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SUFFICIENCY AND DUALITY IN NONDIFFERENTIABLE MULTIOBJECTIVE FRACTIONAL PROGRAMMING PROBLEMS USING (H_p,r)-INVEX FUNCTIONS

KHUSHBOO

UNIVERSITY OF DELHI, DELHI, INDIA

KHUSHBOO

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ABSTRACT

In this paper, we consider a class of nondifferentiable multiobjective fractional programming problems in which each component of the objective function contains a term involving the support function of a compact convex set. We establish sufficient optimality conditions and duality results for weakly efficient solutions of nondifferentiable multiobjective fractional programming problems.

Key Words: Nondifferentiable programming, multiobjective fractional programming, optimality conditions, duality theorems.

1.INTRODUCTION

The fractional optimization problem with multiple objective functions has been subject of intense investigation in the past few years. Many authors have introduced various concepts of generalized convexity and have obtained optimality and duality results for the multiobjective nonlinear (nondifferentiable) fractional programming problems. The areas which have been explored are mainly to weaken the convexity and to relax the differentiability assumption of the functions used in developing optimality and duality of the above programming problems.

Different authors have used different forms of nondifferentiability to obtain optimality conditions and duality theory for fractional programming problem under generalized convexity assumptions. Mond and Schechter [7] considered a class of nondifferentiable multiobjective programming problems in which the objective function contains a support function and derived optimality criteria and discussed duality theory. Based on these results, Yang et al. [8], studied Wolfe-type and Mond-Weir-type dual problems for a class of nondifferentiable multiobjective programming problems. Bector et al. [2], derived Fritz John and Karush-Kuhn-Tucker necessary and sufficient optimality conditions for a class of nondifferentiable convex multiobjective fractional programming problems and established some duality theorems.

Antczak [1] introduced a new class of functions named (*p*,*r*)- invex function, which is an extension of invex function. Liu et al. [6], proposed the concept of ($H_{p,}r$)-invex function and discussed the sufficient optimality conditions to multiple objective programming problem. Jaiswal et al. [3] derived duality theorems for multiobjective fractional programming problems involving ($H_{p,}r$)-invex functions. Khan et al. [4] discussed sufficiency and duality in nondifferentiable minimax fractional programming using ($H_{p,}r$)-invexity.

In this paper, we have considered a class of nondifferentiable multiobjective fractional programming problems in which each component of the objective function contains a term involving the support function of a compact convex set and established sufficient optimality conditions and duality results for weakly efficient solutions of nondifferentiable multiobjective fractional programming problems under the ($H_{p,}r$)-invexity assumptions.

2. Notations and Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x \geq 0\}$ and $\dot{\mathbb{R}}^n_+ = \{x \in \mathbb{R}^n \mid x > 0\}$. Let $x, y \in \mathbb{R}^n$. Then $x \leq y \Leftrightarrow x_i \leq y_i,\ i = 1,2,...,n \text{ and } x \neq y.$

Definition 2.1 [6] A subset $X \subseteq \mathbb{R}^n$ is said to be H_p-invex set, if for any $x, u \in X$, there exists a vector function $H_p : X \times X \times [0,1] \to \mathbb{R}^n$ such that

$$H_p(x,u;0) = e^u, \qquad H_p(x,u;\lambda) \in \dot{\mathbb{R}}^n_+$$
$$l_n(H_p(x,u;\lambda)) \in X,\ \forall \lambda \in [0,1], \qquad p \in \mathbb{R}.$$

Remark 2.1 It is understood that the logarithm and the exponentials appearing in the above definition are taken to be component wise.

