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INTEGRAL POINTS ON THE CONE $3(x^2+y^2)-5xy=47z^2$

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ABSTRACT

The ternary quadratic Diophantine equation $3(x^2+y^2)-5xy=47z^2$ representing a cone is analyzed for non-zero distinct integer points on it. Different patterns of integer solutions to the cone under consideration are presented. A few interesting relations among the solutions are given.

KEYWORDS: Ternary quadratic, homogeneous cone, integer points.

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INTRODUCTION

The ternary quadratic Diophantine equations offer an unlimited field for research due to their variety [1, 2]. For an extensive review of various problems, one may refer [3-21]. This communication concerns with yet another interesting ternary quadratic equation $3(x^2+y^2)-5xy=47z^2$ representing a cone for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

NOTATIONS

- P_n^m - Pyramidal number of rank n with size m
- $T_{m,n}$ - Polygonal number of rank n with size m
- Pr_n - Pronic number of rank n

METHOD OF ANALYSIS

The ternary quadratic equation to be solved for its non-zero distinct integer solutions is

$$3(x^2+y^2)-5xy=47z^2 \tag{1}$$

To start with, it is noted that (1) is satisfied by the following triples of integers:

(x, y, z) : $(1854, 1242, 270)$, $(1760, 1336, 248)$,

$$(14A^2+108A-1242, 10A^2-84A-1854, 2A^2+24A+270),$$

$$(14A^2-80A-1336, 10A^2+104A-1760, 2A^2-20A+248)$$

However we have other patterns of solutions to (1) which are illustrated below:

The substitution of the linear transformation

$$x = u+v, y = u-v \tag{2}$$

where $u \neq v \neq 0$ in (1) leads to

$$u^2 + 11v^2 = 47z^2 \tag{3}$$

Now (3) is solved through different methods to get u, v and z .Thus in view of (2), one obtains different patterns of solutions to (1).

PATTERN 1:

Assume $z = z(a,b) = a^2 + 11b^2$, $a, b \neq 0$ (4)

Write 47 as

$$47 = \left(6 + i\sqrt{11}\right)\left(6 - i\sqrt{11}\right) \tag{5}$$

Using (4) and (5) in (3) and employing the method of factorization, define

$$u + i\sqrt{11}v = \left(6 + i\sqrt{11}\right)\left(a + i\sqrt{11}b\right)^2$$

Equating real and imaginary parts, we get

$$\left.\begin{array}{l} u = 6a^2 - 66b^2 - 22ab \\ v = a^2 - 11b^2 + 12ab \end{array}\right\} \tag{6}$$

Substituting (6) in (2), the values of x and y are given by

$$\left.\begin{array}{l} x = x(a,b) = 7a^2 - 77b^2 - 10ab \\ y = y(a,b) = 5a^2 - 55b^2 - 34ab \end{array}\right\} \tag{7}$$

Thus (4) and (7) represents non-zero distinct integer solutions of (1).

Properties:

1. $x(A,1) + y(A,1) - T_{26,A} \equiv 0 \pmod{33}$
2. $y(1,B) + 55\,\mathrm{Pr}_B \equiv 5 \pmod{21}$
3. $5z(A, A+1) + y(A, A+1) + 68T_{3,A} = 10T_{4,A}$
4. $x(1,B) + T_{156,B} - 7 \equiv 0 \pmod{86}$
5. $z(A,2) - 4z(A,1) + 3T_{4,A} = 0$
6. $y(A(A+1), 2A) + 116P_A^5 + 75\left(T_{6,A} + \mathrm{Pr}_A\right) = 5T_{4,A}^2$
7. $x(A+1, A+1) - y(A+1, A+1)$ is a perfect square.
8. $3A\{x(-A, A) - x(2A, A)\}$ is a cubic integer.
9. Each of the following expressions is a nasty number.
   (i) $z(A, A) - y(A, A)$
   (ii) $2z(A, A)$

PATTERN 2:

