and Beta-Normal in a data set was discussed. Based on the results, it was concluded that models built from Beta-Normal and Kumaraswamy-Normal distributions outperformed the other models according to AIC and gave rise to plots of pdf that were almost indistinguishable when fitted along- side the histogram of the data

Cordeiro (2011). A new family of Kumaraswamy Generalized Poisson (Kw-GP) was proposed by Ramos et al (2015). This distribution was obtained by mixing the minimum order Kumaraswamy- G distribution and Zero Truncated Poisson distribution. Some extensions of the generalized mixture distribution, their pdfs and hazard functions were examined. Poisson distribution was applied by Cordeiro et al (2015) in modeling survival data alongside other Weibull mixtures. It was concluded that Kumaraswamy Weibull Poisson model provided the best model fit. Muhammad H Tahir et al (2016) made a review of some compound classes available in the literature and suggested several new classes of generalized distributions including the Exponentiated Kumaraswamy Generalized Poisson and their cumulative distribution functions. Chakraborty et al (2020) studied a family of distributions based on the mixture of Kumaraswamy-G and Generalized Poisson. This generalization was based on the mixture of random variables of the minimum order Kumaraswamy statistic and Zero Truncated Poisson distribution. Properties of the generalized mixture distribution such as moments, entropy, asymptotes, kurtosis and skewness studied.

3. Methodology

3.1 Kumaraswamy Distribution

Kumaraswamy distribution is continuous defined on the interval [0,1] and has two non- negative shape parameters, a and b. The pdf of this distribution is given by:

$$g(x) = abx^{a-1}(1-x^a)^{b-1}, 0 < x < 1, a, b > 0 \qquad [1]$$

and the cdf is given as

$$G(x) = \int_0^x abx^{a-1}(1-x^a)^{b-1}dx = 1-(1-x^a)^b \qquad [2]$$

3.2 Kumaraswamy-Poisson Distribution

Here we consider the pdf of Kumaraswamy-Poisson mixture distributions based on minimum and maximum order Kumaraswamy statistics. Properties of the distribution ie rth moment, L-moments, quatile function and maximum likelihood estimation are studied.

(1) Kw-P (min) Distribution

The pdf *f(x)* of Kw-P (min) distribution generated from the minimum order Kumaraswamy statistic and Zero-Truncated Poisson distribution is given by:-

$$. f_1(x) = \sum_{n=1}^{\infty} g_1(x\,|\,n)\,h(n)$$

$$= \sum_{n=1}^{\infty} abnx^{a-1}(1-x^a)^{bn-1}\frac{\theta^n e^{-\theta}}{n!\,(1-e^{-\theta})}$$

$$= \frac{ab\theta x^{a-1}(1-x^a)^{b-1}e^{[-\theta(1-(1-x^a)^b)]}}{1-e^{-\theta}} \qquad [3]$$

$$= \frac{\theta g(x)e^{-\theta G(x)}}{1-e^{-\theta}} \qquad [4]$$

Multiplying through [3] by e^{θ}, we get an alternative expression for$f_1(x)$ as

$$\frac{ab\theta x^{a-1}(1-x^a)^{b-1}e^{\theta(1-x^a)^b}}{e^{\theta}-1} \qquad [5]$$

The KwP(min) is a probability distribution with cdf given as

$$F_1(x) = \int_0^x f_1(x)dx = \frac{1 - e^{-\theta(1-(1-x^a)^b}}{1 - e^{-\theta}} \qquad [6]$$

The figure below illustrates plots of pdfs of the distribution for various parameter values.