Definition 2.2 [6] Let *X* be a H_p-invex set, H_p is right differentiable at 0 with respect to the variable λ for each given pair $x, u \in X$ and $f : X \to R$ is differentiable on *X*. If for all $x \in X$, ($x \neq u$) one of the relations

$$\frac{1}{r}e^{rf(x)} \geq \frac{1}{r}e^{rf(u)}\left[1 + \frac{r\nabla f(u)^T}{e^u} H'_p(x,u;0^+)\right](>0), \qquad \text{for } r \neq 0$$

$$f(x) - f(u) \geq \frac{\nabla f(u)^T}{e^u} H'_p(x,u;0^+)(>0), \qquad \text{for } r = 0$$

holds, then *f* is said to be (H_p,*r*)-invex (strictly (H_p,*r*)-invex) at $u \in X$. If the above inequalities are satisfied at any $u \in X$ then *f* is said to be (H_p,*r*)-invex (strictly (H_p,*r*)-invex) on *X*.

We now consider the following nondifferentiable multiobjective fractional programming problem:

(MFP) Minimize $F(x) = \left(\frac{f_1(x) + s(x \mid D_1)}{g_1(x)} .... \frac{f_k(x) + s(x \mid D_k)}{g_k(x)} \right)$

subject to

$$h(x) \leq 0 \qquad (1)$$
$$x \in X \subseteq \mathbb{R}^n,$$

where $X \subseteq \mathbb{R}^n$ is open, $f := (f_1, f_2, ..., f_k) : X \to \mathbb{R}^k$, $g := (g_1, g_2, ..., g_k) : X \to \mathbb{R}^k$, and $h := (h_1, h_2, ..., h_m) : X \to \mathbb{R}^m$ are differentiable functions on a (nonempty) H_p-invex set *X*, for each $i \in \{1, 2, ..., k\}$, D_i is compact convex set in $\mathbb{R}^n$ and $s(x \mid D_i) = \max\{\langle x, y \rangle \mid y \in D_i\}$ denotes the support function of D_i.

Let $X_0 = \{x \in X \mid h(x) \leq 0\}$ be the set of all feasible solutions of (MFP) and

$$f_i(x)+s(x\,|\,D_i)\geq 0;\ g(x)>0,\ \forall i=1,2,..,k,\ \forall x\in X.$$

For any $w=(w_1,w_2,...,w_k)\in\mathbb{R}^n\times\mathbb{R}^n\times....\times\mathbb{R}^n$ and

$$x\in\mathbb{R}^n, x^T w=(x^T w_1, x^T w_2,...,x^T w_k).$$

We now review some known facts about support functions. The support function $s(x\,|\,D)$ of compact convex set $D\subseteq\mathbb{R}^n$, being convex and everywhere finite, has a subgradient at every *x*, that is, there exists $z\in D$ such that

$$s(y\,|\,D)\geq s(x\,|\,D)+z^T(y-x),\quad \forall\ y\in D.$$

Equivalently,

$$z^T x=s(x\,|\,D).$$

The subdifferential of $s(x\,|\,D)$ is given by

$$\partial s(x\,|\,D)=\{z\in D\,|\,z^T x=s(x\,|\,D)\}.$$

Definition2.3 A feasible solution $x^*\in X_0$ is said to be a weakly efficient solution of (MFP) if there does not exist any $x\ \in X_0$ such that

$$F(x)<F(x^*).$$

It can be seen that, if $x^*\in X_0$ is a weakly efficient solution of a multiobjective fractional programming problem (MFP), then the following necessary optimality conditions are satisfied:

Theorem 2.1 [5] (Necessary Optimality Conditions) If x^* is a weakly efficient solution of (MFP) at which a suitable constraint qualification holds then there exist $y^*\in\mathbb{R}^k,\ \mu^*\in\mathbb{R}^m,\ v^*\in\mathbb{R}^k$ and $w_i^*\in\mathbb{R}^n,$ for each *i* = 1,2, ..., *k* such that

$$\sum_{i=1}^{k}\ y_i^*[\nabla f_i(x^*)+w_i^*-v_i^*\nabla g_i(x^*)]+\sum_{j=1}^{m}\ \mu_j^*\nabla h_j(x^*)=0 \tag{2}$$

$$f_i(x^*)+x^{*^T}w_i^*-v_i^*g_i(x^*)=0,\qquad \text{for all } i=1,2,...,k \tag{3}$$

