Vol.11.Issue.4.2023 (Oct-Dec)
©KY PUBLICATIONS

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

http://www.bomsr.com

Email:editorbomsr@gmail.com

INTERNATIONAL STANDARD SERIAL NUMBER 2348-0580

RESEARCH ARTICLE

A QUEUEING SYSTEM WITH CATASTROPHE, STATE DEPENDENT SERVICE AND ENVIRONMENTAL CHANGE

DARVINDER KUMAR
Associate Professor, Department of Statistics
PGDAV College (University of Delhi)
DOI:10.33329/bomsr.11.4.9

ISSN:2348-0580 BOMSR BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH www.bomsr.com

ABSTRACT

In this paper, a finite capacity queueing system with state dependent service operating in different environments with catastrophes is studied. The service rate increases (decreases) according as n, the number of units in the system, is less (greater) than N, a pre-assigned number. We undertake the transient analysis of a limited capacity queueing system with two environmental states in the presence of catastrophes. Transient state solution is obtained by using the technique of probability generating function. The steady state results of the model is obtained by using the property of Laplace transform. Finally, some particular cases of the queueing model are also derived and discussed.

Keywords: Catastrophes, Environment, Service rate, Probability generating function, Laplace transform.

1. Introduction

From the last few decades, the M/M/1 queue has been the object of systematic and through investigations. In recent years, the attention has been focused on certain extensions that include the effect of catastrophes. In this connection, a special reference may be made to the paper by A. Di Crescenzo et al. [6]. In paper [6], the authors have recognized the role played by the notion of catastrophe in various areas of science and technology, in particular birth and death models. This consists of adding to the standard assumptions the hypothesis that the number of customers are instantly reset to zero at certain random times. The catastrophes occur at the service- facility as a

Poisson process with rate ξ. Whenever a catastrophe occurs at the system, all the customers there are destroyed immediately, the server gets inactivated momentarily, and the server is ready for service when a new arrival occurs.

A large number of research papers have appeared dealing with population processes under the influence of catastrophes (see. e.g., Brockwell [2], Brockwell et al., [3] and Bartoszynski et al., [3]). These works are also concerned with various quantities of interest, such as transition probabilities, the stationary probabilities and the time to extinction. It is also well known that computer networks with a virus may be modeled by queueing networks with catastrophes [5]. Jain and Kanethia [8] studied the transient analysis of a queue with environmental and catastrophic effects.

A. Di Crescenzo et al. [6] proved that the M/M/1 catastrophized processes may be suitable to approach a current hot topic of great biological relevance, concerning the interaction between myosin heads and actin filaments that is responsible for force generation during muscle contraction. However, the force of contraction may rise on changing other conditions like a change in temperature or pH or a slight stretching of the fiber. Now, in the present paper, we have added another factor of environmental change, i.e. the change in the environment affects the state of the queueing system. In other words, the state of the queueing system is a function of environmental change factors.

The direct application of the model can be ascribed to a biological phenomenon that there are many creatures such as cockroaches, ants, mosquitoes etc whose movement is restricted with the change of temperature (environment). As the temperature drops below a critical temperature say T_0, the movement (production) of such like creatures becomes almost zero. On the other hand, as the temperature goes higher than T_0 the movement becomes normal. The catastrophes may occur with these creatures in both the environmental states i.e., spray etc which make them zero instantaneously. Then the number of such like creatures present in any area can be estimated by using the described queueing model with environmental change and catastrophes.

The layout of this paper is as follows. In the next section we present the assumptions and definitions of the model. Section 3 provides a detailed analysis of the main model, which is used in section 4 in proving some particular cases. Steady state results are also derived and shown in section 5.

2. Assumptions and Definitions:

(i) The customers arrive in the system one by one in accordance with a Poisson process at a single service station. The arrival pattern is non-homogeneous, i.e. there may exist two arrival rates, namely λ_1 and 0 of which only one is operative at any instant.

(ii) The customers are served one by one at the single channel. The service time is exponentially distributed. Further, it has been assumed that corresponding to arrival rate λ_1 the Poisson service rate is a_n and the service rate corresponding to the arrival rate 0 is b_n. The state of the queueing system when operating with arrival rate λ_1 and service rate a_n is designated as E whereas the other with arrival rate 0 and service rate b_n is designated as F.