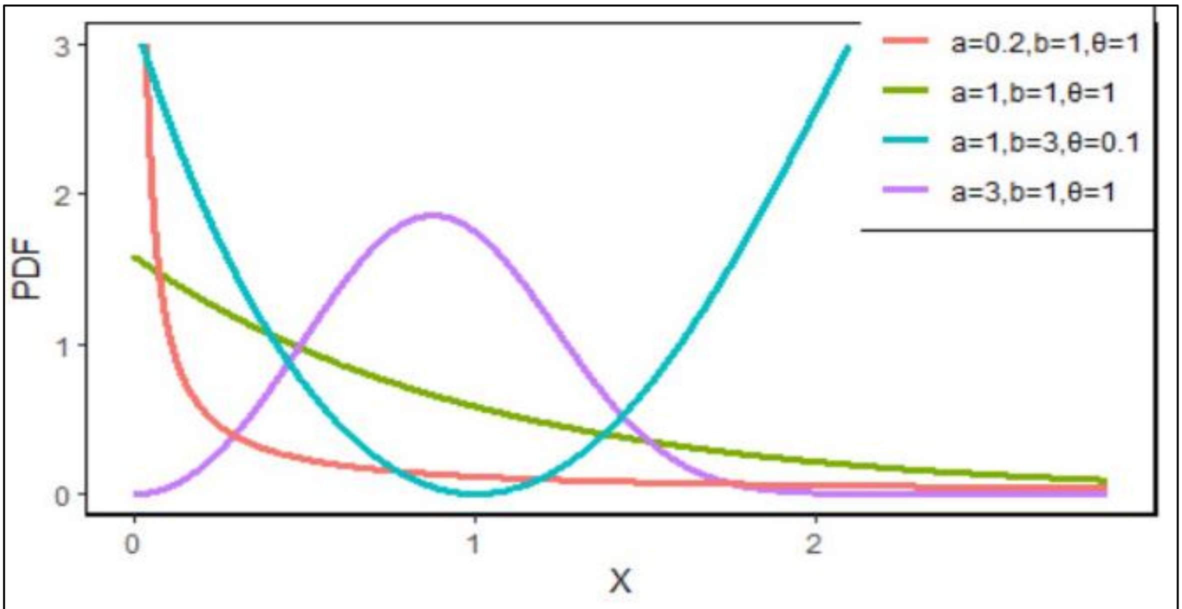

Figure 3.1. pdf of Kw-P(min)

For parameter values a= 3,b = 1, θ = 1, the pdf is unimodal while for a= 1,b = 3, θ = 0.1, the pdf is antimodal. The pdfs when both a= 1,b = 1, θ = 1 and a= 0.2,b = 1, θ = 1 are non-increasing, though at different rates.

Survival and Hazard Functions

The survival function $S_1(x)$ and hazard function $h_1(x)$ of Kw-P(min) are given by

$$S_1(x) = 1 - F_1(x)$$
$$= \frac{e^{-\theta}(e^{1-(1-x^a)^b} - 1)}{1-e^{-\theta}} \qquad [7]$$

and

$$h_1(x) = \frac{f_1(x)}{S_1(x)}$$

$$= \frac{ab\theta x^{a-1}(1-x^a)^{b-1}e^{\theta(1-x^a)^b}}{e^{1-(1-x^a))^b} - 1} \qquad [8]$$

Moment

The , r^{th} moment of Kw-P(min) is given by

$$E(x^r) = \int_0^1 x^r f_1(x)\,dx$$

$$= \frac{b}{e^{\theta}-1} \sum_{n=0}^{\infty} \frac{\theta^{n+1}}{n!} B\left(1 + \frac{r}{a}, b(n+1)\right) \qquad [9]$$

Substituting r=1 in the above equation the expected value will be

$$E(X) = \frac{b}{e^{\theta}-1}\sum_{n=0}^{\infty}\frac{\theta^{n+1}}{n!}B\left(1+\frac{1}{a}, b(n+1)\right) \quad [10]$$

and the variance is

$$Var(X) = \frac{b}{e^{\theta}-1}\sum_{n=0}^{\infty}\frac{\theta^{n+1}}{n!}B\left(1+\frac{2}{a}, b(n+1)\right) - \frac{b^2}{(e^{\theta}-1)^2}\left[\sum_{n=0}^{\infty}\frac{\theta^{n+1}}{n!}B\left(1+\frac{1}{a}, b(n+1)\right)\right]^2 \quad [11]$$

L-Moment

Probability Weighted Moment used to approximate L-moment is given by

$$b_r = \int_0^1 x\left[\frac{e^{\theta}(1-e^{(1-x^a)^b})}{e^{\theta}-1}\right]^r \frac{ab\theta x^{a-1}(1-x^a)^{b-1}e^{\theta(1-x^a)^b}}{e^{\theta}-1}dx$$

$$= \frac{ab\theta x^{r\theta}}{(e^{\theta}-1)^{r+1}}\sum_{n=0}^{\infty}\sum_{k=0}^{r}(-1)^k\binom{r}{k}\frac{(k+\theta)^n}{n!}B(b(n+1), 1+1/a) \quad [12]$$

This can be used to determine L-Skewness and L-Kurtosis as appropriate.

Quantile

Starting from the cdf *F₁(x)* of Kw-P(min) distribution, the quantile function can be easily obtained by letting

$$u = F_1(x) = \frac{1-e^{-\theta(1-(1-x^a)^b)}}{1-e^{-\theta}}$$

$$log[1-(1-e^{-\theta})u] = -\theta(1-(1-x^a)^b)$$

$$1-[\frac{-1}{\theta}log(1-(1-e^{-\theta})u)] = (1-x^a)^b$$

$$x^a = [1-\{1-(-\frac{1}{\theta}log(1-(1-e^{-\theta})u))\}^{\frac{1}{b}}]$$

$$x = [1-\left\{1-\left(-\frac{1}{\theta}log(1-\left(1-e^{-\theta}\right)u\right)\right\}^{\frac{1}{b}}]^{\frac{1}{a}} \quad [13]$$

Substituting u = 0.5, we can easily obtain the median as

$$M = \left\{1-\left\{1-(-\frac{1}{\theta}log(e^{-\theta}(1-0.5(1-e^{-\theta}))\right\}^{\frac{1}{b}}\right\}^{\frac{1}{a}} \quad [14]$$

Random variable X can be generated using the quantile function in equation [13] where $u \sim U(0,1)$ and parameters a, b, θ are preset.