Equation (3) can be written as

$$u^2 + 11v^2 = 47z^2 * 1 \tag{8}$$

Write 1 as

$$1=\frac{\left(5+i\sqrt{11}\right)\left(5-i\sqrt{11}\right)}{36} \tag{9}$$

Using (4), (5) and (9) in (8) and employing the method of factorization, define

$$u+i\sqrt{11}v=\left(6+i\sqrt{11}\right)\left(a+i\sqrt{11}b\right)^2\left(\frac{5+i\sqrt{11}}{6}\right)$$

Equating real and imaginary parts, we get

$$\left.\begin{aligned} u&=\frac{1}{6}\left[19\left(a^2-11b^2\right)-242ab\right]\\ v&=\frac{1}{6}\left[11\left(a^2-11b^2\right)+38ab\right]\end{aligned}\right\} \quad 10)$$

Substituting (10) in (2), we have

$$\left.\begin{aligned} x&=5a^2-55b^2-34ab\\ y&=\frac{1}{3}\left[4a^2-44b^2-140ab\right]\end{aligned}\right\} \tag{11}$$

Replacing a by $3a$ and b by $3b$ in (4) and (11), the corresponding integer solutions to (1) are given by

$$x=x(a,b)=45a^2-495b^2-306ab$$
$$y=y(a,b)=12a^2-132b^2-420ab$$
$$z=z(a,b)=9a^2+99b^2$$

Properties:

1. $y(A,1)-T_{26,A}\equiv -132(\mathrm{mod}409)$
2. $z(A,1)+y(A,1)-T_{44,A}\equiv -33(\mathrm{mod}400)$
3. $z(A(A+1),A)=36P_A^5+90T_{4,A}+9T_{4,A}^2$
4. $x(A,2)-T_{92,A}\equiv -276(\mathrm{mod}568)$ d
5. $x(A^2+1,A)-45T_{4,A}^2+612P_A^5+99\Pr_A\equiv 45(\mathrm{mod}207)$
6. $y(A-1,-A)-100T_{8,A}\equiv 0(\mathrm{mod}4)$
7. $x(2A,1)-2T_{182,A}\equiv -61(\mathrm{mod}434)$
8. $3\left(z(1,A)-99T_{4,A}\right)$ is a cubic integer.
9. Each of the following expressions is a nasty number.
   (i) $y(A,-A)-150T_{4,A}$
   (ii) $x(-2A,A)-3T_{4,A}$

PATTERN 3:

Write 47 as

$$47=\frac{\left(41+i\sqrt{11}\right)\left(41-i\sqrt{11}\right)}{36} \tag{12}$$

Substituting (4) and (12) in (3) and employing the method of factorization, define

$$u+i\sqrt{11}v=\left(\frac{41+i\sqrt{11}}{6}\right)\left(a+i\sqrt{11}b\right)^2$$

Following the procedure as in pattern 1, the corresponding solutions to (1) are obtained as

$$x = x(a,b) = 63a^2 - 693b^2 + 90ab$$
$$y = y(a,b) = 60a^2 - 660b^2 - 156ab$$
$$z = z(a,b) = 9a^2 + 99b^2$$

Properties:

1. $y(A,1) - 6T_{22,A} \equiv -48 \pmod{102}$
2. $x(A,1) - 63\Pr_A \equiv -18 \pmod{27}$
3. $z(A+3, A-2) - 12T_{20,A} \equiv 231 \pmod{246}$
4. Each of the following expressions is a perfect square.
   (i) $x(3A, A)$
   (ii) $x(A,1) + 7z(A,1) - 90\Pr_A$
5. $x(3A, A) + 6T_{4,A}$ is a nasty number.