(iii) The Poisson service rate a_n is assumed to depend on the number waiting in the queue, including the one in service in such a manner that whenever this number (say n) is equal to some fixed number (say N), we have some normal rate as μ_1 and for number of units greater than N, the rate is higher and for number of units less than N it is lower than the normal rate. We therefore, define

$$a_n = \mu_1[1+\varepsilon(n-N)] \quad \text{with} \quad n \geq N - \frac{1}{\varepsilon}$$

$$\text{and} \quad 0 \leq N - \frac{1}{\varepsilon} \leq n \leq M$$

Where M denotes the size of the waiting space and ε is a positive number $\geq \frac{1}{N}$. This restriction on M is necessary to avoid a negative value of a_n. Similarly, the Poisson service rate b_n is defined as

$$b_n = \mu_2[1+\varepsilon(n-N)] \quad \text{with} \quad n \geq N - \frac{1}{\varepsilon}$$

$$\text{and} \quad 0 \leq N - \frac{1}{\varepsilon} \leq n \leq M$$

(iv) The Poisson rates at which the system moves from environmental states F to E and E to F are denoted by α and β respectively.

(v) When the system is not empty, catastrophes occur according to a Poisson process with rate ξ. The effect of each catastrophe is to make the queue instantly empty. Simultaneously, the system becomes ready to accept the new customers.

(vi) The queue discipline is first- come- first- served.

(vii) The capacity of the queueing system is limited to M. i.e., if at any instant there are M units in the queue then the units arriving at that instant will not be permitted to join the queue, it will be considered lost for the system.

3. Formulation of Model and Analysis (Time Dependent Solution):

Define,

P_n (t) = Joint probability that at time t the system is in state E and n units are in the queue, including the one in service.

Q_n(t) = Joint probability that at time t the system is in state F and n units are in the queue, including the one in service.

R_n(t) = The probability that at time t there are n units in the queue, including the one in service.

Obviously,

$$R_n(t) = P_n(t) + Q_n(t)$$

Let us reckon time t from an instant when there are zero customers in the queue and the system is in the environmental state E so that the initial conditions associated with P_n(t) and Q_n(t) becomes,

$$P_n(0) = \begin{cases} 1 & ; \quad n = 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$Q_n(0) = 0$; for all n.

The differential -difference equations governing the system are:

$$\frac{d}{dt}P_0(t) = -(\lambda_1 + \beta + \xi)P_0(t) + a_1P_1(t) + \alpha Q_0(t) + \xi\sum_{n=0}^{M}P_n(t);\ n = 0 \quad \text{..... (1)}$$

$$\frac{d}{dt}P_n(t) = -(\lambda_1 + a_n + \beta + \xi)P_n(t) + a_{n+1}P_{n+1}(t) + \lambda_1 P_{n-1}(t) + \alpha Q_n(t);\ 0 < n < M \quad \text{.... (2)}$$

$$\frac{d}{dt}P_M(t) = -(a_M + \beta + \xi)P_M(t) + \lambda_1 P_{M-1}(t) + \alpha Q_M(t);\quad n = M \quad \text{.... (3)}$$

$$\frac{d}{dt}Q_0(t) = -(\alpha + \xi)Q_0(t) + b_1Q_1(t) + \beta P_0(t) + \xi\sum_{n=0}^{M}Q_n(t);\quad n = 0 \quad \text{.... (4)}$$

$$\frac{d}{dt}Q_n(t) = -(b_n + \alpha + \xi)Q_n(t) + b_{n+1}Q_{n+1}(t) + \beta P_n(t);\ 0 < n < M \quad \text{..... (5)}$$

$$\frac{d}{dt}Q_M(t) = -(b_M + \alpha + \xi)Q_M(t) + \beta P_M(t);\ n = M \quad \text{..... (6)}$$

Define, the Laplace Transform as

$$\text{L.T. } [f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s) \quad \text{.....(7)}$$

