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RESEARCH ARTICLE

Analysis of the M/M/2 Queue with Heterogeneous Servers, System Disaster, Server Repair, Customers' Impatience, Balking and Reneging

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Abstract

Heterogeneity of service is a major aspect of many real multi-server queueing situations such as banks, hospitals, telecommunication networks, manufacturing systems and several busi- ness organisations. In this paper we consider a two-server heterogeneous queueing system with disastrous breakdowns, repair, balking and reneging and customer impatience where customers became impatient when the system is down. The system size probabilities in tran- sient state are obtained in modified Bessel function closed form using generating functions and the identity of the confluent hypergeometric function. Further, the steady-state system size probabilities are derived in closed form using identities and hypergeometric function. Numerical illustrations are also shown to visualize the system effect.

Keywords: Heterogeneous servers - System disaster - Server repair - Customers' impatience - Hypergeometric - Moments - Steady-state probabilities.

1. Introduction

In most studies on queueing systems, the customers always wait in the system until service is completed. In many practical systems, such as telephone switchboard customers, hospital emergency rooms' handling of critical patients, and perishable goods storage inventory systems, the customers may become impatient and leave (i.e., balk or renege) the

system without getting services when the waiting time is intolerable. Balking is a widespread effect of not joining a queue because the arriving customer estimates the queue to be too long. The concept of balking finds its applications not only in daily life, but in computer communication, production systems and hospital management

Queueing with customer impatience finds its applications in various areas like call centers, packet-switched communication networks, hospitals, perishable inventory systems etc. Queueing systems with customer impatience can be treated for relating some types of perishable inventory systems as there is an correspondence between queueing systems with reneging and perishable inventory systems.

The on hand inventory can be regarded as a queue, the demand completion as completion of service, the products pending in the form of replenishment as arrivals to the queueing system, and the life time of a product as the impatience (reneging) time. Customer reneging and product perishing are analogous event. A customer whose patience time expires leaves the queue whereas a product made to stock whose lifetime expires is removed from the inventory.

In supply chains, the perishable items like vegetables, fruits etc. in the conges- tion situations become worthless if they are not reached to the vendors (customers) at appropriate time as they may get damaged i.e. the perishable items can be modeled as the reneged customers.

In the call centres, a calling customer generally hangs up before service agent and thus gets reneged. In packet switched communication networks with time critical traffic, a packet loses its value if it is not transmitted within a given time interval.

The patients (customers) who leave the emergency rooms in hospitals without been observed are also regarded as reneged customers. Kidney transplant waiting system can be considered as a queue with reneging, where reneging occurs because a customer that is waiting for a kidney may die.

For further exploration of the topic, interested individuals can consult the works of [1] and [2]. Conversely, heterogeneity of service is a common feature of many real multi-server queueing situations. Multi-server queueing systems with heterogeneous servers are useful for analyzing a wide variety of real systems, including manufacturing systems, service systems, telecommunica- tions and computer systems.

Heterogeneous servers are the servers who have varying individual service rates based on their efficiency of providing service. In real life situations, the service rates of servers are hetero- geneous in nature. Service stations which are not mechanically controlled like checkout counters, grocery stores, banks and departments etc. have heterogeneous service because one cannot ex- pect human servers to work at constant rate.. The significance of quality and service performance plays a pivotal role in shaping customer perspectives, necessitating companies to allocate spe- cific focus on these elements during the formulation and execution of their operational strategies. Due to this factor, queues with heterogeneous servers have garnered substantial interest within the literature. In [3] the authors have derived the transient solution for the probabilities in the two-server queueing system subject to catastrophes, where one server is faster than the other, by defining a suitable probability

generating function. In [4], the author studied an M/M/2 queueing system with balking and two heterogeneous servers extends the work of Krishnamoor- thy [11] (1963)[11] on two heterogeneous servers by involving balking and revealed that the heterogeneous system is better than the corresponding homogeneous system . In [5], an M/M/2 queueing system with two heterogeneous servers under a variant vacation policy, where the two servers may take together at most J vacations when the system is empty was studied. In [6], an M/M/2 queuing framework featuring heterogeneous servers, wherein one server remains inactive while the other enters a vacation mode when no waiting customers are present was discussed. Transient solution for the probabilities in the two server subject to balking and reneging was investigated in [7]. Two Heterogeneous Server Queuing Model with Intermittently Obtainable Server using Matrix Geometric Method has analysed by [8]. The authors analysed the queueing system with arbitrary distributed service time and inter-arrival time, which is functioning within a random time period [9]. [10] obtained the steady state results of a two heterogeneous server Markovian queue with the second server has a threshold for service.