$$\sum_{j=1}^{m}\ \mu_j^*h_j(x^*)=0 \tag{4}$$

$$x^{*T}w_i^*=s(x^*\,|\,D_i),\ \ w_i^*\in D_i\ ,\text{for all } i=1,2,..,k \tag{5}$$

$$(y^*,\mu^*)\geq 0,\ \ y^*\neq 0. \tag{6}$$

Proof. Let x^* be a weakly efficient solution of (MFP) at which a suitable constraint qualification holds. Then there exist $\lambda^*\in\mathbb{R}^k,\mu^*\in\mathbb{R}^m$ and $w_i^*\in\mathbb{R}^n,$ for each $i\in\{1,2,....,k\}$ such that

$$\sum_{i=1}^{k}\ \lambda_i^*\nabla\left(\frac{f_i(x^*)+x^{*^T}w_i^*}{g_i(x^*)}\right)+\sum_{j=1}^{m}\ \mu_j^*\nabla h_j(x^*)=0 \tag{7}$$

$$\sum_{j=1}^{m}\ \mu_j^*h_j(x^*)=0$$

$$x^{*T}w_i^*=s(x^*\,|\,D_i),\ w_i^*\in D_i,\qquad \text{for each } i\in\{1,2,....,k\}$$

$$(\lambda^*,\mu^*)\geq 0,\quad \lambda^*\neq 0.$$

Hence (4), (5), and (6) hold. Now (7) can be written as

$$\sum_{i=1}^{k} \lambda_i^* \left[\frac{g_i(x^*)(\nabla f_i(x^*) + w_i^*) - \nabla g_i(x^*)(f_i(x^*) + x^{*T} w_i^*)}{g_i(x^*)^2} \right] + \sum_{j=1}^{m} \mu_j^* \nabla h_j(x^*) = 0$$

i.e. $$\sum_{i=1}^{k} y_i^* [\nabla f_i(x^*) + w_i^* - v_i^* \nabla g_i(x^*)] + \sum_{j=1}^{m} \mu_j^* \nabla h_j(x^*) = 0$$

where, $$y_i^* = \frac{\lambda_i^*}{g_i(x^*)},$$

$$v_i^* = \frac{f_i(x^*) + x^{*T} w_i^*}{g_i(x^*)},$$

$$\Rightarrow f_i(x^*) + x^{*T} w_i^* - v_i^* g_i(x^*) = 0.$$

Hence (2) and (3) hold.

Remark 2.2 All the theorems in the subsequent parts of this paper will be proved only in the case when $r \neq 0$. The proof in the case when $r = 0$ is easier than in this one since the difference arise only the form of inequality. Moreover, without less of generality, we shall assume that $r > 0$ because in the case when $r < 0$, the direction of some of the inequalities in the proof of the theorems should be changed to the opposite one.

3. Parametric Duality

We consider the following dual of (MFP) as follows:

(D) Maximize $v = (v_1, v_2, ..., v_k)$

subject to

$$\sum_{i=1}^{k} y_i [\nabla f_i(u) + w_i - v_i \nabla g_i(u)] + \sum_{j=1}^{m} \mu_j \nabla h_j(u) = 0 \tag{8}$$

$$f_i(u) + u^T w_i - v_i g_i(u) \geq 0, \quad \text{for each } i \in \{1,2, ..., k\} \tag{9}$$

$$\sum_{j=1}^{m} \mu_j h_j(u) = 0 \tag{10}$$

$$u^T w_i = s(u \mid D_i),\ w_i \in D_i, \quad \text{for each } i \in \{1,2, ..., k\} \tag{11}$$

$$(y, \mu, w) \geq 0,\ y \neq 0. \tag{12}$$

Theorem 3.1 (Weak Duality) Let $x^* \in X_0$ be a feasible solution for (MFP) and let (u, y, μ, v, w) be a feasible solution for (D). Moreover, we assume that any one of the following conditions holds:

(a) $S(\cdot) = \sum_{i=1}^{k} y_i [f_i(\cdot) + (\cdot)^T w_i - v_i g_i(\cdot)] + \sum_{j=1}^{m} \mu_j h_j(\cdot)$ is strictly (H_p,*r*)-invex at *u*.