Now, taking the Laplace transforms of equations (1)–(6) and using the initial conditions, we get

$$(s + \lambda_1 + \beta + \xi)\overline{P}_0(s) - 1 = a_1\overline{P}_1(s) + \alpha\overline{Q}_0(s) + \xi\sum_{n=0}^{M}\overline{P}_n(s) \quad \text{`.... (8)}$$

$$(s + \lambda_1 + a_n + \beta + \xi)\overline{P}_n(s) = a_{n+1}\overline{P}_{n+1}(s) + \lambda_1\overline{P}_{n-1}(s) + \alpha\overline{Q}_n(t) \quad \text{.... (9)}$$

$$(s + a_M + \beta + \xi)\overline{P}_M(s) = \lambda_1\overline{P}_{M-1}(s) + \alpha\overline{Q}_M(s) \quad \text{... (10)}$$

$$(s + \alpha + \xi)\overline{Q}_0(s) = b_1\overline{Q}_1(s) + \beta\overline{P}_0(s) + \xi\sum_{n=0}^{M}\overline{Q}_n(s) \quad \text{.... (11)}$$

$$(s + b_n + \alpha + \xi)\overline{Q}_n(s) = b_{n+1}\overline{Q}_{n+1}(s) + \beta\overline{P}_n(s) \quad \text{.... (12)}$$

$$(s + b_M + \alpha + \xi)\overline{Q}_M(s) = \beta\overline{P}_M(s) \quad \text{.... (13)}$$

Define, the probability generating functions

$$P(z,s) = \sum_{n=0}^{M}\overline{P}_n(s)z^n \quad \text{.... (14)}$$

$$Q(z,s) = \sum_{n=0}^{M}\overline{Q}_n(s)z^n \quad \text{.... (15)}$$

$$R(z,s) = \sum_{n=0}^{M}\overline{R}_n(s)z^n \quad \text{.... (16)}$$

where

$$\overline{R}_n(s) = \overline{P}_n(s) + \overline{Q}_n(s)$$

Multiplying equations (8)–(10) by the suitable powers of z, summing over all n and using equations (14)–(16), we have.

$$z(z-1)\mu_1\varepsilon P'(z,s) + [zs + \mu_1(z-1)(1-\varepsilon N) + \lambda_1 z(1-z) + \beta z + \xi z]P(z,s) - \alpha\, zQ(z,s)$$
$$= z + \mu_1(1-\varepsilon N)(z-1)\overline{P}_0(s) + \lambda_1 z^{M+1}(1-z)\overline{P}_M(s) + \xi z\sum_{n=0}^{M}\overline{P}_n(s) \quad \text{.... (17)}$$

Similarly, from equations (11)–(13) and using equations (14)–(16), we have

$$z(z-1)\mu_2\varepsilon Q'(z,s) + [zs + \mu_2(z-1)(1-\varepsilon N) + \alpha z + \xi z]Q(z,s) - \beta z P(z,s)$$
$$= \mu_2(1-\varepsilon N)(z-1)\overline{Q}_0(s) + \xi z\sum_{n=0}^{M}\overline{Q}_n(s) \quad \text{... (18)}$$

In order to obtain P(z,s) and Q(z,s) from equations (17) and (18), we use **Iteration Method.**

If we assume that the parameter β is small, then we can use it in the series solution as follows:

$$P(z,s) = P_0(z,s) + \beta P_1(z,s) + \cdots\cdots\cdots \quad \text{..... (19)}$$

$$Q(z,s) = Q_0(z,s) + \beta Q_1(z,s) + \cdots\cdots\cdots \quad \text{..... (20)}$$

Where the non-written terms are of higher order of β (i.e., we limit ourselves to the first approximation). Substituting values of P(z,s) and Q(z,s) from equations (19) and (20) in equations (17) and (18) and identifying terms with like powers of β . We obtain thus the zero order (i.e., terms not containing β) and one order (i.e., terms containing first power of β) approximations:

$$P_0'(z,s) + \eta_1(z)P_0(z,s) - \frac{\alpha}{\mu_1\varepsilon(z-1)}Q_0(z,s) = z_1 \quad \text{.... (21)}$$

$$Q_0'(z,s) + \eta_2(z)Q_0(z,s) = z_2 \quad \text{.... (22)}$$

$$P_1'(z,s) + \frac{1}{(z-1)\mu_1\varepsilon}P_0(z,s) + \eta_1(z)P_1(z,s) - \frac{\alpha}{(z-1)\mu_1\varepsilon}Q_1(z,s) = 0 \quad \text{.... (23)}$$

$$Q_1'(z,s) + \eta_2(z)Q_1(z,s) - \frac{1}{(z-1)\mu_2\varepsilon}P_0(z,s) = 0 \quad \text{... (24)}$$

where,

$$\eta_1(z) = \frac{zs + \mu_1(z-1)(1-\varepsilon N) + \lambda_1 z(1-z) + \xi z}{z(z-1)\mu_1\varepsilon}$$