Maximum Likelihood Estimation

Given the pdf $f(x; a, b, \theta)$ of Kw-P(min) distribution, the maximum likelihood estimators can be obtained by

$$L_1(x_i; a, b, \theta) = \prod_{i=1}^{n} f_1(x_i; a, b, \theta)$$

$$= \left(\frac{ab\theta}{1-e^{-\theta}}\right)^n \prod_{i=1}^{n} x_i^{a-1}(1-x_i^a)^{b-1}e^{-\theta(1-(1-x_i^a)^b)}$$

Taking partial derivatives, we have

$$\frac{\partial l}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} \log x_i - (b-1)\sum_{i=1}^{n}\frac{x_i^a \log x_i.}{1-x_i^a} - b\theta \sum_{i=1}^{n} x_i^a (1-x_i)^{b-1}\log x_i \qquad [15a]$$

$$\frac{\partial l}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \log(1-x_i^a) + \theta \sum_{i=1}^{n}(1-x_i^a \ )^b \log(1-{x_i}^a) \qquad [15b]$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - \frac{ne^{-\theta}}{1-e^{-\theta}} - \sum_{i=1}^{n}(1-(1-{x_i}^a)^b) \qquad [15c]$$

$\hat{a}, \hat{b}$and $\hat{\theta}$ can be obtained by solving numerically non-linear equations above set to zero.

(2} Kw-P (max) Distribution

The pdf *f(x)* of Kw-P (max) distribution generated from the maximum order Kumaraswamy statistic and Zero-Truncated Poisson distribution is given by:-

$$f_2(x) = \sum_{i=1}^{\infty} g_n\left(\frac{x}{n}\right)h(n) = \frac{\theta g(x)e^{-\theta(1-G(x))}}{1-e^{-\theta}} \qquad [16]$$

The KwP(max) is a probability distribution with cdf given as

$$F_2(x) = \frac{e^{-\theta}\left(e^{(1-x^a)^b}-1\right)}{1-e^{-\theta}} \qquad [17]$$

The figure below illustrates plots of pdfs of the distribution for various parameter values.

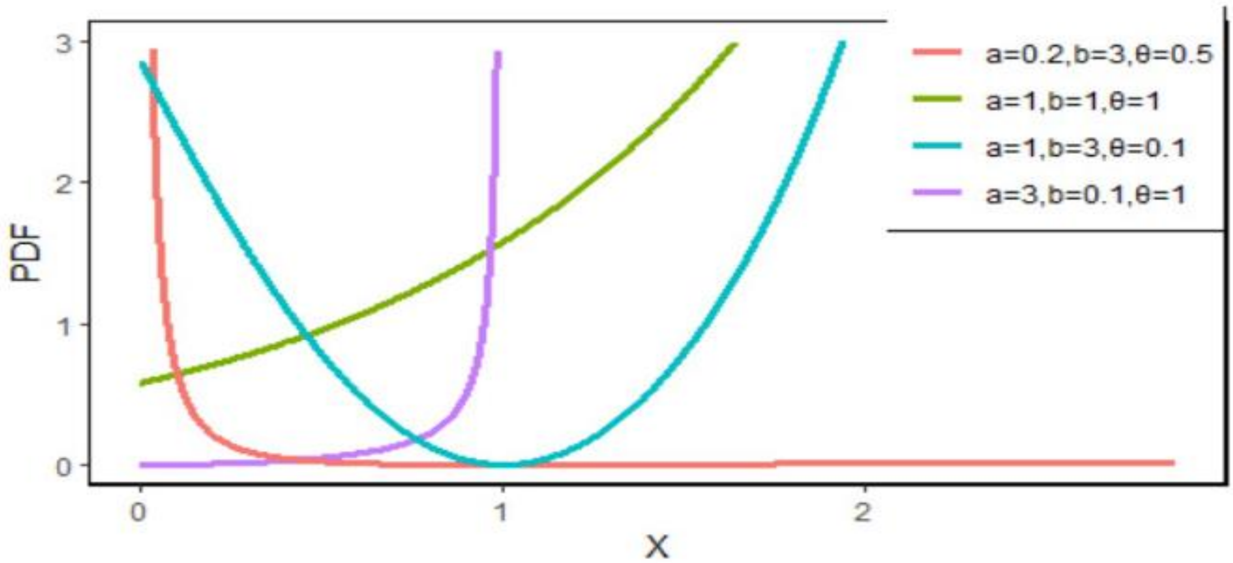

Figure 3.2. pdf of Kw-P(max)

For parameter values a= 1,b = 3,θ = 0.1, the pdf is antimodal. As for a= 0.2,b = 3, θ = 0.5, the pdf is non-ncreasing. The pdfs when a= 3,b = 0.1, θ = 1 and a= 1,b = 1, θ = 1 are non-decreasing, though at different rates.

