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RESEARCH ARTICLE

(α, β) - Quasi A - Normal Operators in Semi–Hilbert Spaces

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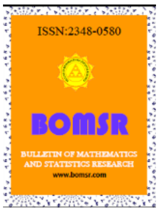

ABSTRACT

Let H be a Hilbert space and A be a positive bounded operator on H and the semi inner product $\langle u|v\rangle_A = \langle Au|v\rangle,\ u, v \in H$ induces a semi-norm $\|.\|_A$ on H and to make H into a Semi-Hilbertian space. In this paper, we introduce a new class of operators called $(\alpha, \beta) -$ quasi A - normal operators in semi – Hilbertian spaces including some key characteristics of this operator, as well as a discussion of various theorems related to this operator.

Mathematics Subject Classification: 47B20, 47B37 , 47B38.

Keywords: Operator, Hilbert space, Normal, (α, β) - Quasi Normal, (α, β) - Normal Operator, (α, β) - Quasi Normal Operator, Semi – Hilbertian Spaces, A - Normal Operator, A - Selfadjoint Operator, and A – Positive Operator.

1. Introduction

There have been some thorough research done on the class of normal operators on Hilbert spaces. J. B. Conway and C. R. Putnam in particular looked into the theory of these operators in their respective works [7] and [17]. Several authors have presented the classes of quasi normal, hyponormal, isometric, partly isometric, and m-isometric operators on Hilbert spaces. In the works [2], [3], [15], [16], [18] and other papers, these classes of operators have just lately been generalized in Semi-Hilbertian spaces. We are interested in introducing the (α,β) - quasi A-normal operators, a novel notion of normality in semi-Hilbertian spaces, in this context. We demonstrate that numerous findings from [8] and [11] still apply to this new class.

2. Preliminaries

First, various notations are introduced. A complex Hilbert space with the inner products $\langle . | . \rangle$ and $\|.\|$ represents by the sign H. The Banach algebra of all bounded linear operators on H is denoted by B(H), with I = I_H serving as the identity operator. The cone of positive (semi-definite) operators is denoted by the symbol $B(H)^+$, where

$$B(H)^+ = \{A \in B(H) : \langle Au|u \rangle \geq 0, \forall\, u \in H\}.$$

Definition 2.1[2]

If we assume that T ∈ B(H), then an operator S ∈ B(H) is said to be an A-adjoint of T if and only if for every u, v ∈ H then $\langle Tu|v \rangle_A = \langle u|Sv \rangle_A$, i.e., AS = T^*A where T is an A – adjoint of itself, then T is called an A – selfadjoint operator. The possibility that an operator T does not have an A-adjoint was noted in [2], and if S is an A-adjoint of T, we may find multiple A-adjoints; In reality, S + R is an A-adjoint of T when AR = 0 for some R∈ B(H).$B_A(H)$ stands for the collection of all A-bounded operators that allow an A-adjoint. Using the Douglas Theorem, we can say that

$$B_A(H) = \{T \in B(H) / \ R(T^*A) \subset R(A)\}$$

Definition 2.2[21]

If T, A ∈ $B(H)$ and A ≥ 0, we call T is an A – isometry if $T^*AT = A$.

Definition 2.3[2, 18]

Any operator T ∈ $B_A(H)$ then

i) A – normal if $TT^* = T^*T$

ii) A – unitary if $T^*T = TT^* = P_{\overline{R(A)}}$ where closure of $R(A)$ will be denoted by $\overline{R(A)}$.