$$\eta_2(z) = \frac{zs + \mu_2(z-1)(1-\varepsilon N) + \alpha z + \xi z}{z(z-1)\mu_2\varepsilon}$$

$$z_1 = \frac{z + \mu_1(1-\varepsilon N)(z-1)\overline{P}_0(s) + \lambda_1 z^{M+1}(1-z)\overline{P}_M(s) + \xi z\sum_{n=0}^{M}\overline{P}_n(s)}{z(z-1)\mu_1\varepsilon}$$

$$z_2 = \frac{\mu_2(1-\varepsilon N)(z-1)\overline{Q}_0(s) + \xi z \sum_{n=0}^{M} \overline{Q}_n(s)}{z(z-1)\mu_2\,\varepsilon}$$

On solving equation (22), we have

$$Q_0(z,s) = \frac{M(z)\ \overline{Q}_0(s) + N(z)\ \sum_{n=0}^{M} \overline{Q}_n(s)}{A(z)} \qquad \text{... (25)}$$

where

$$A(z) = z^{\frac{1-\varepsilon N}{\varepsilon}} \cdot (z-1)^{\frac{(s+\alpha+\xi)}{\mu_2 \varepsilon}}$$

$$M(z) = \frac{1-\varepsilon N}{\varepsilon} \int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}-1} \cdot (z-1)^{\frac{(s+\alpha+\xi)}{\mu_2 \varepsilon}}\ dz$$

$$N(z) = \frac{\xi}{\mu_2 \varepsilon} \int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}} \cdot (z-1)^{\frac{(s+\alpha+\xi)}{\mu_2 \varepsilon}-1}\ dz$$

Solving equations (25) and (21) for $P_0(z,s)$, we have

$$P_0(z,s) = \frac{\overline{Q}_0(s)K_1(z) + \overline{P}_0(s)K_2(z) + \overline{P}_M(s)K_3(z) + \sum_{n=0}^{M} \overline{Q}_n(s)K_4(z) + \sum_{n=0}^{M} \overline{P}_n(s)K_5(z) + K_6(z)}{B(z)} \qquad \text{... (26)}$$

where

$$B(z) = z^{\frac{1-\varepsilon N}{\varepsilon}} \cdot (z-1)^{\frac{(s+\xi)}{\mu_1 \varepsilon}} \cdot e^{\frac{-\lambda_1}{\mu_1 \varepsilon} \cdot z}$$

$$K_1(z) = \frac{\alpha}{\mu_1 \varepsilon} \int_0^z (z-1)^{\frac{s+\xi}{\mu_1 \varepsilon} - \left\{\frac{(s+\alpha+\xi)}{\mu_2 \varepsilon}+1\right\}} \cdot e^{\frac{-\lambda_1}{\mu_1 \varepsilon} \cdot z} \cdot M(z)\, dz$$

$$K_2(z) = \frac{1-\varepsilon N}{\varepsilon} \int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}-1} \cdot (z-1)^{\frac{(s+\xi)}{\mu_1 \varepsilon}} \cdot e^{\frac{-\lambda_1}{\mu_1 \varepsilon} \cdot z}\ dz$$

$$K_3(z) = -\frac{\lambda_1}{\mu_1 \varepsilon} \int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}+M} \cdot (z-1)^{\frac{(s+\xi)}{\mu_1 \varepsilon}} \cdot e^{\frac{-\lambda_1}{\mu_1 \varepsilon} \cdot z}\ dz$$

$$K_4(z) = \frac{\alpha}{\mu_1 \varepsilon} \int_0^z (z-1)^{\frac{s+\xi}{\mu_1 \varepsilon} - \left\{\frac{(s+\alpha+\xi)}{\mu_2 \varepsilon}+1\right\}} \cdot e^{\frac{-\lambda_1}{\mu_1 \varepsilon} \cdot z} \cdot N(z)\ dz$$

$$K_5(z) = \frac{\xi}{\mu_1 \varepsilon} \int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}} \cdot (z-1)^{\frac{s+\xi}{\mu_1 \varepsilon}-1} \cdot e^{\frac{-\lambda_1}{\mu_1 \varepsilon} \cdot z}\ dz$$

$$K_6(z)=\frac{1}{\mu_1\varepsilon}\int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}}\cdot(z-1)^{\frac{(s+\xi)}{\mu_1\varepsilon}-1}\cdot e^{\frac{-\lambda_1}{\mu_1\varepsilon}\cdot z}\,dz$$