Survival and Hazard Functions

The survival function$S_2(x)$ *of* Kw-P(max) is given by

$$S_2(x) = \frac{1 - e^{-\theta^{(1-x^a)^b}}}{1 - e^{-\theta}} \quad [18]$$

and hazard function $h_2(x)$ of Kw-P(max) are given by

$$h_2(x) = \frac{ab\theta x^{a-1}(1-x^a)^{b-1}e^{-\theta(1-x^a))^b}}{1 - e^{-\theta(1-x^a))^b}} \quad [19]$$

Moment

The r^{th} moment of Kw-P(max) distribution are given by

$$E[X^r] = \int_0^1 x^r \frac{ab\theta x^{a-1}(1-x^a)^{b-1}e^{-\theta(1-x^a)^b}}{1-e^{-\theta}} dx$$

$$= \frac{b}{1-e^{-\theta}} \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{n+1}}{n!} B\left(1 + \frac{r}{a}, b(n+1)\right) \quad [20]$$

The expected value thus becomes

$$E[X] = \frac{b}{1-e^{-\theta}} \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{n+1}}{n!} B\left(1 + \frac{1}{a}, b(n+1)\right) \quad [21]$$

And the vaeiance is given by

$$Var[X] = \frac{b}{1-e^{-\theta}} \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{n+1}}{n!} B\left(1 + \frac{2}{a}, b(n+1)\right) - \frac{b^2}{(1-e^{-\theta})^2}$$

$$\left[\sum_{n=0}^{\infty} \frac{(-1)^n \theta^{n+1}}{n!} B\left(1 + \frac{1}{a}, b(n+1)\right)\right]^2 \quad [22]$$

Probability Weighted Moment used to approximate L-moments is given as

$$a_r = \int_0^1 x \left[\frac{1-e^{-\theta(1-x^a)^b}}{1-e^{-\theta}}\right]^r \frac{ab\theta x^{a-1}(1-x^a)^{b-1}e^{-\theta(1-x^a)^b}}{1-e^{-\theta}} dx$$

$$= \frac{b}{(1-e^{-\theta})^{r+1}} \sum_{n=0}^{\infty} \sum_{k=0}^{r} (-1)^{n-k} \binom{r}{k} \frac{(k+1)^n \theta^{n+1}}{n!} B\left(b(n+1), 1 + \frac{1}{a}\right) \quad [23]$$

Quantiles

Starting from the cdf $F_2(x)$ of Kw-P(max) distribution, the quantile function can be easily obtained by

$$x = \left\{1 - \{log[ue^{\theta}(1-e^{-\theta}) + 1]\}^{1/b}\right\}^{1/a} \quad [24]$$

Substituting u = 0.5, we obtain the median

$$M = \left\{1 - \{log[0.5e^{\theta}(1-e^{-\theta}) + 1]\}^{1/b}\right\}^{1/a} \quad [25]$$

Random variable X can be generated using the quantile function in equation [24] where $u \sim U(0,1)$ and parameters $a, b, \theta$are preset.

Maximum Likelihood Estimation

Given the pdf of Kw-P(max) ,$f_2(x; a, b, \theta)$ the maximum likelihood estimators can be obtained by

$$L_2(x_i; a, b, \theta) = \prod_{i=1}^{n} f_2(x_i; a, b, \theta)$$

$$= \left(\frac{ab\theta}{1-e^{-\theta}}\right)^n \prod_{i=1}^{n} x_i^{a-1}(1-x_i^a)^{b-1} e^{-\theta(1-x_i^a)^b} \qquad [26]$$

Taking partial derivatives, we have

$$\frac{\partial l}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} log\, x_i - (b-1)\sum_{i=1}^{n} \frac{x_i^a\, log\, x_i}{1-x_i^a} + b\theta \sum_{i=1}^{n} x_i^a\,(1-x_i)^{b-1}\, log\, x_i \qquad [27a]$$

$$\frac{\partial l}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} log(\,1-x_i^a - \theta \sum_{i=1}^{n} (1-x_i^a)^b\, log(\,1-x_i^a) \qquad [27b]$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - \frac{ne^{-\theta}}{1-e^{-\theta}} - \sum_{i=1}^{n}(1-x_i^a) \qquad [27c]$$

$\hat{a}, \hat{b}$and $\hat{\theta}$ can be obtained by solving numerically non-linear equations above set to zero.