Sudhesh et al. [12] performed transient and steady state analysis of a two heterogeneous servers queue with system disaster where customers become impatient while the system is down. Two-heterogeneous server Markovian queueing model with discouraged arrivals, reneging and retention of reneged customers was discussed in [13]. Liou analysed a Markovian queue optimisa- tion analysis with an unreliable server subject to working breakdowns and impatient customers. In [14], the authours investigated the concept of a Markovian queueing model with heteroge- neous, intermittently available servers with feedback under a hybrid vacation policy. Seenivasan et al. [15] studied a two-heterogeneous-server queueing model with an intermittently obtainable server by using the matrix geometric method. A queueing model based on k sequential het- erogeneous service steps and vacations was recently proposed by Mohammadi and Salehi rad [16].

2   Model Description

We consider a two-server heterogeneous queueing system with disastrous breakdowns,repair, balking and reneging and customer impatience where customers became impatient when the system is down. Customers arrive according to a Poisson Process with rate λ. Service is exponentially distributed where the two servers provide heterogeneous service with different service rates $\mu 1$ and $\mu 2$ such that $\mu 1 > \mu 2$.Each customer needs only one server for service and the customers select the servers on fastest server first (FSF) basis.

A customer who on arrival finds at least two customers in the system, either decides to enter the queue with probability p or balk with probability $1 - p$. Let $\lambda p = \lambda p$. After joining the queue, each customer will wait a certain length of time T for service to begin. If it has not begun by then, he will get impatient and leave the queue without getting service. This time T is assumed to be distributed according to an exponential distribution with mean $1/\delta$. Since the arrival and the departure of the impatient customers without service are independent, the reneging rate when there are n customers is $(n - 2)\delta$.

When the system is idle or busy, disaster occurs according to a Poisson process of rate γ. Whenever a disaster occurs at the system,all present customers (waiting and served) are flushed out from the system and both the servers become inactivated. A repair process then starts immediately and the repair time of the system is exponentially distributed with mean

η−1.When the system is down, inoperative, and undergoing a repair process,new arrivals become impatient. Each individual customer, upon arrival, activates a random-duration impatience timer with parameter ξ. If the timer expires before the system is repaired, the customer abandons the queue and never return.

Let $\{(X(t), Y(t)), t \geq 0\}$be a two-dimensional continuous-time Markov chain, where $X(t)$denotes the number of customers in the system at time $t$and $Y(t)$represents the state of the system at time t, with state space

$$S = \{(n, j): n = 0,1,2, \dots; \ j = 0,1,2,3,4\}.$$

The state transition diagram of the system is given in **Figure 1**. If $Y(t) = 3$, the system is functioning and both the servers are serving customers, whereas if $Y(t) = 4$, the system is down and undergoing a repair process. The pictorial representation of the model is shown in **Fig. 1**.

The state $(0, 0)$represents that the system is empty and the servers are in the 'ON' state. The state $(1, 1)$represents that there is one customer in the system being served by the faster server, and $(1, 2)$represents that there is one customer in the system being served by the slower server. The state $(n, 3)$, $n = 2,3,4, \dots$, represents that there are $n$customers in the system when the system is in the working state, and the state $(n, 4)$, $n = 0,1,2, \dots$, represents that there are $n$customers in the system when the system is in the failure state.