Proposition 2.4[2, 3]

Let T ∈ $B_A(H)$ then the following statements hold

i) $T^* \in B_A(H)$ then $(T^*)^* = P_{\overline{R(A)}} T P_{\overline{R(A)}}$ and $((T^*)^*)^* = T^*$.

ii) If S ∈ $B_A(H)$ then TS ∈ $B_A(H)$ and $(TS)^* = S^*T^*$.

iii) T^*T and TT^* are A – selfadjoint.

iv) $\|T\|_A = \|T^*\|_A = \|T^*T\|_A^{1/2} = \|TT^*\|_A^{1/2}$.

v) $\|S\|_A = \|T^*\|_A$ for every S ∈ B(H) which is an A-adjoint of T.

vi) If S ∈ $B_A(H)$ then $\|TS\|_A = \|ST\|_A$

3 Properties of (α, β) - Quasi A - Normal Operators

Here we define the concept of (α, β) - quasi A - normal operators according to Semi-Hilbertian space structures and we will discuss some of their properties. Let $(\alpha, \beta) \in R^2$ such that $0 \leq \alpha \leq 1 \leq \beta$, an operator T ∈ $B(H)$ is said to be (α, β) - quasi normal and if T satisfies

$$T[\alpha^2 T^*T] \leq [TT^*]T \leq T[\beta^2 T^*T]$$

which is equivalent to the condition that

$$T[\alpha\|Tu\|] \leq \|T^*u\|T \leq T[\beta\|Tu\|]$$

for all u $\in B(H)$. When $\alpha = 1$, we can see from the left inequality that T^* is hyponormal, and when $\beta = 1$, we can infer this from the right inequality that T is hyponormal.

Theorem 3.1

Let T $\in B_A(H)$ and $(\alpha, \beta) \in R^2$ such that $0 \leq \alpha \leq 1 \leq \beta$ then T is an (α, β) quasi A - normal operators if and only if T satisfies the following conditions.

$$\begin{cases} T[\lambda^2 TT^* + 2\alpha^2\lambda T^*T + TT^*] \geq_A 0, & \text{for all } \lambda \in R \quad (1) \\ \text{and} & \\ T[\lambda^2 T^*T + 2\lambda TT^* + \beta^4 T^*T] \geq_A 0, & \text{for all } \lambda \in R \quad (2) \end{cases}$$

Proof

Assume that T satisfies the condition (1) and (2) and prove that T is (α, β) quasi A - normal operators. Then

$$T[\lambda^2 TT^* + 2\alpha^2\lambda T^*T + TT^*] \geq_A 0$$

$$\Leftrightarrow \langle T[\lambda^2 TT^* + 2\alpha^2\lambda T^*T + TT^*]u|u\rangle_A \geq 0, \ \forall\, u \in H, \forall\, \lambda \in R$$

$$\Leftrightarrow T[\lambda^2\|T^*u\|_A^2 + 2\alpha^2\lambda\|Tu\|_A^2 + \|T^*u\|_A^2] \geq 0, \ \forall\, u \in H, \ \forall\, \lambda \in R$$

$$\Leftrightarrow T[\alpha\|Tu\|_A] \leq \|T^*u\|_A T \quad \forall\, u \in H. \tag{3}$$

Similarly

$$T[\lambda^2 T^*T + 2\lambda TT^* + \beta^4 T^*T] \geq_A 0$$

$$\Leftrightarrow \langle T[\lambda^2 T^*T + 2\lambda TT^* + \beta^4 T^*T]u|u\rangle_A \geq 0, \ \forall\, u \in H, \forall\, \lambda \in R$$

$$\Leftrightarrow T[\lambda^2\|Tu\|_A^2 + 2\lambda\|T^*u\|_A^2 + \beta^4\|Tu\|_A^2] \geq 0, \ \forall\, u \in H, \ \forall\, \lambda \in R$$

$$\Leftrightarrow [\|T^*u\|_A]T \leq T[\beta\|Tu\|_A] \quad \forall\, u \in H. \tag{4}$$

From (3) and (4), we get

$$T[\alpha\|Tu\|_A] \leq \|T^*u\|_A T \leq T[\beta\|Tu\|_A] \quad \forall\, u \in H.$$

Therefore, T is (α, β) - quasi A - normal operator.

We expand this result to (α, β) – quasi A - normal operators in the following proposition, where it is widely known that an operator T $\in B(H)$ is normal if and only if T^* is normal.

Proposition 3.2

Let T $\in B_A(H)$ such that $N(A)$ is a reducing subspace for T and let $(\alpha, \beta) \in R^2$ such that $0 \leq \alpha \leq 1 \leq \beta$. Then T is an (α, β) - quasi A - normal if and only if T^* is $\left(\frac{1}{\beta}, \frac{1}{\alpha}\right)$ - quasi A - normal operator.