4. Exponentiated Kumaraswamy Poisson Distribution

In this section we construct generalizations of Kw-P(min) and Kw-P(max) to EKw-P(min) and EKw-P(max) respectively and study their properties.

(1) EKw=P(min) Distribution

The cdf $F_3(x)$ of EKw-P(min) generated from the cdf of Kw-P(min) distribution is

$$F_3(x) = \frac{1-e^{i\theta[G(x)]^c}}{1-e^{-\theta}}$$

$$= \frac{1-e^{-\theta(1-(1-x^a)^b)^c}}{1-e^{-\theta}} \qquad [28]$$

and the corresponding pdf is

$$f_3(x) = \frac{d}{dx}\left(\frac{1-e^{-\theta[G(x)]^c}}{1-e^{-\theta}}\right)$$

$$= \frac{abc\theta x^{a-1}(1-x^a)^{b-1}(1-(1-x^a)^b)^{c-1} e^{-\theta(1-(1-x^a)^b)^c}}{1-e^{-\theta}} \qquad [29]$$

Thus EKw-P(min) is a continuous probability distribution with parameters a,b,c, θ; x>0. When c = **1,** [28] and [29] will reduce to the cdf and pdf of Kw-P(min) distribution respectively.

The figure below illustrates plots of pdfs of the distribution for various parameter values.

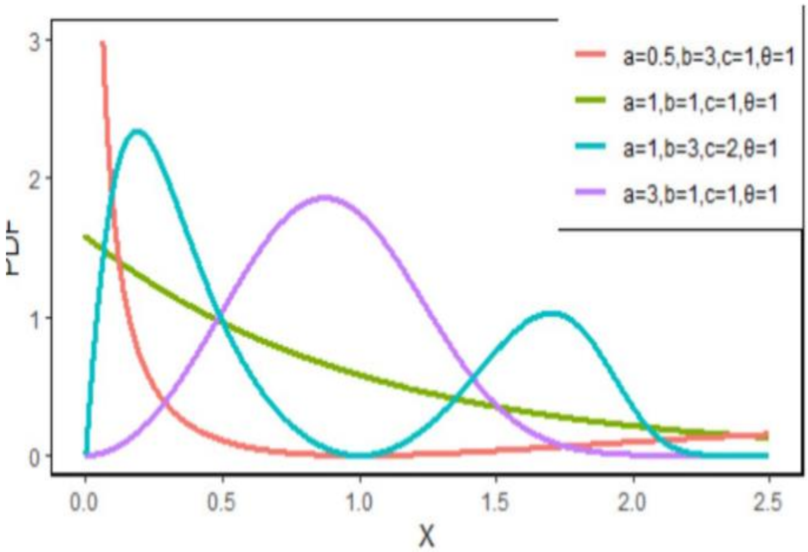

Figure 4.1. pdf of EKw-P(max)

For parameter values $a=3, b=1, c=1, \theta=1$, the pdf is unimodal while for $a=1, b=3, c=2$, $\theta=1$, the pdf is bimodal. The pdfs when $a=1, b=1, c=1, \theta=1$ and $a=0.5, b=3,, c=1, \theta=1$ are both non-increasing, though at different rates.

Survival and Hazard Functions

The survival function $S(x)$ and hazard function $h(x)$ of EKw-P(min) are given by

$$S_3(x) = \frac{1-e^{-\theta}-1+e^{-\theta[G(x)]^c}}{1-e^{-\theta}}$$

$$= \frac{e^{-\theta}[e^{[(1-(1-x^a)^b)^c}-1]}{1-e^{-\theta}} \qquad [30]$$

And

$$h_3(x) = \frac{abc\theta x^{a-1}(1-x^a)^{b-1}(1-(1-x^a)^b)^{c-1}e^{-\theta(1-(1-x^a)^b)^c}}{e^{-\theta}(e^{[(1-(1-x^a)^b)^c}-1)}$$

$$= \frac{abc\theta x^{a-1}(1-x^a)^{b-1}(1-(1-x^a)^b)^{c-1}e^{(1-(1-x^a)^b)^c}}{(e^{(1-(1-x^a)^b)^c}-1)} \qquad [31]$$

Moment

The r^{th} moment of EKw-P(min) distribution is given by

$$E[x^r] = \int_0^1 x^r \frac{abc\theta x^{a-1}(1-x^a)^{b-1}(1-(1-x^a)^b)^{c-1}e^{-\theta(1-(1-x^a)^b)^c}}{1-e^{-\theta}}dx \qquad [32]$$

Substituting appropriately, the expected value **will** be

$$E[x] = \frac{bc}{1-e^{-\theta}}\sum_{n=0}^{\infty}\sum_{k=0}^{s}(-1)^{n-k}\binom{s}{k}\frac{\theta^{n+1}}{n!}B\left(1+\frac{1}{a}, bk+1\right) \qquad [33]$$

and the variance is

$$Var[x] = \frac{bc}{1-e^{-\theta}} \sum_{n=0}^{\infty} \sum_{k=0}^{s} (-1)^{n-k} \binom{s}{k} \frac{\theta^{n+1}}{n!} B(1+2/a, bk+1)$$
$$- \left[\frac{bc}{1-e^{-\theta}} \sum_{n=0}^{\infty} \sum_{k=0}^{s} (-1)^{n-k} \binom{s}{k} \frac{\theta^{n+1}}{n!} B\left(1+\frac{1}{a}, bk+1\right) \right]^2 \quad [34]$$

When c=1 [32],[33] and [34] will reduce to the rth moment, expectation and variance of Kw-P(mi) distribution.