Let $P_{nj}(t)$denote the time-dependent system size probabilities where there are $n$customers in the system at time t, and $j$takes the values 0,1,2,3,and 4. Mathematically,

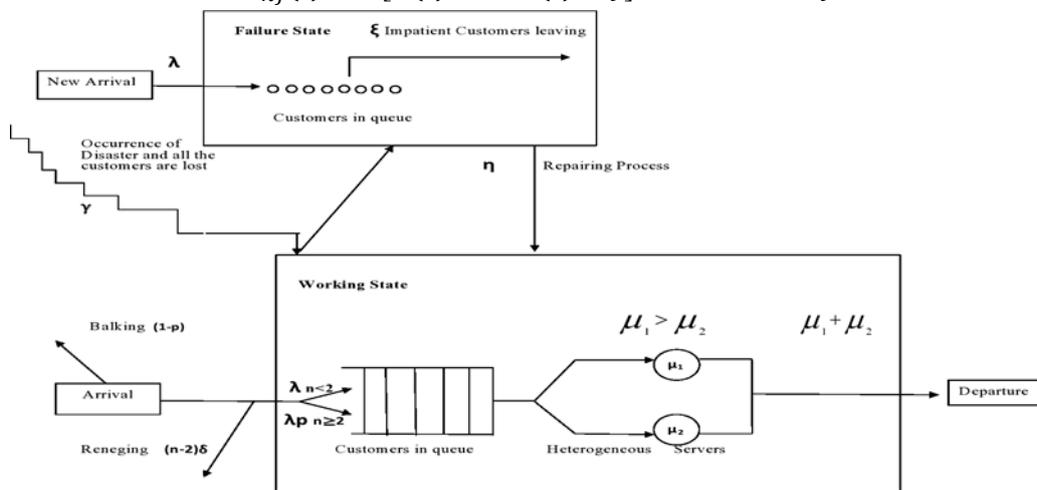$$P_{nj}(t) = P[X(t) = n, \ Y(t) = j], n = 0,1,2, \dots; \ j = 0,1,2,3,4.$$

Figure 1: Diagrammatic Representation of the Model

2.1 Governing Equations

With the underlying assumptions, the behaviour of the resulting system is described by a set of **Chapman–Kolmogorov forward equations**, which can be written as:

$$P'_{0,0}(t) = -(\lambda + \gamma)P_{0,0}(t) + \mu_1 P_{1,1}(t) + \mu_2 P_{1,2}(t) + \eta P_{0,4}(t) \qquad (2.1)$$

$$P'_{1,1}(t) = -(\lambda + \mu_1 + \gamma)P_{1,1}(t) + \lambda P_{0,0}(t) + \mu_2 P_{2,3}(t) + \eta P_{1,4}(t) \qquad (2.2)$$

$$P'_{1,2}(t) = -(\lambda + \mu_2 + \gamma)P_{1,2}(t) + \mu_1 P_{2,3}(t) \qquad (2.3)$$

$$P'_{2,3}(t) = -(\lambda p + \mu + \gamma)P_{2,3}(t) + \lambda P_{1,1}(t) + \lambda P_{1,2}(t) + (\mu + \delta)P_{3,3}(t) + \eta P_{2,4}(t) \qquad (2.4)$$

$$P'_{n,3}(t) = -(\lambda p + \mu + (n-2)\delta + \gamma)P_{n,3}(t) + \lambda p P_{n-1,3}(t) + (\mu + (n-1)\delta)P_{n+1,3}(t) + \eta P_{n,4}(t), n \geq 3 \qquad (2.5)$$

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$$P'_{0,4}(t) = -(\lambda + \eta)P_{0,4}(t) + \xi P_{1,4}(t) + \gamma \left[1 - \sum_{n=0}^{\infty} P_{n,4}(t)\right] \qquad (2.6)$$

$$P'_{n,4}(t) = -(\lambda + \eta + n\xi)P_{n,4}(t) + \lambda P_{n-1,4}(t) + (n+1)\xi P_{n+1,4}(t), n \geq 1 \qquad (2.7)$$

where

$$\mu = \mu_1 + \mu_2$$

We assume that the number of customers present initially is random with probability p_r, $r = 0,1,2, ...$, where r represents the number of customers in the system initially.