Proof

Assume that T is (α,β) – quasi A - normal operator and we have to prove that T^* is $\left(\frac{1}{\beta},\frac{1}{\alpha}\right)$ - quasi A - normal operator. Infact, we have

$$T[\alpha\|Tu\|_A] \leq \|T^*u\|_A T \leq T[\beta\|Tu\|_A] \quad \forall\, u \in H$$

Then it follows that

$\frac{1}{\beta}T[\|T^*u\|_A\,] \leq \|Tu\|_A T$ and

$\|Tu\|_A T \leq \frac{1}{\alpha}T[\|T^*u\|_A\,]$

On the other hand, $N(A)$ is a reducing subspace for T, By proposition (3.2) we have,

$TP_{\overline{R(A)}} = P_{\overline{R(A)}}T$ and $AP_{\overline{R(A)}} = P_{\overline{R(A)}}A = A$.

In the view of proposition(3.2),we can write

$\frac{1}{\beta}T[\|T^*u\|_A\,] \leq [\|(T^*)^*\,u\|_A\,]T \leq \frac{1}{\alpha}T[\|T^*u\|_A\,]$ for all u $\in H$ and it follows that

$$\frac{1}{\beta}T[\|T^*u\|_A\,] \leq [\|T\,u\|_A\,]T \leq \frac{1}{\alpha}T[\|T^*u\|_A\,] \text{ for all u} \in H.$$

Hence

$$T[\alpha\|Tu\|_A] \leq \|T^*u\|_A T \leq T[\beta\|Tu\|_A] \quad \forall\, u \in H$$

Thus, T is (α,β) - quasi A - normal operator.

Hence the proof.

Theorem 3.3

Let T $\in B_A(H)$ such that $N(A)$ is a reducing subspace for T and $0 \leq \alpha \leq 1 \leq \beta$.Then the following statements hold.

i) If T is (α,β) quasi A - normal operator then λT is (α,β) - quasi A – normal for $\lambda \epsilon C$.

ii) If T is (α,β) quasi A - normal operator then $T+\lambda$ for $\lambda \epsilon C$ is (α,β) - quasi A – normal then the following condition hold,

1) $\mu_A^1(\bar{\lambda}\,T) \geq 0$.

Proof

i) Since $N(A)$ is a reducing subspace for T, then we have

$TP_{\overline{R(A)}} = P_{\overline{R(A)}}T$ and $AP_{\overline{R(A)}} = P_{\overline{R(A)}}A = A$.

Assume that T is (α,β) quasi A – normal. If $\lambda = 0$ then there is nothing to prove.

If $\lambda \neq 0$ then by known definition, it follows that

$T[\beta^2T^*T] \geq_A [TT^*]T \geq_A T[\alpha^2TT^*]$

$$\Leftrightarrow T[\beta^2|\lambda|^2T^*T] \geq_A [|\lambda|^2TT^*]T \geq_A T[|\lambda|^2\alpha^2T^*T]$$

$$\Leftrightarrow T[\beta^2A\bar{\lambda}T^*\lambda T] \geq \left[A\lambda T\bar{\lambda}T^*\right]T \geq T[\alpha^2A\bar{\lambda}T^*\lambda T]$$

$$\Leftrightarrow T[\beta^2AP_{\overline{R(A)}}\bar{\lambda}T^*\lambda T] \geq [AP_{\overline{R(A)}}\lambda T\bar{\lambda}T^*]T \geq T[\alpha^2AP_{\overline{R(A)}}\bar{\lambda}T^*\lambda T]$$

$$\Leftrightarrow T[\beta^2 A(\lambda T)^*(\lambda T)] \geq [A(\lambda T)(\lambda T)^*]T \geq T[\alpha^2 A(\lambda T)^*(\lambda T)]$$

$$\Leftrightarrow T[\beta^2 (\lambda T)^*(\lambda T)] \geq_A [(\lambda T)(\lambda T)^*]T \geq_A T[\alpha^2 (\lambda T)^*(\lambda T)]$$

Hence, λT is (α, β) - quasi A – normal operator.

ii) Assume that T is (α, β) -q uasi A – normal operator and the above condition (1) holds.

