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ON SOME FATED CLASS OF INCONSISTENCY ANALOGOUS TO p-VALENT SUBCLASSES

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ABSTRACT

Various p-valent subclasses of meromorphic, complex, and integral operator functions are used to derive a class of inequalities in this communication. The distortion property for the identical function $M_p(\beta)$, is demonstrated by the inequality given for it.

Keywords and Phrases: Meromorphic functions, p-valent functions, differential subordination, integral operators etc.

1. INTRODUCTION

Numerous well-known univalent functions, such as the starlike functions, the functions convex in one direction, the functions starlike concerning symmetrical points [5], and the functions with a derivative of positive real part in the unit circle, are included in the set of close-to-convex univalent functions introduced by Kaplan [2] and Umezawa [6]. But it doesn't have any spiral-shaped ones.

Ogawa [3] recently presented a broader adequate condition for univalence that also contains the weakest sufficient requirement for spiral-likeness, and it was simultaneously extended to the case of p-valence.

The class of functions $f(z)$ of the form $f(z) = z^p + \sum_{n=0}^{\infty} a_n z^n \ \forall\ p \in N$ is denoted by $V(p)$. Which are analytical and multivalent in the open unit disc $\boldsymbol{D} = z: |z| < 1$. Let p-valent convex functions and the well-known class of p-valent starlike functions, respectively, be denoted by ϵ_p and

ω_p. For $f(z) \in V(p)$ as previously said and $g(z) \in V(p)$ as stated by $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n \; \forall \; p \in N$, $(f.g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n \; z^n \forall \; z \in \boldsymbol{D}$ gives the Hadamard product or convolution of $f(z)$ and $g(z)$.

A generalization of univalent functions is multivalent functions, particularly p-valent functions. The existence of a univalent mapping from one domain to another is one of the key issues in the study of univalent functions. Such a mapping must meet the requirement and have equal connectivity levels. This condition is likewise adequate, and the issue is reduced to the task of mapping a given domain onto a disc if and is simply connected to domains whose boundaries contain more than one point.

In this regard, the class of functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, z \in D$ that are regular and univalent on the unit disc $\boldsymbol{D} = z\text{: } |z| < 1$, normalized by the constraints $f(0) = 0, f'(0) = 1$, and possessing the expansion play a special significance in the theory of univalent functions on simply-connected domains.

When it comes to multiply-connected domains, it is examined how a specific multiply-connected domain maps onto so-called canonical domains. P-valent functions, in particular, can be defined as follows:

Let A_{p} (p is a positive integer) stand for the class of analytic functions in the unit disc $\boldsymbol{D} = z\text{: } |z| < 1$ of form $f(z) = z^p + \sum_{n=0}^{\infty} a_{n+p} z^{n+p}$. If the function $f(z) \in A_{\mathrm{p}}$ assumes no value more than p times in $\boldsymbol{D}$, it is said to be p-valent in $\boldsymbol{D}$.

Lemma 1.1 [7] If a, b and c $(c \neq 0, -1, -2, \dots)$ be any real or complex numbers then

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-st)^{-a} dt = \frac{\Gamma b \; \Gamma(c-b)}{\Gamma c} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n} \; ; Re\,(c) > Re\,(b) > 0.$$

Lemma 1.2 [4] Let $\omega(z) = 1 + b_1 z + b_2 z^2 + \cdots$ be the class of function. Then,

$$Re\,\big(\omega(z)\big) - 2\gamma + 1 \geq \frac{2(1-\gamma)}{1+|z|}, 0 \leq \gamma < 1; z \in U.$$

Later, in 1980 Goel and Sohi [1] introduced a differential operator $D^{\delta+p-1}$ for p-valent analytic functions given by

$D^{\delta+p-1} f(z) = \frac{z}{(1-z)^{\delta+p}} . f(z) = z^p + \sum_{n=p+1}^{\infty} \varphi_n(\delta) a_n z^n$ with $\delta > -p$ and $\varphi_n(\delta) = \frac{(\delta+p)_{n-p}}{(n-p)!}$.