(b) $P(\cdot) = \sum_{i=1}^{k} y_i [f_i(\cdot) + (\cdot)^T w_i - v_i g_i(\cdot)]$ and $Q(\cdot) = \sum_{j=1}^{m} \mu_j h_j(\cdot)$ are strictly (H_p,*r*)-invex at *u*.

Then $F(x^*) \not\leq v$.

Proof. Suppose the condition (a) holds. If $x^* \neq u$ then strict (H_p,*r*) invexity of $S(\cdot)$ at *u* implies that

$$\frac{1}{r} e^{rS(x^*)} > \frac{1}{r} e^{rS(u)} \left[1 + r \frac{\nabla S(u)^T}{e^u} H_p'(x^*, u; 0^+) \right].$$

Using the fundamental property of exponential functions, the above inequality together with (8), gives

$$S(x^*) > S(u). \tag{13}$$

Now suppose contrary to the result that $F(x^*) \leq v.$ Then

$$\frac{f_i(x^*)+s(x^*\,|\,D_i)}{g_i(x^*)} \le v_i \quad \text{for each } i \in \{1,2,\ldots,k\}$$

and
$$\frac{f_t(x^*)+s(x^*\,|\,D_t)}{g_t(x^*)} < v_t \quad \text{for some } t \in \{1,2,\ldots,k\}.$$

Using the fact that $s(x^*\,|\,D_i) \ge x^{*T}w_i$, for each $i \in \{1,2, \ldots, k\}$, we get

$f_i(x^*)+x^{*T}w_i \le v_i g_i(x^*)$ for each $i \in \{1,2, \ldots, k\}$ and $f_t(x^*)+x^{*T}w_t < v_t g_t(x^*),$ for some $t \in \{1,2, \ldots, k\}$. Now using (9), we get

$$f_i(x^*)+x^{*T}w_i - v_i g_i(x^*) \le 0 \le f_i(u)+u^T w_i - v_i g_i(u), \quad \text{for each } i \in \{1,2,\ldots,k\}$$

and
$$f_t(x^*)+x^{*T}w_t - v_t g_t(x^*) < 0 \le f_t(u)+u^T w_t - v_t g_t(u) \quad \text{for some } t \in \{1, 2, \ldots,k\}.$$

The above inequalities along with (12) give

$$\sum_{i=1}^{k} y_i[f_i(x^*)+x^{*T}w_i - v_i g_i(x^*)] \le \sum_{i=1}^{k} y_i[f_i(u)+u^T w_i - v_i g_i(u)]. \tag{14}$$

By the feasibility of x^* and from (10) and (12) we obtain

$$h_j(x^*) \le 0 \qquad \forall\, j=1,2,\ldots,m$$

$$\Rightarrow \quad \mu_j h_j(x^*) \le 0 \quad \forall\, j=1,2,\ldots,m$$

$$\Rightarrow \quad \sum_{j=1}^{m} \mu_j h_j(x^*) \le 0 = \sum_{j=1}^{m} \mu_j h_j(u). \tag{15}$$

On adding (14) and (15), we get

$$\sum_{i=1}^{k} y_i[f_i(x^*)+x^{*T}w_i - v_i g_i(x^*)] + \sum_{j=1}^{m} \mu_j h_j(x^*)$$

$$\le \sum_{i=1}^{k} y_i[f_i(u)+u^T w_i - v_i g_i(u)] + \sum_{j=1}^{m} \mu_j h_j(u)$$

$$\Rightarrow \quad S(x^*) \le S(u)$$

which contradicts (13). Hence $F(x^*) \not\le v$.