Solving equations (26) and (24) for $Q_1(z,s)$, we have

$$Q_1(z,s)=\frac{\overline{Q}_0(s)K_7(z)+\overline{P}_0(s)K_8(z)+\overline{P}_M(s)K_9(z)+\sum_{n=0}^{M}\overline{Q}_n(s)K_{10}(z)+\sum_{n=0}^{M}\overline{P}_n(s)K_{11}(z)+K_{12}(z)}{A(z)} \quad ..(27)$$

where,

$$K_{i+6}(z)=\frac{1}{\mu_2\varepsilon}\int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}}\cdot(z-1)^{\frac{(s+\alpha+\xi)}{\mu_2\varepsilon}-1}\cdot K_i(z)\,dz \qquad ;\ i=1, 2, 3, 4, 5, 6.$$

Again solving equations (27), (26) and (23) for $P_1(z,s)$, we have

$$P_1(z,s)=\frac{\overline{Q}_0(s)K_{13}(z)+\overline{P}_0(s)K_{14}(z)+\overline{P}_M(s)K_{15}(z)+\sum_{n=0}^{M}\overline{Q}_n(s)K_{16}(z)+\sum_{n=0}^{M}\overline{P}_n(s)K_{17}(z)+K_{18}(z)}{B(z)} \quad ..(28)$$

where

$$K_{i+j+(6-j)}(z)=\frac{1}{\mu_1\varepsilon}\int_0^z\frac{1}{z-1}\left[\frac{\alpha B(z)}{A(z)}K_i(z)-K_{j+1}(z)\right]dz \quad ;$$
$$\left[(j,i):\{(0,7),(1,8),(2,9),(3,10),(4,11)\ (5,12)\}\right]$$

Thus by putting the values of $P_0(z,s),\ P_1(z,s),\ Q_0(z,s),\ Q_1(z,s)$ in equations (19) and (20) we have the final approximate solutions for $P(z,s)$ and $Q(z,s)$

$$P(z,s)=\frac{\begin{array}{l}[K_1(z)+\beta K_{13}(z)]\overline{Q}_0(s)+[K_2(z)+\beta K_{14}(z)]\overline{P}_0(s)+[K_3(z)+\beta K_{15}(z)]\overline{P}_M(s)\\ +[K_4(z)+\beta K_{16}(z)]\sum_{n=0}^{M}\overline{Q}_n(s)+[K_5(z)+\beta K_{17}(z)]\sum_{n=0}^{M}\overline{P}_n(s)+[K_6(z)+\beta K_{18}(z)]\end{array}}{B(z)} \quad ....(29)$$

$$Q(z,s)=\frac{\begin{array}{l}[M(z)+\beta K_7(z)]\overline{Q}_0(s)+\beta K_8(z)\overline{P}_0(s)+\beta K_9(z)\overline{P}_M(s)+[N(z)+\beta K_{10}(z)]\sum_{n=0}^{M}\overline{Q}_n(s)\\ +\beta K_{11}(z)\sum_{n=0}^{M}\overline{P}_n(s)+\beta K_{12}(z)\end{array}}{A(z)} \quad ....(30)$$

where

$$P(1,s)=\sum_{n=0}^{M}\overline{P}_n(s)=\frac{s+\alpha+\xi}{s(s+\alpha+\beta+\xi)}$$

$$Q(1,s)=\sum_{n=0}^{M}\overline{Q}_n(s)=\frac{\beta}{s(s+\alpha+\beta+\xi)}$$

On adding relations (29) and (30), we have

$$R(z,s)=\frac{\begin{array}{l}[A(z)\{K_1(z)+\beta K_{13}(z)\}+B(z)\{M(z)+\beta K_7(z)\}]\overline{Q}_0(s)+[A(z)\{K_2(z)+\beta K_{14}(z)\}+\beta B(z)K_8(z)]\overline{P}_0(s)+\\ [A(z)\{K_3(z)+\beta K_{15}(z)\}+\beta B(z)K_9(z)]\overline{P}_M(s)+[A(z)\{K_4(z)+\beta K_{16}(z)\}+B(z)\{N(z)+\beta K_{10}(z)\}]\sum_{n=0}^{M}\overline{Q}_n(s)\\ +[A(z)\{K_5(z)+\beta K_{17}(z)\}+\beta B(z)K_{11}(z)]\sum_{n=0}^{M}\overline{P}_n(s)+[A(z)\{K_6(z)+\beta K_{18}(z)\}+\beta B(z)K_{12}(z)]\end{array}}{A(z)\,B(z)} \quad ...(31)$$