2. OUR CLAIMS

Claim 2.1 If $M_p(\beta)$ is the subclass of, class of meromorphic p-valent function $\sum_p'$ in the unit disc with $0 < |z| < r \leq 1, 0 < \beta \leq p \; \forall p \in N$. Then demonstrate the inequality $r^p - r^p \frac{p+\beta}{p-\beta} \leq |f(z)| \leq r^p + r^p \frac{p+\beta}{p-\beta}$.

Proof: Taking an assumption that $f \in M_p(\beta)$ then

$$\sum_{n=0}^{\infty} \left((n+p)\left(1-\frac{1}{\alpha}\right) - \beta\right) a_{n+p} \leq p + \beta$$

We achieve $(p+\beta)\sum_{n=0}^{\infty} a_{n+p} \leq \sum_{n=0}^{\infty} \left((n+p)\left(1-\frac{1}{\alpha}\right) - \beta\right) a_{n+p} \leq p + \beta$ that means $\sum_{n=0}^{\infty} a_{n+p} \leq \frac{p+\beta}{p-\beta}$. Now, $|f(z)| = \left|z^p + \sum_{n=0}^{\infty} a_{n+p} z^{(n+p)\left(1-\frac{1}{\alpha}\right)}\right|$ and this implies that

$$|f(z)| \leq |z|^p + \sum_{n=0}^{\infty} a_{n+p} \, |z|^{(n+p)\left(1-\frac{1}{\alpha}\right)}$$

$$\leq r^p + \sum_{n=0}^{\infty} a_{n+p} r^{(n+p)\left(1-\frac{1}{\alpha}\right)}$$

$$\leq r^p + r^p \frac{p+\beta}{p-\beta}$$

Again, since $|f(z)| = \left|z^p + \sum_{n=0}^{\infty} a_{n+p} z^{(n+p)\left(1-\frac{1}{\alpha}\right)}\right|$ and this implies that

$$|f(z)| \geq |z|^p - \sum_{n=0}^{\infty} a_{n+p} |z|^{(n+p)\left(1-\frac{1}{\alpha}\right)}$$

$$\geq r^p - \sum_{n=0}^{\infty} a_{n+p} r^{(n+p)\left(1-\frac{1}{\alpha}\right)}$$

$$\geq r^p - r^p \frac{p+\beta}{p-\beta}.$$

Hence, $r^p - r^p \frac{p+\beta}{p-\beta} \leq |f(z)| \leq r^p + r^p \frac{p+\beta}{p-\beta}$ for $0 \leq |z| \leq 1$. Hence the proof is completed.

Claim 2.2 If $p(z)$ be analytic within unit disc $\boldsymbol{D} = z: |z| < 1$ with $p(0) = 1$ and $Re\, p(z) > 0, \forall\, z \in \boldsymbol{D}$ and $f \in u\Delta_p^{\epsilon}(c, \tau, a, b)$. Then demonstrate the inequality $\frac{|A_{k+p}|}{|\tau|} \prod_{j=1}^{k-2} \frac{|a-1|j}{|a-1|j+|\tau||\eta|(c+p)}$ $\leq \frac{(c+p)|\eta|}{k|a-1|\psi_{k+p}(c)}$.

Proof: On using the fact

$$Re\, e^{i\epsilon}\left(\frac{\tau D^{c+p-1} f(z) - 2D^{c+p-1} f(z) + 2D^{c+p} f(z)}{bD^{c+p-1} f(z)}\right) > \left(\frac{1-2a+b}{1-a}\right) \cos \epsilon \;\forall\, z \in D$$

Now, setting $p(z)$ by $e^{i\epsilon}\left(\frac{\tau D^{c+p-1} f(z) - 2D^{c+p-1} f(z) + 2D^{c+p} f(z)}{bD^{c+p-1} f(z)}\right) = \frac{1}{1-a}\big((1-\tau)p(z)\cos\epsilon + (b - a)\cos\epsilon\,\big) + i \sin \epsilon$. Since, $p(z)$ is analytic in D with $p(0) = 1$ and $Re\, p(z) > 0, \forall\, z \in \boldsymbol{D}$. Let