We have to prove that

$$\begin{cases} T[\alpha^2 \langle (T+\lambda)^*(T+\lambda)u, u\rangle]_A \leq [\langle (T+\lambda)(T+\lambda)^* u, u\rangle] T_A \\ [\langle (T+\lambda)(T+\lambda)^* u, u\rangle] T_A \leq T[\beta^2 \langle (T+\lambda)^*(T+\lambda)u, u\rangle]_A \end{cases} \quad (5)$$

Inequality (5) becomes

$T[\alpha^2 \langle (T+\lambda)^*(T+\lambda)u, u\rangle]_A$

$$= T[\alpha^2 \left\{ (T^*Tu, u)_A + (\lambda T^*Tu, u)_A + \left(\bar{\lambda} P_{\overline{R(A)}} Tu, u\right)_A + |\lambda|^2 (P_{\overline{R(A)}} u, u)_A \right\}]$$

$$= T[\alpha^2 \left\{ (T^*Tu, u)_A + 2Re\left(\bar{\lambda} Tu, u\right)_A + |\lambda|^2 \|u\|^2)_A \right\}]$$

$$\leq T[\alpha^2 (T^*Tu, u)_A + \alpha^2 \left\{ 2Re\left(\bar{\lambda} Tu, u\right)_A + |\lambda|^2 \|u\|^2)_A \right\}]$$

By using condition (i) implies that $2Re\left(\bar{\lambda} Tu, u\right)_A \geq 0$ and then we have

$$T[\alpha^2 \langle (T+\lambda)^*(T+\lambda)u, u\rangle]_A \leq \left\{ (T^*Tu, u)_A + 2Re\left(\bar{\lambda} Tu, u\right)_A + |\lambda|^2 \|u\|^2)_A \right\} T$$
$$\leq T[\beta^2 \langle (T+\lambda)^*(T+\lambda)u, u\rangle]_A$$

And hence $T + \lambda$ is (α, β) - quasi A – normal operator.

Corollary 3.4

Let $T \in B_A(H)$ be an (α, β) - quasi A – normal operator with $0 \leq \alpha \leq 1 \leq \beta$.

The following statements hold

(i) If $\mu_A^1(\bar{\lambda}\, T) \geq 0$, then $T + \lambda$ is (α, β) - quasi A – normal operator for every $\lambda > 0$.

(ii) If $\mu_A^2(\bar{\lambda}\, T) \leq 0$, then $T + \lambda$ is (α, β) - quasi A – normal operator for every $\lambda < 0$.

Proof

(i) For every $\lambda > 0$ we have $\mu_A^1(\bar{\lambda} T) = \mu_A^1(\lambda T) = \mu_A^1(T) \geq 0$. By using Previous theorem we deduce that $T + \lambda$ is (α, β) - quasi A – normal operator.

(ii) For every $\lambda < 0$ we have $\mu_A^1(\bar{\lambda} T) = -\mu_A^2(T) \geq 0$. By using Previous theorem we deduce that $T + \lambda$ is (α, β) - quasi A – normal operator.

Proposition 3.5

Let $T, S \in B_A(H)$ such that T is (α, β) quasi A – normal and S is A-selfadjoint. If $T^*S = ST^*$, then TS is (α, β) - quasi A – normal operator.

Proof

Assume that T is (α, β) - quasi A – normal we have for $u \in H$,

$$T[\alpha \|TSu\|_A] \leq [\|T^*Su\|_A] T \leq T\,[\beta \|TSu\|_A]$$

On the other hand, we notice that

$$\|T^*Su\|_A^2 = \langle T^*Su|T^*Su\rangle_A = \langle AST^*u|ST^*u\rangle = \langle (TS)^*u|(TS)^*u\rangle_A = \|(TS)^*u\|_A^2$$

This implies that

$$T[\alpha\|TSu\|_A] \leq [\|T^*Su\|_A]T \leq T[\beta\|TSu\|_A]$$

Consequently, TS is (α,β) - quasi A – normal operator.