Since,

$$\sum_{n=0}^{M}\overline{R}_n(s)=\sum_{n=0}^{M}\overline{P}_n(s)+\sum_{n=0}^{M}\overline{Q}_n(s)=\frac{1}{s} \quad ...\ (32)$$

Thus relation (31), for z=1 gives

$$R(1,s)=\frac{1}{s}=\lim_{z\to 1}R(z,s) \quad ...\ (33)$$

$$P(0,s)=\overline{P}_0(s)=\lim_{z\to 0}P(z,s) \quad ......\ (34)$$

and $$Q(0,s)=\overline{Q}_0(s)=\lim_{z\to 0}Q(z,s) \quad .....\ (35)$$

The relations (33), (34), and (35) on solution gives the values of $\overline{P}_0(s), \overline{Q}_0(s), \overline{P}_M(s)$..

4. Particular Case:

Letting $\alpha\to\infty$, $\beta\to 0$ and setting $\varepsilon=1,\ N=1$ and $\mu_1=\mu_2=\mu$ (i.e., when the departure rate is $n\mu$), in relation (31), we have

$$r(z,s)=\frac{L_1(z)\,\overline{P}_M(s)+L_2(z)\frac{1}{s}+L_3(z)}{K(z)} \quad ....\ (36)$$

where

$$L_1(z)=-\frac{\lambda_1}{\mu}\int_0^z z^M\cdot(z-1)^{\frac{s+\xi}{\mu}}\cdot e^{\frac{-\lambda_1}{\mu}\cdot z}\,dz$$

$$L_2(z)=\frac{\xi}{\mu}\int_0^z (z-1)^{\frac{s+\xi}{\mu}-1}\cdot e^{\frac{-\lambda_1}{\mu}\cdot z}\,dz$$

$$L_3(z)=\frac{1}{\mu}\int_0^z (z-1)^{\frac{s+\xi}{\mu}-1}\cdot e^{\frac{-\lambda_1}{\mu}\cdot z}\,dz$$

$$K(z)=(z-1)^{\frac{(s+\xi)}{\mu}}\cdot e^{\frac{-\lambda_1}{\mu}\cdot z}$$

The value of the unknown quantity $\overline{P}_M(s)$ can be obtained by solving the equation $\lim_{z\to 1} r(z,s)=\frac{1}{s}$.

5. Steady State Results:

This can at once be obtained by the well -known property of the Laplace transform given below:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\ \bar{f}(s),$$ If the limit on the left hand side exists.

Then

$$R(z)=\lim_{s\to 0}\left[s\ R\ (z,s)\right]$$

By employing this property, we have from relation (31).

$$R(z)=\frac{\begin{aligned}&Q_0\left[A'(z)\left\{K_1'(z)+\beta K_{13}'(z)\right\}+B'(z)\left\{M'(z)+\beta K_7'(z)\right\}\right]+P_0\left[A'(z)\left\{K_2'(z)+\beta K_{14}'(z)\right\}+\beta B'(z)K_8'(z)\right]\\&+P_M\left[A'(z)\left\{K_3'(z)+\beta K_{15}'(z)\right\}+\beta B'(z)K_9'(z)\right]+\sum_{n=0}^{M}Q_n\left[A'(z)\left\{K_4'(z)+\beta K_{16}'(z)\right\}+B'(z)\left\{N'(z)+\beta K_{10}'(z)\right\}\right]\\&+\sum_{n=0}^{M}P_n\left[A'(z)\left\{K_5'(z)+\beta K_{17}'(z)\right\}+\beta B'(z)K_{11}'(z)\right]+C\end{aligned}}{A'(z)\ B'(z)} \quad \text{...(37)}$$

where,

$$A'(z)=z^{\frac{1-\varepsilon N}{\varepsilon}}\cdot(z-1)^{\frac{(\alpha+\xi)}{\mu_2\varepsilon}}$$