L-Moments

Probability Weighted Moments used to compute L-moments using special case of $b_r$is given by

$$b_r^{'} = \int_0^1 x \left[\frac{1-e^{-\theta(1-(1-x^a)^b)^c}}{1-e^{-\theta}} \right]^r \frac{abc\theta x^{a-1}(1-x^a)^{b-1}(1-(1-x^a)^b)^{c-1} e^{-\theta(1-(1-x^a)^b)^c}}{1-e^{-\theta}} dx \quad [35]$$

which can be simplified to

$$\frac{bc}{(1-e^{-\theta})^{r+1}} \sum_{n=0}^{\infty} \sum_{j=0}^{r} \sum_{k=0}^{s} (-1)^{n-j-k} \binom{r}{j} \binom{s}{k} \frac{(j+1)^n \theta^{n+1}}{n!} B\left(b(k+1), \frac{1}{1+a}\right) \quad [36]$$

When c=1, [36] reduces to L-moments for Kw-P(min) distribution.

Quantiles

Starting from the cdf $F_3(x)$ of EKw-P(min) distribution, the quantile function can be easily obtained as

$$x = \left[1 - \left[1 - \left[-\theta^{-1} \log(1-(1-e^{-\theta})u \right]^{1/c} \right]^{1/b} \right]^{1/a} \quad [37]$$

Substituting u = 0.5, we obtain the median as

$$M = \left[1 - \left[1 - \left[-\theta^{-1} \log(1-(1-e^{-\theta})0.5 \right]^{1/c} \right]^{1/b} \right]^{1/a} \quad [38]$$

Random variable X can be generated using the quantile function in equation [37] where $u \sim U(0,1)$ and parameters $a, b, c, \theta$are preset.

Maximum Likelihood Estimation

Given the pdf $f_3(x; a, b, c, \theta)$ of EKw-P(min) distribution, the maximum likelihood estimators can be obtained by

$$L_3(x_i; a, b, c, \theta) = \prod_{i=1}^{n} \frac{abc\theta x_i^{a-1}(1-x_i^a)^{b-1}(1-(1-x_i^a)^b)^{c-1} e^{-\theta(1-(1-x_i^a)^b)^c}}{1-e^{-\theta}} \quad [39]$$

Taking partial derivatives, we have

$$\frac{\partial l}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} \log x_i + b(c-1) \sum_{i=1}^{n} \frac{\log(1-x_i^a)^{b-1} x_i^a \log x_i}{1-(1-x_i^a)^b} - (b-1) \sum_{i=1}^{n} \frac{x_i^a \log x_i}{1-x_i^a}$$

$$-bc\theta\sum_{i=1}^{n}(1-(1-x_i^a)^b)^{c-1}(1-x_i)^{b-1}x_i^a\log x_i \qquad [40a]$$

$$\frac{\partial l}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}log(1-x_i^a)+(c-1)\sum_{i=1}^{n}\frac{(1-x_i^a)^b\,log(1-x_i^a)}{1-(1-x_i^a)^b}$$

$$-c\theta\sum_{i=1}^{n}(1-(1-x_i^a)^b)^{c-1}(1-x_i)^b x_i^a\,log(1-log\,x_i^a) \qquad [40b]$$

$$\frac{\partial l}{\partial c}=\frac{n}{c}+\sum_{i=1}^{n}(1-(1-x_i^a)^b)-\theta\sum_{i=1}^{n}(1-(1-x_i^a)^b)^c\,log(1-(1-x_i^a)^b) \qquad [40c]$$

$$\frac{\partial l}{\partial \theta}=\frac{n}{\theta}-\frac{ne^{-\theta}}{1-e^{-\theta}}-\sum_{i=1}^{n}(1-(1-x_i^a)^b)^c \qquad [40d]$$

$\hat{a},\hat{b},\hat{c}$and $\hat{\theta}$ can be obtained by solving numerically non-linear equations above set to zero.