PATTERN 4:

Write 1 as

$$1 = \frac{(5 + i\sqrt{11})(5 - i\sqrt{11})}{36} \tag{13}$$

Using (4), (12) and (13) in (8) and employing the method of factorization, define

$$u + i\sqrt{11}v = \left(\frac{41 + i\sqrt{11}}{6}\right)(a + i\sqrt{11}b)^2\left(\frac{5 + i\sqrt{11}}{6}\right)$$

Following the procedure as in pattern 2, the corresponding integer solutions to (1) are given by

$$x = x(a,b) = 60a^2 - 660b^2 - 156ab$$
$$y = y(a,b) = 37a^2 - 407b^2 - 350ab$$
$$z = z(a,b) = 9a^2 + 99b^2$$

Properties:

1. $x(A,1) - y(A,1) - 23\Pr_A \equiv -82 \pmod{171}$
2. $z(A,1) - T_{20,A} \equiv 3 \pmod{8}$
3. $z(2A,1) - 9T_{10,A} \equiv 18 \pmod{27}$
4. Each of the following expressions is a perfect square.
   (i) $y(2A, -A)$
   (ii) $-y(-A, A) - 4T_{4,A}$

PATTERN 5:

Introducing the transformations

$$\left.\begin{aligned} z &= X + 11T \\ v &= X + 47T \\ u &= 6U \end{aligned}\right\} \tag{14}$$

in (3), it becomes

$$X^2 = U^2 + 517T^2$$

which is satisfied by

$$\left.\begin{aligned} T &= 2ab \\ U &= 517a^2 - b^2 \\ X &= 517a^2 + b^2 \end{aligned}\right\} \quad (15)$$

From (2), (14) and (15), the integer solutions to (1) are found to be

$$x = x(a,b) = 3619a^2 - 5b^2 + 94ab$$
$$y = y(a,b) = 2585a^2 - 7b^2 - 94ab$$
$$z = z(a,b) = 517a^2 + b^2 + 22ab$$

NOTE:

Instead of (14), if we introduce the transformations

$$\left.\begin{aligned} z &= X - 11T \\ v &= X - 47T \\ u &= 6U \end{aligned}\right\}$$

Then the corresponding solutions to (1) are given by

$$x = x(a,b) = 3619a^2 - 5b^2 - 94ab$$
$$y = y(a,b) = 2585a^2 - 7b^2 + 94ab$$
$$z = z(a,b) = 517a^2 + b^2 - 22ab$$

REMARKABLE OBSERVATIONS

A: If the non-zero integer triples (X_0, Y_0, Z_0) is any solution of (1), then each of the following two triples represented by $(X_0, 80X_0 - 95Y_0 + 376Z_0, 20X_0 - 24Y_0 + 95Z_0)$ and $(-95X_0 + 80Y_0 - 376Z_0, Y_0, 24X_0 - 20Y_0 + 95Z_0)$ also satisfies (1).

B: Employing the solution (x, y, z) of (1) each of the following expressions among the special polygonal and pyramidal numbers are observed.

1. $\frac{1}{47}\left\{3\left[\left(\frac{P_x^5}{T_{3,x}}\right)^2 + \left(\frac{3P_{y-2}^3}{T_{3,y-2}}\right)^2\right] - 5\left(\frac{P_x^5}{T_{3,x}}\right)\left(\frac{3P_{y-2}^3}{T_{3,y-2}}\right)\right\}$ is a perfect square.

2. $3\left[\left(\frac{3P_{x-2}^3}{T_{3,x-2}}\right)^2 + \left(\frac{P_y^5}{T_{3,y}}\right)^2\right] - 15\left(\frac{P_{x-2}^3}{T_{3,x-2}}\right)\left(\frac{P_y^5}{T_{3,y}}\right) = 47\left(\frac{6P_{z-1}^4}{T_{3,2(z-1)}}\right)^2$

3. $3\left[\left(\frac{3P_{x-2}^3}{T_{3,x-2}}\right)^2 + \left(\frac{P_y^5}{T_{3,y}}\right)^2\right] - 15\left(\frac{P_{x-2}^3}{T_{3,x-2}}\right)\left(\frac{P_y^5}{T_{3,y}}\right) \equiv 0 \pmod{47}$

CONCLUSION

In this paper, we have presented a few choices of integral points on the cone $3(x^2 + y^2) - 5xy = 47z^2$. One may search for other patterns of solutions and their corresponding properties.

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