3 Steady-State Probabilities

In this section, the **steady-state probabilities** are derived by replacing the left-hand side of equations $(2.1) - (2.7)$ by zero. From these equations we obtain:

$$P_{0,0} = \frac{(a + c + \mu_1)\mu_1\mu_2}{X} P_{1,1} + \frac{\eta(a + \mu_1)}{X} P_{1,4} + \frac{\eta\mu_1}{X} P_{0,4} \qquad (3.1)$$

$$P_{1,1} = \frac{\lambda\mu_1\mu_2 + ac\mu_2}{X} P_{2,3} + \frac{\lambda\eta}{X} P_{1,4} + \frac{a\eta}{X} P_{0,4} \qquad (3.2)$$

$$P_{1,2} = \frac{\mu_1}{c} P_{2,3} \qquad (3.3)$$

$$P_{2,3} = \frac{\lambda^2\eta c}{X} P_{0,4} + \frac{\lambda\eta ac}{X} P_{1,4} + \frac{\eta c(a^2 + \gamma\mu_1)}{X} P_{2,4} + \frac{c(\mu + \delta)(a^2 + \gamma\mu_1)}{X} P_{3,3} \qquad (3.4)$$

where

$$a = \lambda + \gamma$$
$$c = \lambda + \gamma + \mu_2$$
$$X = (a^2 + \gamma\mu_1)cd - \lambda\mu_2(ac + \lambda\mu_1) - \mu_1(a^2 + \gamma\mu_1)$$

Using the identity from **Lorentzen and Waadeland [18]**, we obtain for $n \geq 1$:

$$P_{n,4} = \psi_n P_{0,4} \qquad (3.5)$$

where

$$\psi_n = \left(\frac{\lambda}{\xi}\right)^n \frac{F(n + 1; \eta + n + 1; -\lambda/\xi)}{\prod_{i=1}^{n}(\eta + i)/\xi \ F(1; \eta + 1; -\lambda/\xi)}$$

and

$$P_{0,4} = \gamma \left[\lambda + \eta + \gamma - \xi\psi_1 + \gamma \sum_{n=1}^{\infty} \psi_n\right]^{-1} \qquad (3.6)$$

To find $P_{n,3}$, $n \geq 3$, define the **probability generating function**

$$G(z) = \sum_{n=1}^{\infty} P_{n+2,3} \, z^n \qquad (3.7)$$

and

$$G'(z) = \sum_{n=1}^{\infty} n \, P_{n+2,3} z^{n-1}$$

Using Eq. (2.5), we obtain the differential equation

$$G'(z) - \frac{\lambda pz - \mu + \frac{\gamma}{z}}{\delta(1-z)}G(z) - \frac{\mu+\delta}{\delta(1-z)}P_{3,3} + \frac{\lambda pz}{\delta(1-z)}P_{2,3} + \frac{\eta}{\delta(1-z)}\sum_{n=1}^{\infty}P_{n+2,4}\,z^n = 0 \qquad (3.8)$$

To solve the above first-order linear differential equation, an **integrating factor** is

$$I.F = \exp\left(\ \int \frac{\lambda pz - \mu + \frac{\gamma}{z}}{\delta(1-z)}dz\right) = z^{-\mu/\delta}(1-z)^{-\gamma/\delta}e^{\lambda pz/\delta}$$

The general solution of the differential equation becomes

$$G(z) = z^{-\mu/\delta}(1-z)^{-\gamma/\delta}\left[(\delta+1)P_{3,3}A(z) - \lambda pP_{2,3}B(z) - \frac{\eta}{\delta}\sum_{n=1}^{\infty}P_{n+2,4}\,C(z)\right] \qquad (3.9)$$

where

$$A(z) = \int_0^z e^{\frac{\lambda p(z-\chi)}{\delta}}\chi^{\frac{\mu}{\delta}-1}(1-\chi)^{\frac{\gamma}{\delta}-1}d\chi \qquad (3.10)$$

$$B(z) = \int_0^z e^{\frac{\lambda p(z-\chi)}{\delta}}\chi^{\frac{\mu}{\delta}}(1-\chi)^{\frac{\gamma}{\delta}-1}d\chi \qquad (3.11)$$

$$C(z) = \int_0^z e^{\frac{\lambda p(z-\chi)}{\delta}}\chi^{\frac{\mu}{\delta}}(1-\chi)^{\frac{\gamma}{\delta}}d\chi \qquad (3.12)$$

To obtain $P_{3,3}$ in terms of $P_{2,3}$, let us determine (3.10)–(3.12) as $z \to 1$.