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \qquad \forall\, z \in D$$

Then, both equalities gives us

$$\frac{\tau D^{c+p-1} f(z) - 2D^{c+p-1} f(z) + 2D^{c+p} f(z)}{bD^{c+p-1} f(z)} = \frac{\left((1-b)\cos\epsilon - i(a-1)\sin\epsilon\right)\sum_{n=1}^{\infty} p_n z^n - e^{i\epsilon}(a-1)}{e^{i\epsilon}(a-1)}$$

$i.e.\quad (D^{c+p-1} - D^{c+p}) f(z) e^{i\epsilon}(a-1)(\eta D^{c+p-1} f(z) \sum_{n=1}^{\infty} p_n z^n)^{-1} = 2^{-1}\tau$

$i.e.\quad \frac{\eta \sum_{n=1}^{\infty} p_n z^n}{\left(pD^{c+p-1} f(z) - z\left(D^{c+p-1} f(z)\right)'\right)} = \frac{2e^{i\epsilon}(\alpha-1)}{b(c+p)D^{c+p-1} f(z)}$

$i.e.\quad \frac{z^p \sum_{n=1}^{\infty} p_n z^n}{\sum_{k=p+1}^{\infty} (K-p)\psi_k(c) z^n A_k z^k} + \frac{\sum_{k=p+1}^{\infty} \psi_k(c) A_k z^k \sum_{n=1}^{\infty} p_n z^n}{\sum_{k=p+1}^{\infty} (K-p)\psi_k(\delta) z^n A_k z^k} = \frac{2e^{i\lambda}(\alpha-1)}{b(\delta+p)\eta}$

On comparing the coefficient of z^{n+p-1} on both sides we achieve

$$e^{ia} \frac{\psi_{n+p-1}(c)_{n+p-1}}{p_1 A_{n+p-2}\psi_{n+p-2}(c) + \cdots + p_{n-1}} = \frac{\tau\eta}{2}\left\{\frac{c(1-a)^{-1}}{(n-1)} - \frac{p(a-1)^{-1}}{(n-1)}\right\}$$

Taking absolute on both sides and applying the coefficient estimates we get,

$$\frac{|A_{n+p-1}|}{1 + \psi_{p+1}(c)|A_{p+1}| + \cdots + \psi_{n+p-2}(c)|A_{n+p-2}|} \leq \frac{|\tau|(c+p)|\eta|}{(n-1)|a-1\psi_{n+p-2}(c)|}.$$

Applying the same argument, then we get by mathematical induction $\frac{|\tau|}{|A_{p+1}|} \leq \frac{|a-1|}{|\eta|}$ which is valid for $n = 2$, similarly $\frac{|A_{p+1}|}{|\tau|(1+\psi_{p+1}(c)|A_{p+1}|)} \leq \frac{|\eta|(c+p)}{2|a-1|\psi_{p+2}(c)}$ is valid for $n = 3$.

Also, $\frac{|A_{k+p-1}|}{|\tau|}\prod_{j=1}^{k-2}\frac{|a-1|j}{|a-1|j+|\tau||\eta|(c+p)} \le \frac{|\eta|(c+p)}{(k-1)|a-1|\psi_{k+p-1}(c)}$ is valid for $n = k$. Taking, $\frac{|A_{k+p}|}{|\tau|}\prod_{j=1}^{k-2}\frac{|a-1|j}{|a-1|j+|\tau||\eta|(c+p)} \le \frac{|\eta|(c+p)}{k|a-1|\psi_{k+p}(c)}\left\{\frac{|a-1|+|\tau||\eta|(c+p)}{|a-1|} + \frac{|\tau|(c+p)|\eta|}{2|a-1|}.\frac{|a-1|+|\tau||\eta|(c+p)}{|a-1|} + \cdots\right.$

$\left.\ldots\frac{|\tau|(c+p)|\eta|}{(k-1)|1-a|}\right\}$. Which implies that $\frac{|A_{k+p}|}{|\tau|}\prod_{j=1}^{k-2}\frac{|a-1|j}{|a-1|j+|\tau||\eta|(c+p)} \le \frac{(c+p)|\eta|}{k|a-1|\psi_{k+p}(c)}$ and this completes the proof.

