ON THE TERNARY CUBIC DIOPHANTINE EQUATION

\[ X^2 + Y^2 - XY = 12^{2n} Z^3 \]

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ABSTRACT

The ternary cubic Diophantine equation is analyzed for its infinitely many non-zero distinct integral solutions. A few interesting properties among the solutions are presented.

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INTRODUCTION

Integral solutions for the cubic homogeneous or non-homogeneous Diophantine equations are an interesting concept, as it can be seen from [1, 2, 3]. In [4-17] a few special cases of ternary cubic Diophantine equations are studied. In this communication, we present the integral solutions of yet another ternary cubic equation \[ x^2 + y^2 - xy = 12^{2n} z^3 \]. A few interesting relations between the solutions are obtained.

NOTATIONS:

\( T_{m,n} \) : Polygonal number
\( P_m \) : Pyramidal number
\(Pr_n\) : Pronic number

**METHOD OF ANALYSIS:**

The cubic equation under consideration is
\[x^2 + y^2 - xy = 12^{2n} z^3\]  (1)

Assuming \(x = u + v\), \(y = u - v\), \(u \neq v\) in (1), it is written as
\[u^2 + 3v^2 = 12^{2n} z^3\]  (3)

Here, we present two different choices of solutions of (3) and hence, obtain two different patterns of solutions to (1).

Assume \(z = (a, b) = a^2 + 3b^2\), \(a, b \neq 0\)  (4)

**PATTERN: 1**

Write 12 as
\[12 = (3 + i\sqrt{3}) (3 - i\sqrt{3})\]  (5)

Using (4) and (5) in (3) and employing the method of factorization, define
\[(u + i\sqrt{3}v) = (3 + i\sqrt{3})^{2n} (a + i\sqrt{3}b)^3\]  (6)

Where in we write
\[3 + i\sqrt{3} = (\alpha + i\beta)\]  (7)

On comparing real and imaginary parts on both sides of (6), we get
\[u = \alpha(a^3 - 9ab^2) + \beta(9b^3 - 9a^2b)\]
\[v = \alpha(3a^2b - 3b^3) + \beta(a^3 - 9ab^2)\]

Substituting the values of \(u\) and \(v\) in (2), we get
\[x = x(a, b) = (a^3 - 9ab^2)(\alpha + \beta) + (3a^2b - 3b^3)(\alpha - 3\beta)\]
\[y = y(a, b) = (a^3 - 9ab^2)(\alpha - \beta) + (3b^3 - 3a^2b)(\alpha + 3\beta)\]

For simplicity, taking \(n = 1\) in (7), the corresponding integer solutions to (1) are found to be
\[x = x(a, b) = 12a^3 - 108ab^2 - 36a^2b + 12b^3\]
\[y = y(a, b) = 72b^3 - 72a^2b\]
along with (4).

**Properties:**

- \(x(a, 1) - y(a, 1) - 72P_a^3 \equiv 0 (mod 6)\)
- \(x(a, 1) - 72P_a^3 + 144T_{3,a} \equiv 0 (mod 12)\)
- \(y(1,b) = 432P_{v-1}\)
- \(y(a,-1) - 72T_{4,a} \equiv 0 (mod 72)\)
- \(z(a,1) - y(a,1) - 73T_{4,a} \equiv 0 (mod 3)\)

**PATTERN: 2**

Here, instead of (7), assume
\[(3 + i\sqrt{3}) = r(cos \theta + i sin \theta)\]  (8)

Following the procedure as in pattern: 1, employing the method of factorization, define
\[(u + i\sqrt{3}v) = 12^n (cos \frac{n\pi}{3} + i sin \frac{n\pi}{3})(a + i\sqrt{3}b)^3\]  (9)

On comparing the real and imaginary parts, we get
\[ u = 12^n \cos \frac{n\pi}{3} \left( a^3 - 9ab^2 \right) + 3\sqrt{3} \left( 12^n \sin \frac{n\pi}{3} \right) \left( b^3 - a^2b \right) \]
\[ v = 12^n \sin \frac{n\pi}{3} \left( \frac{3a^3}{3} - 3\sqrt{3}ab^2 \right) + (12^n \cos \frac{n\pi}{3}) \left( 3a^2b - 3b^3 \right) \]

Substituting the values of \( u \) and \( v \) in (2), we get
\[
x = x(a,b) = 12^n \cos \frac{n\pi}{3} \left[ a^3 - 9ab^2 - 3b^3 + 3a^2b \right] + 12^n \sin \frac{n\pi}{3} \left[ 3\sqrt{3}b^3 - 3\sqrt{3}a^2b + \frac{3a^3}{3} - 3\sqrt{3}ab^2 \right]
\]
\[
y = y(a,b) = 12^n \cos \frac{n\pi}{3} \left[ a^3 - 9ab^2 + 3b^3 - 3a^2b \right] + 12^n \sin \frac{n\pi}{3} \left[ 3\sqrt{3}b^3 - 3\sqrt{3}a^2b - \frac{3a^3}{3} + 3\sqrt{3}ab^2 \right]
\]

For simplicity, putting \( n=3 \) in the above equations, the corresponding integer solutions to (1) are given by
\[
x = x(a,b) = -1728a^3 + 15552ab^2 + 5184b^3 - 5184a^2b
\]
\[
y = y(a,b) = -1728a^3 + 15552ab^2 - 5184b^3 + 5184a^2b
\]
along with (4)

Properties:
- \( 3\{ y(a,1) - x(a,1) \} \) is a nasty number.
- \( x(-a,1) - 3456P_a^4 + 13824\frac{T_{3a}}{3} \equiv 0 \pmod{12} \)
- \( y(-1,b) - 15552P_b^4 + 23328P_{3b} \equiv 0 \pmod{2} \)
- \( x(-a,1) - 1728P_a^4 + 10368P_{3b} \equiv 0 \pmod{12} \)
- \( z(2,b) - 3T_{4b} \equiv 0 \pmod{4} \)

**CONCLUSION:** To conclude, we may search for other patterns of solutions to (1) along with their properties.

**REFERENCES**