Using the identity from **Abramowitz and Stegun [17]**, for
$\text{Re}(u) > 0, \text{Re}(v) > 0$:

$$\int_0^w \chi^{v-1}(w-\chi)^{u-1}e^{\beta\chi}\,d\chi = B(u,v)w^{u+v-1}{}_1F_1(v;u+v;\beta w) \qquad (3.13)$$

with

$$B(b,c) = \int_0^1 t^{b-1}(1-t)^{c-1}dt, b > 0,\ c > 0$$

the **Beta function**;

$${}_1F_1(\alpha;\beta;z) = \sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\beta)_k}\frac{z^k}{k!}$$

the **degenerate hypergeometric function**, and

$$\left(\alpha\right)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$$

the **Pochhammer symbol**.

Substituting $z = 1$, $1 - \chi = t$, and using identity (3.13) in Eqs. (3.10)–(3.12), we obtain

$$A(1) = K \boxed{\cdot} \left(\frac{\lambda p}{\delta}, \frac{\gamma}{\delta}, \frac{\mu}{\delta} + 1 \right) \qquad (3.14)$$

$$B(1) = K \boxed{\cdot} \left(\frac{\lambda p}{\delta}, \frac{\gamma}{\delta}, \frac{\mu}{\delta} + 2 \right) \qquad (3.15)$$

$$C(1) = K \boxed{\cdot} \left(\frac{\lambda p}{\delta}, \frac{\gamma}{\delta}, \frac{\mu}{\delta} + n + 1 \right) \qquad (3.16)$$

where

$$K(a, b, c) = B(b, c) \quad {}_1F_1(c; b + c; a)$$

Now determining $G(z)$ for the limit $z \to 1$ yields

$$\lim_{z \to 1} G(z) = G(1)$$

$$= \left(\frac{\mu}{\delta} + 1 \right) P_{3,3} A(1) - \frac{\lambda p}{\delta} P_{2,3} B(1) - \frac{\eta}{\delta} \sum_{n=1}^{\infty} P_{n+2,4} \, C(1)$$

Since

$$0 \le G(1) = \sum_{n=1}^{\infty} P_{n+2,3} \le 1$$

and

$$\lim_{z \to 1} (1 - z)^{-\gamma/\delta} = \infty,$$

we must have

$$\left(\frac{\mu}{\delta} + 1 \right) P_{3,3} A(1) - \frac{\lambda p}{\delta} P_{2,3} B(1) - \frac{\eta}{\delta} \sum_{n=1}^{\infty} P_{n+2,4} \, C(1) = 0$$

and this condition gives

$$P_{3,3} = \frac{\lambda p}{\delta} \frac{B(1)}{A(1)} P_{2,3} + \frac{\eta}{\delta} \frac{C(1)}{A(1)} \sum_{n=1}^{\infty} P_{n+2,4} \qquad (3.17)$$

Using the above relation in (3.9), we obtain

$$G(z) = z^{-\mu/\delta} (1 - z)^{-\gamma/\delta} \left[\frac{\lambda p B(1)}{\delta A(1)} P_{2,3} + \frac{\eta C(1)}{\delta A(1)} \sum_{n=1}^{\infty} P_{n+2,4} \right] \qquad (3.18)$$

Series Expansion of $A(z)$, $B(z)$, and $C(z)$

To express $G(z)$ as a series, the integrals $A(z)$, $B(z)$, and $C(z)$ are expanded.