$$B'(z)=z^{\frac{1-\varepsilon N}{\varepsilon}}\cdot(z-1)^{\frac{\xi}{\mu_2\varepsilon}}\cdot\ e^{\frac{-\lambda_1}{\mu_1\varepsilon}\cdot z}$$

$$M'(z)=\frac{1-\varepsilon N}{\varepsilon}\int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}-1}\cdot(z-1)^{\frac{(\alpha+\xi)}{\mu_2\varepsilon}}\cdot dz$$

$$N'(z)=\frac{\xi}{\mu_2\varepsilon}\int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}}\cdot(z-1)^{\frac{(\alpha+\xi)}{\mu_2\varepsilon}-1}\ dz$$

$$K_1'(z)=\frac{\alpha}{\mu_1\varepsilon}\int_0^z (z-1)^{\frac{\xi}{\mu_1\varepsilon}-\left\{\frac{(\alpha+\xi)}{\mu_2\varepsilon}+1\right\}}\cdot e^{\frac{-\lambda_1}{\mu_1\varepsilon}\cdot z}\cdot M'(z)\ dz$$

$$K_2'(z)=\frac{1-\varepsilon N}{\varepsilon}\int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}-1}\cdot(z-1)^{\frac{\xi}{\mu_1\varepsilon}}\cdot e^{\frac{-\lambda_1}{\mu_1\varepsilon}\cdot z}\ dz$$

$$K_3'(z)=-\frac{\lambda_1}{\mu_1\varepsilon}\int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}+M}\cdot(z-1)^{\frac{\xi}{\mu_1\varepsilon}}\cdot e^{\frac{-\lambda_1}{\mu_1\varepsilon}\cdot z}\ dz$$

$$K_4'(z)=\frac{\alpha}{\mu_1\varepsilon}\int_0^z (z-1)^{\frac{\xi}{\mu_1\varepsilon}-\left\{\frac{(\alpha+\xi)}{\mu_2\varepsilon}+1\right\}}\cdot e^{\frac{-\lambda_1}{\mu_1\varepsilon}\cdot z}\cdot N'(z)\ dz$$

$$K_5'(z)=\frac{\xi}{\mu_1\varepsilon}\int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}}\cdot(z-1)^{\frac{\xi}{\mu_1\varepsilon}-1}\cdot e^{\frac{-\lambda_1}{\mu_1\varepsilon}\cdot z}\ dz$$

$$K_{i+6}'(z)=\frac{1}{\mu_2\varepsilon}\int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}}\cdot(z-1)^{\frac{(\alpha+\xi)}{\mu_2\varepsilon}-1}\cdot K_i'(z)\ dz \qquad ;\ i = 1, 2, 3, 4, 5.$$

$$K'_{i+j+(6-j)}(z) = \frac{1}{\mu_1 \varepsilon} \int_0^z \frac{1}{z-1} \left[\frac{\alpha B'(z)}{A'(z)} K'_i(z) - K'_{j+1}(z) \right] dz \quad ;$$

$$\left[(j, i): \{(0,7), (1,8), (2,9), (3,10), (4,11)\} \right]$$

C = the constant of integration.

The unknown quantities P_0, Q_0, P_M, $\sum_{n=0}^{M} Q_n$ and $\sum_{n=0}^{M} P_n$ can be evaluated as before.

Particular Case:

Relation (36), on applying the theory of Laplace transforms gives

$$r(z) = \frac{L'_1(z) P_M + L'_2(z) + C'}{K'(z)} \quad \text{..... (38)}$$

Where

$$r(z) = \lim_{s \to 0} s \cdot r(z, s)$$

$$L'_1(z) = -\frac{\lambda_1}{\mu} \int_0^z z^M \cdot (z-1)^{\frac{\xi}{\mu}} \cdot e^{\frac{-\lambda_1}{\mu} \cdot z} \, dz$$

$$L'_2(z) = \frac{\xi}{\mu} \int_0^z (z-1)^{\frac{\xi}{\mu}-1} \cdot e^{\frac{-\lambda_1}{\mu} \cdot z} \, dz$$

$$K'(z) = (z-1)^{\frac{\xi}{\mu}} \cdot e^{\frac{-\lambda_1}{\mu} \cdot z}$$

C' = the constant of integration.

The unknown quantity of equation (38) can be evaluated as before.