Proposition 3.6

Let $T, S \in B_A(H)$ such that T is (α,β) - quasi A – normal and S is A-unitary. If $TS = ST$ and $N(A)$ is a reducing subspace for T ,then TS is (α,β) - quasi A – normal.

Proof

Since $N(A)$ is a reducing subspace for T we notice that

$TP_{\overline{R(A)}} = P_{\overline{R(A)}}T$ and $T^*P_{\overline{R(A)}} = P_{\overline{R(A)}}T^*$

Since S is A – unitary, then $S^*S = SS^* = P_{\overline{R(A)}}$

Now we have

$$T[\beta^2((TS)^*(TS))] = T[\beta^2(T^*S^*ST)] = T[\beta^2(T^*P_{\overline{R(A)}}T)] = T[\beta^2 P_{\overline{R(A)}}T^*TP_{\overline{R(A)}})]$$

By using the condition that T is (α,β) - quasi A – normal, then

$$T[\beta^2((TS)^*(TS))] \geq_A [(P_{\overline{R(A)}}T\,T^*P_{\overline{R(A)}})]T \geq_A T[\alpha^2(P_{\overline{R(A)}}T^*TP_{\overline{R(A)}})] \qquad (6)$$

Inequality (6) gives

$[P_{\overline{R(A)}}T\,T^*P_{\overline{R(A)}}]T = [TP_{\overline{R(A)}}\,T^*]T = [TSS^*T^*]T = [TS(TS)^*]T$ and

$[P_{\overline{R(A)}}\,T^*TP_{\overline{R(A)}}]T = [T^*P_{\overline{R(A)}}\,T]T = [T^*S^*ST]T = [(TS)^*TS]T$. Then

$$T[\beta^2(TS)^*TS] \geq_A [TS(TS)^*]T \geq_A T[\alpha^2(TS)^*TS]$$

Hence TS is (α,β) - quasi A – normal.

Theorem 3.7

Let $T, S \in B_A(H)$ such that T is (α,β) - quasi A – normal $0 \leq \alpha \leq 1 \leq \beta$ and S is (α',β') quasi A – normal $0 \leq \alpha' \leq 1 \leq \beta'$. Then the following statements hold

(i) If $T^*S = ST^*$, then TS is $(\alpha\alpha',\beta\beta')$ - quasi A – normal operator.

(ii) If $S^*T = TS^*$, then ST is $(\alpha\alpha',\beta\beta')$ - quasi A – normal operator.

Proof

Since T is (α,β) - quasi A – normal and S is (α',β') - quasi A – normal such that $T^*S = ST^*$, it follows that for all $u \in H$

$$T[\alpha\alpha'\|TSu\|_A] \leq \|S^*T^*u\|_A]T \leq T[\beta\beta'\|TSu\|_A]$$

$$T[\alpha\alpha'\|STu\|_A] \leq \|(TS)^*u\|_A]T \leq T[\beta\beta'\|STu\|_A]$$

Hence, TS is $(\alpha\alpha',\beta\beta')$ - quasi A – normal operator. The proof of the second condition is proved as the same way as the first condition.

Example 3.8

Consider the operators Let $A = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $S = T + I = \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix} \in B(R^2)$. Hence easily to check that T is (α, β) quasi A – normal but S is not (α, β) quasi A – normal.

Example 3.9

(i) Consider $T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ which are (α, β) $I_3 -$ quasi A – normal and their product is (α, β) $I_3 -$ quasi A – normal.

(ii) Consider $T = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ and $S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ which are (α, β) $I_2 -$ quasi A – normal whereas their product is not (α, β) $I_2 -$ quasi A – normal.

Conclusion

We defined as the (α, β) - quasi A - normal operators in semi – Hilbert spaces are relatively new. We attempted to prove some properties of (α, β) - quasi A - normal operators in semi – Hilbert spaces in complex Hilbert space. The results of this paper will be accessible for further research to develop application side of Functional Analysis.

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