The function $A(z)$ can be written using the **incomplete Beta function**

$$B(x; a, b) = \int_0^x t^{a-1} (1 - t)^{b-1} dt$$

and the **Gauss hypergeometric function**

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(\ (}{(c)_k k!} z^k$$

Thus Eq. (3.10) becomes

$$A(z) = e^{\lambda p z/\delta} z^{-\mu/\delta} (1 - z)^{-\gamma/\delta} \sum_{n=0}^{\infty} H(n) z^{n+1} \qquad (3.19)$$

where

$$H(n) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \left(\frac{\lambda p}{\delta}\right)^k \frac{1}{n - k + \frac{\mu}{\delta} + \frac{\gamma}{\delta} + 1}$$

Similarly,

$$B(z) = e^{\lambda p z/\delta} z^{-\mu/\delta} (1 - z)^{-\gamma/\delta} \sum_{n=0}^{\infty} J(n) z^{n+2} \qquad (3.20)$$

and

$$C(z) = e^{\lambda p z/\delta} z^{-\mu/\delta} (1 - z)^{-\gamma/\delta} \sum_{n=0}^{\infty} K(n) z^{2n+1} \qquad (3.21)$$

Substituting (3.19)–(3.21) into (3.18) and expanding the exponential terms gives

$$G(z) = \sum_{n=1}^{\infty} P_{n+2,3} z^n \qquad (3.22)$$

Using the **Cauchy product** of power series, we obtain

$$G(z) = \sum_{n=1}^{\infty} \left[\sum_{r=1}^{n} \frac{(\lambda p)^{n-r}}{\delta(n-r)!} H(r - 1) \right] z^n$$
$$\times \left[\frac{\lambda p}{\delta} \frac{B(1)}{A(1)} P_{2,3} + \frac{\eta}{\delta} \frac{C(1)}{A(1)} \sum_{k=1}^{\infty} P_{k+2,4} \right] \qquad (3.23)$$

and this is the solution of the differential equation (3.8).

Thus the probabilities $P_{n,3}$, $n \geq 3$, are

$$P_{n,3} = \sum_{r=1}^{n} \frac{(\lambda p)^{n-r}}{\delta(n-r)!} H(r - 1) \left[\frac{\lambda p}{\delta} \frac{B(1)}{A(1)} P_{2,3} + \frac{\eta}{\delta} \frac{C(1)}{A(1)} \sum_{k=1}^{\infty} P_{k+2,4} \right] \qquad (3.24)$$

Thus **Eqs. (3.1)–(3.6) and (3.24)** completely determine the steady-state system size probabilities.

4 Transient Behaviour

Define the probability generating function

$$G(t,z) = R(t) + \sum_{n=0}^{\infty} P_{n+3,3}(t) z^{n+1} \qquad (4.1)$$

where

$$R(t) = P_{0,0}(t) + P_{1,1}(t) + P_{1,2}(t) + P_{2,3}(t)$$

with initial condition

$$G(0,z) = \sum_{r=0}^{\infty} p_r\, z^r$$

The system of Eqs. (2.1)–(2.5) yields the partial differential equation

$$\frac{\partial G}{\partial t} = -\gamma R(t) + \eta \left[\sum_{i=0}^{2} P_{i,4}(t) + \sum_{n=1}^{\infty} P_{n+2,4}(t) z^n \right] + \lambda p(z-1)P_{2,3}(t) + \cdots \qquad (4.2)$$

Integrating gives

$$G(t,z) = \int_0^t \exp\{-(\lambda p + \gamma + \mu + \delta)(t-y)\}\exp\{(\lambda pz + \frac{\mu+\delta}{z})(t-y)\}R(y)\,dy + \cdots \qquad (4.3)$$

If

$$\alpha = 2\sqrt{\lambda p(\mu+\delta)}, \beta = \sqrt{\frac{\lambda p}{\mu+\delta}}$$

then

$$\exp\{(\lambda pz + \frac{\mu+\delta}{z})(t-y)\} = \sum_{n=-\infty}^{\infty} (\beta z)^n I_n[\alpha(t-y)]$$

where $I_n(\cdot)$ is the **modified Bessel function of the first kind**.