(2) EKw=P(max) Distribution

The cdf $F_4(x)$ of EKw-P(max) generated from the cdf of Kw-P(max) distribution is

$$F_4(x)=\frac{e^{-\theta}[e^{-\theta[1-G(x)]^c}-1]}{1-e^{-\theta}}$$

$$=\frac{e^{-\theta}[e^{-\theta((1-x^a)^b)^c}-1]}{1-e^{-\theta}} \qquad [41]$$

and the corresponding pdf is

$$f_4(x)=\frac{d}{dx}\left[\frac{e^{-\theta}[e^{-\theta[1-G(x)^c]}]}{1-e^{-\theta}}\right]$$

$$=\frac{abc\theta x^{a-1}(1-x^a)^{b-1}((1-x^a)^b)^{c-1}e^{-\theta((1-x^a)^b)^c}}{1-e^{-\theta}} \qquad [42]$$

Thus EKw-P(max) distribution is a continuous with parameters a,b,c,θ and $x>0$.

The figure below illustrates plots of pdfs of the distribution for various parameter values.

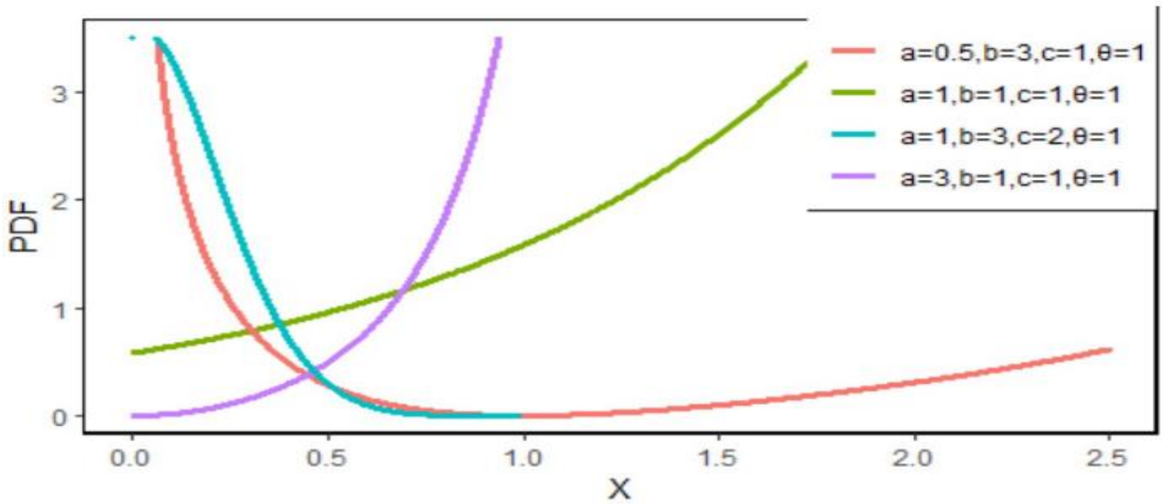

Figure 4.2. pdf of E-KwP(max)

For parameter values a = 1,b = 3, c = 2, θ = 1, the pdf is non-increasing. As for a= 0.5,b = 3,c = 1, θ = 1, the pdf increases then decreases. The pdfs when a= 1,b = 3,c = 2, θ = 1 and a = 1,b = 1,c = 1, θ = 1 are non-decreasing, though at different rates.

Survival and Hazard Functions

The survival function *S(x)* and hazard function *h(x)* of EKw-P(max) distribution are given by

$$S_4(x) = \frac{1 - e^{-\theta} - e^{-\theta[1-G(x)]^c} + e^{-\theta}}{1 - e^{-\theta}} \quad [43]$$

and

$$h_4(x) = \frac{abc\theta x^{a-1}(1-x^a)^{b-1}((1-x^a)^b)^{c-1}e^{-\theta((1-x^a)^b)^c}}{1-e^{-\theta((1-x^a)^b)^c}} \quad [44]$$

Moments

The r^{th} moment of EKw-P(max) distribution is given by

$$E[x^r] = \int_0^1 x^r \frac{abc\theta x^{a-1}(1-x^a)^{b-1}((1-x^a)^b)^{c-1}e^{-\theta((1-x^a)^b)^c}}{1-e^{-\theta}} dx$$

$$= \frac{bc}{1-e^{-\theta}} \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{n+1}}{n!} B\left(1 + \frac{r}{a}, bc(n+1)\right) \quad [45]$$

Substituting appropriately, the expectation will be

$$E[X] = \frac{bc}{1-e^{-\theta}} \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{n+1}}{n!} B\left(1 + \frac{1}{a}, bc(n+1)\right) \quad [46]$$

and the variance is

$$Var[X] = \frac{bc}{1-e^{-\theta}} \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{n+1}}{n!} B\left(1 + \frac{2}{a}, bc(n+1)\right)$$

$$-\left[\frac{bc}{1-e^{-\theta}} \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{n+1}}{n!} B\left(1 + \frac{1}{a}, bc(n+1)\right)\right]^2 \quad [47]$$

When c=1, [45], [46] and [47] will reduce to rth moment, expectation an variance of Kw-P(max) distribution.