Claim 2.3 If $b > 0$ and $f_k(z) \in \sum_p$ then show that the inequality $Re\left(\tau_0(z)\right) > b\delta^{-1}\int_0^1 x^{b\delta^{-1}-1}\left(2c_3 - 1\frac{2(1-c_3)}{1+u}\right)dx.$

Proof: Suppose that each of the functions $f_k(z) \in \sum_p$ for $k = 1, 2$ satisfies the condition $\tau_k(z) = z^p\left[\delta P_{b,p}^{a-1}f_k(z)+(1-\delta)P_{b,p}^{a-1}f_k(z)\right]$ for $k = 1, 2$ we have $\tau_k(z) \in P(c_k)$ where $c^{-1} = \frac{1-Y_k}{1-X_k}$; $k = 1, 2$. Making use of the identity $P_{b,p}^{a-1}f_k(z) = \frac{b}{\delta}z^{-\frac{b+\delta p}{\delta}}\int_0^z x^{-\frac{b+\delta p}{\delta}}\tau_k(x)dx$ for $k = 1, 2$. Now, taking the help of $J(z) = P_{b,p}^{a-1}(g_1.g_2)(z)$ we get the following result $P_{b,p}^{a-1}f_k(z)$

$= \left(\frac{b}{\delta}z^{-\frac{b+\delta p}{\delta}}\int_0^z x^{-\frac{b+\delta p}{\delta}}\tau_1(x)dx\right).\left(\frac{b}{\delta}z^{-\frac{b+\delta p}{\delta}}\int_0^z x^{-\frac{b+\delta p}{\delta}}\tau_2(x)dx\right)$. Which leads us the result $P_{b,p}^{a-1}f_k(z) = b\delta^{-1}z^{-\frac{b+\delta p}{\delta}}\int_0^z x^{-\frac{b+\delta p}{\delta}}\tau_0(x)dx$ here $\tau_0(z) = z^p\left[\delta P_{b,p}^{a-1}J(z)+(1-\delta)P_{b,p}^{a-1}J(z)\right]$ and is equal to $z^p\left(\delta P_{b,p}^{a-1} + (1-\delta)P_{b,p}^{a-1}\right)J(z) = b\delta^{-1}z^{-b\delta^{-1}}\int_0^z x^{b\delta^{-1}-1}(\tau_1.\tau_2)dx$. But, $\tau_1(z) \in P(c_1)$ and $\tau_2(z) \in P(c_2)$ it follows that $(\tau_1.\tau_2)(z) \in P(c_3)\left(c_3 = 1 - 2(1-\tau_1)(1-\tau_2)\right)$. Now, applying Lemma 1.2 we achieve $Re\ (\tau_1.\tau_2)(z) - 2c_3 + 1 \ge \frac{2(1-c_3)}{1+|z|}$. Now, by using this inequality in the value of $\tau_0(z)$ and then applying the Lemma 1.1 we have $Re\left(\tau_0(z)\right) = b\delta^{-1}\int_0^1 x^{b\delta^{-1}-1}Re\ (\tau_1.\tau_2)(zx)dx \ge b\delta^{-1}\int_0^1 x^{b\delta^{-1}-1}\left(2c_3 - 1\frac{2(1-c_3)}{1+u|z|}\right)dx$ and we can also write $Re\left(\tau_0(z)\right) > b\delta^{-1}\int_0^1 x^{b\delta^{-1}-1}\left(2c_3 - 1\frac{2(1-c_3)}{1+u}\right)dx$ and this completes the proof of the theorem.

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