4.1 Evaluation of $P_{n+2,3}(t)$

Comparing coefficients of z^n in (4.3) yields

$$P_{n+2,3}(t) = \beta^{-n} \int_0^t e^{-(\lambda p + \gamma + \mu + \delta)(t-y)} [\lambda p\beta I_{n-1} - \lambda p I_n]P_{2,3}(y)dy + \cdots \qquad (4.4)$$

Using the Bessel property

$$I_{-n}(x) = I_n(x)$$

we obtain

$$P_{n+2,3}(t) = \lambda p \beta \int_0^t e^{-(\lambda p + \gamma + \mu + \delta)(t-y)} [I_{n-1} - I_{n+1}] P_{2,3}(y) dy + \cdots \qquad (4.6)$$

4.2 Evaluation of $P_{2,3}(t)$

Eqs. (2.1)–(2.3) can be written as

$$\frac{dH(t)}{dt} = BH(t) + \eta P_{0,4}(t)e_1 + (\mu_2 P_{2,3}(t) + \eta P_{1,4}(t))e_2 + \mu_1 P_{2,3}(t)e_3 \qquad (4.7)$$

where

$$H(t) = (P_{0,0}(t), P_{1,1}(t), P_{1,2}(t))^T$$

$$B = \begin{bmatrix} -(\lambda + \gamma) & \mu_1 & \mu_2 \\ \lambda & -(\lambda + \mu_1 + \gamma) & 0 \\ 0 & 0 & -(\lambda + \mu_2 + \gamma) \end{bmatrix}$$

Taking Laplace transforms gives

$$\hat{H}(s) = (sI - B)^{-1} [H(0) + \eta \hat{P}_{0,4}(s)e_1 + \mu_2 \hat{P}_{2,3}(s)e_2 + \eta \hat{P}_{1,4}(s)e_2 + \mu_1 \hat{P}_{2,3}(s)e_3] \qquad (4.8)$$

Further manipulation yields

$$\hat{P}_{2,3}(s) = \left[s + \gamma - \frac{1}{2}(w - \sqrt{w^2 - \alpha^2} - 2\lambda) + c^*(s) \right]^{-1} (\cdots) \qquad (4.18)$$

After Laplace inversion we obtain the explicit expression

$$P_{2,3}(t) = \sum_{m=0}^{\infty} \left(\frac{\mu}{\lambda}\right)^m \sum_{k=0}^{m} (-1)^k \int_0^t c_2^{*k}(t-u) e^{-(\lambda+\mu+\gamma)u} [\cdots] du \qquad (4.19)$$

4.3 Evaluation of $P_{0,0}(t), P_{1,1}(t), P_{1,2}(t)$

Using inverse transforms we obtain

$$P_{0,0}(t) = \int_0^t \sum_{j=1}^{3} b_{1j}(u) P_j(0) b_{11}(t-u) du + \cdots \qquad (4.23)$$

$$P_{1,1}(t) = \int_0^t \sum_{j=1}^{3} b_{2j}(u) P_j(0) b_{21}(t-u) du + \cdots \qquad (4.24)$$

$$P_{1,2}(t) = \int_0^t \sum_{j=1}^{3} b_{3j}(u) P_j(0) b_{31}(t-u) du + \cdots \qquad (4.25)$$

4.4 Evaluation of $P_{n,4}(t)$

Using the method of **Sudhesh et al. [12]**, the probabilities are

$$P_{n,4}(t) = \phi_n(t) * P_{0,4}(t) \qquad (4.26)$$

where

$$P_{0,4}(t) = \gamma \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^t y^k\, e^{-(\lambda+\eta+\gamma)y} \left[\sum_{i=1}^{\infty} (\gamma - \delta_1 i\xi)\phi_i(y) \right]^{*k} dy \qquad (4.27)$$

and

$$a_k(t) = \frac{1}{\xi^{k-1}} \sum_{r=1}^{k} \frac{(-1)^{k-r}}{(r-1)!\,(k-r)!} e^{-(\eta+r\xi)t}$$

$$b_k(t) = \sum_{i=1}^{k-1} (-1)^{i-1} a_i(t) * b_{k-i}(t),\, b_1(t) = a_1(t) \qquad (4.28)$$

where $*$ denotes **convolution**.

Thus Eqs. (4.6), (4.19), (4.23)–(4.25), (4.26), (4.27), and (4.28) completely determine all the time-dependent system size probabilities.

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