L-Moments

Probabilty Weighted Moments of EKw-P(max) distribution will be used to approximate its L-moments using the special case of a_r ie

$$a_r' = \int_0^1 x \left[\frac{1 - e^{-\theta((1-x^a)^b)^c}}{1-e^{-\theta}}\right]^r \frac{abc\theta x^{a-1}(1-x^a)^{b-1}(1-x^a)^{b(c-1)}e^{-\theta(1-x^a)^{bc}}}{1-e^{-\theta}} dx[\quad 48]$$

which can be simplified to

$$\frac{abc}{(1-e^{-\theta})^{r+1}} \sum_{k=0}^{r} (-1)^k \binom{r}{k} \sum_{n=1}^{\infty} \frac{(-1)^n (k+1)^n \theta^{n+1}}{n!} B\left(bc(n+1), 1 + \frac{1}{a}\right) \quad [49]$$

When c=1,[49] will reduce Lmoments of Kw-P(max) distribution.

Quantiles

Let $F_4(x) = u,$ the quantile function will be given by:

$$u = \frac{e^{-\theta}[e^{\theta((1-x^a)^b)^c} - 1]}{1-e^{-\theta}}$$

After mathematical manipulation this becomes

$$x = \left[1 - \left[\left[\log(1+(1-e^{-\theta})ue^{\theta}\right]^{1/c}\right]^{1/b}\right]^{1/a} \qquad [50]$$

Substituting u = 0.5, we can easily obtain the median as

$$M = \left[1 - \left[\left[\log(1+(1-e^{-\theta})0.5e^{\theta}\right]^{1/c}\right]^{1/b}\right]^{1/a} \qquad [51]$$

Random variable X can be generated using quantile function given in [50] where $u \sim U(0,1)$ and parameters $a, b, c, \theta$are preset.

Maximum Likelihood Estimation

Given the pdf *f(x;a,b,c,O)* of EKw-P(max) distribution, the maximum likelihood function is given by

$$L(x; a, b, \theta) = \prod_{i=1}^{n} \frac{abc\theta x_i^{a-1}(1-x_i^a)^{b(c-1)}e^{-\theta(1-x_i^a)^{bc}}}{1-e^{-\theta}} \qquad [52]$$

Taking logs and partial derivatives, we have

$$\frac{\partial l}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} log\, x_i - (b-1)\sum_{i=1}^{n}\frac{x_i^a\, log\, x_i}{1-x_i^a} + bc\theta\sum_{i=1}^{n}(1-x_i^a)^{bc-1}\, x_i^a\, log\, x_i \; [53a]$$

$$\frac{\partial l}{\partial b} = \frac{n}{b} + (c-1)\sum_{i=1}^{n} log(1-x_i^a) - c\theta\sum_{i=1}^{n}(1-x_i^a)^{b(c-1)}(1-x_i^a)^b\, log(1-x_i^a) [53b]$$

$$\frac{\partial l}{\partial c} = \frac{n}{c} + b\sum_{i=1}^{n}(1-x_i^a) - b\theta\sum_{i=1}^{n}(1-x_i^a)^{c(b-1)}(1-x_i^a)^c\, log(1-x_i^a) [53c]$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - \frac{ne^{-\theta}}{1-e^{-\theta}} - \sum_{i=1}^{n}(1-x_i^a)^{bc} \qquad [53d]$$

$\hat{a}, \hat{b}, \hat{c}$and $\hat{\theta}$ can be obtained by solving numerically non-linear equations above set to zero.

5. Conclusion

Kumaraswamy-Poisson mixture distributions are continuous with random variablex > 0. This allows for a wider application of the mixtures compared to Kumaraswamy distribution whose domain is O < x < I. Kw-P(min) and Kw-P(max) distributions are generalized by exponentiating part of their cdfs involving the cdf and sf of the parent distribution respectively to obtain the cdfs of EKw-P(min) and EKw-P(max) distributions. When the exponent c is equated to 1, EKw-P(min) and EKw-P(max) distributions reduce to Kw-

P(min) and Kw-P(max) distributions respectively. Plots of the pdfs of the mixtures under different parameter values are versatile implying the distributions could be applied in modelling data sets of a wide variety.The mixture distributions have the benefit of quantile functions that are simple to derive, just like their parent distribution. This makes it easy to conduct quantile-based statistical analysis and in studying the properties of the mixtures through data simulations. The median, on the other hand, can be easily used as a measure of central tendency. The expectations and variances of the mixtures are versatile and resemble those of Ku- maraswamy distribution. This is an improvement on Poisson distribution which assumes equidispersion. Skewness and kurtosis can be easily defined using the derived PWMs and their numerical computations done using programming languages like R.

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