When no catastrophe is allowed i.e., $\xi = 0$ then relation (38), gives

$$e^{\frac{-\lambda_1}{\mu} z} r(z) = C_1 - \frac{\lambda_1}{\mu} P_M \int z^M e^{\frac{-\lambda_1 z}{\mu}} \, dz \quad \text{.... (39)}$$

where

C_1 = the constant of integration.

The unknown quantity P_M can be evaluated as before.

For unlimited waiting space, the relation (39) becomes, If Max (ρ, |z|) < 1.

$$e^{\frac{-\lambda_1 z}{\mu}} r(z) = C_1$$

Which for z = 1 gives, $C_1 = e^{\frac{-\lambda_1}{\mu}}$

Hence, $r(z)=e^{\frac{-\lambda_1}{\mu}(1-z)}$ ..... (40)

which is a well- known result.

Steady-state probabilities of the M/M/1 queueing model:

In [9], Kumar, B. K., and Arivudainambi, D. have studied the transient solution of an M/M/1 queue with catastrophes. They have also obtained the steady- state probabilities and mean & variance of the M/M/1 queue with catastrophes.

When a catastrophe occurs at the service facility i.e. $\xi >0$, the steady- state distribution $\{p_n;\ n \geq 0\}$ of the M/M/1 queue with catastrophes corresponds to

$$p_0 = (1-\rho)\ ;\ \ n = 0 \qquad ....(41)$$

$$p_n = (1-\rho)\rho^n\ ;\ \ n = 1, 2, 3, ... \qquad ....(42)$$

where

$$\rho = \frac{(\lambda+\mu+\xi)-\sqrt{\lambda^2+\mu^2+\xi^2+2\lambda\xi+2\mu\xi-2\lambda\mu}}{2\mu} \qquad ....(43)$$

Thus equations (41)-(43) provide the steady- state distribution for the queueing system. Obviously, the steady state distribution exists if and only if $\rho < 1$.

Note: The steady- state probability of this Markov process exists if and only if $\xi > 0$ or $\xi = 0$ and $\lambda > \mu$. It is also observed that the results of equations (41)-(43) agree with the model discussed above and with [5] by Chao, X.

6. Conclusion:

In the present paper, we have established a queueing system with catastrophes, state dependent service and environmental change. We have also obtained some particular cases with (without) catastrophes and steady state results in detail.

References

[1]. Bartoszynski, R., Buhler, W. J., Chan Wenyan and Pearl, D.K. (1989) Population processes under the influence of disasters occurring independently of population size, J. Math. Bio. Vol. 27, 179-190.

[2]. Brockwell, P. J. (1985) The Extinction time of a birth, death and catastrophe process and of a related diffusion model, Adv. in Appl. Probab. Vol.17, 42-52.

[3]. Brockwell, P.J.,Gani, J. M., and Resnick, S. I. (1982) Birth immigration and catastrophe processes, Adv. Appl. Probab. Vol.14, 709-731.

[4]. Chao, X., and Zheng, Y. (2003) Transient analysis of immigration birth-death processes with total catastrophes, Probl. Engrg. Inform. Science, Vol.17, 83-106.

[5]. Chao, X. (1995) A queueing network model with catastrophes and product form solution, O.R. letters Vol.18, 75-79.

[6]. Crescenzo, A. Di, Giorno, V., Nobile, A. G., and Ricciardi, L. M. (2003) On the M/M/1 queue with catastrophes and its continuous approximation, Queueing Systems, Vol.43, 329-347.

[7]. Gripenberg, G. (1983) A Stationary distribution for the growth of a population subject to random catastrophes, J. Math. Bio. Vol.17, 371-379.

[8]. Jain, N.K and Kanethia, D.K (2006) Transient Analysis of a Queue with Environmental and Catastrophic Effects, International Journal of Information and ManagementSciences Vol. 17 No.1, 35-45.

[9]. Kumar, B. K., and Arivudainambi, D. (2000) Transient solution of an M/M/1 queue with catastrophes, Computational and Mathematics with Applications Vol.40, 1233-1240 .

[10]. Swift, R. J. (2001) Transient probabilities for simple birth- death immigration process under the influence of total catastrophes, Inter. Jour. Math. Math. Sci. Vol.25, 689-692.

[11]. Vinodhini, G. Arul Freeda and Vidhya, V. (2016) Computational Analysis of Queues with Catastrophes in a Multiphase Random Environment, Math. Problems in Engineering, Article ID 2917917, 7 pages.