If $x^* = u$ then from (9) we have

$$f_i(x^*)+x^{*T}w_i - v_i g_i(x^*) \ge 0 \quad \forall\, i=1,2,\ldots,k$$

$$\Rightarrow \quad f_i(x^*)+x^{*T}w_i \ge v_i g_i(x^*) \qquad \forall\, i=1,2,\ldots,k$$

$$\Rightarrow \quad \frac{f_i(x^*)+x^{*T}w_i}{g_i(x^*)} \ge v_i \qquad \forall\, i=1,2,\ldots,k$$

$$\Rightarrow \quad F(x^*) \ge v$$

that is, $F(x^*) \not\le v$.

Suppose that condition (b) holds. If $x^* \ne u$ then from the strict (H_p, r) invexity of $Q(\cdot)$ at u, we have

$$\frac{1}{r}e^{rQ(x^*)} > \frac{1}{r}e^{rQ(u)}\left[1+\frac{r\nabla Q(u)^T}{e^u}H_p'(x^*,u;0^+)\right]$$

$$\Rightarrow \quad \frac{\nabla Q(u)^T}{e^u}H_p'(x^*,u;0^+) < \frac{1}{r}\left[e^{rQ(x^*)} - e^{rQ(u)}\right]$$

$$= \frac{1}{r}\left[e^{rQ(x^*)} - 1\right]$$

$$\Rightarrow \quad \frac{\nabla Q(u)^T}{e^u} H_p'(x^*, u; 0^+) < 0 . \tag{16}$$

Now from the strict (H_p, r) invexity of $\mathrm{P}(\cdot)$ at u, we have

$$\frac{1}{r} e^{r\mathrm{P}(x^*)} > \frac{1}{r} e^{r\mathrm{P}(u)} \left[1 + \frac{r\nabla \mathrm{P}(u)^T}{e^u} H_p'(x^*, u; 0^+)\right]$$

$$\Rightarrow \quad \frac{\nabla \mathrm{P}(u)^T}{e^u} H_p'(x^*, u; 0^+) < \frac{1}{r}\left[e^{r\mathrm{P}(x^*)} - e^{r\mathrm{P}(u)}\right]. \tag{17}$$

Now, using (8) and (16), we get

$$0 = \frac{[\nabla \mathrm{P}(u) + \nabla Q(u)]^T}{e^u} H_p'(x^*, u; 0^+)$$

$$< \frac{\nabla \mathrm{P}(u)^T}{e^u} H_p'(x^*, u; 0^+)$$

$$\Rightarrow \quad \frac{\nabla \mathrm{P}(u)^T}{e^u} H_p'(x^*, u, 0^+) > 0. \tag{18}$$

Using (17) and (18), we obtain

$$\frac{1}{r}[e^{r\mathrm{P}(x^*)} - e^{r\mathrm{P}(u)}] > 0.$$

Then the fundamental property of exponential functions implies that

$$\mathrm{P}(x^*) > \mathrm{P}(u).$$

That is,

$$\sum_{i=1}^{k} y_i[f_i(x^*) + x^{*T} w_i - v_i g_i(x^*)] > \sum_{i=1}^{k} y_i[f_i(u) + u^T w_i - v_i g_i(u)] \tag{19}$$

Again if, $F(x^*) \le v$ then we get (14) in the same way. But (14) contradicts (19). Therefore, $F(x^*) \not\le v.$ If $x^*=u$ then on the same lines we can prove that $F(x^*) \not\le v.$

Theorem 3.2 (Strong Duality) Let x^* be a weakly efficient solution for (MFP) at which a suitable constraint qualification holds. Then there exist $y^* \in \mathbb{R}^k$, $\mu^* \in \mathbb{R}^m$, $v^* \in \mathbb{R}^k$ and $w_i^* \in \mathbb{R}^n$, for each i = 1,2,...,k such that $(x^*, y^*, \mu^*, v^*, w^*)$ is feasible for (D). Also, if the weak duality Theorem 3.1 holds for all feasible solutions of the problems (MFP) and (D), then $(x^*, y^*, \mu^*, v^*, w^*)$ is a weakly efficient solution for (D) and the two objectives are equal at these points.

Proof. Since x^* is a weakly efficient solution of (MFP) therefore by Theorem 2.1, there exist $y^* \in \mathbb{R}^k$, $\mu^* \in \mathbb{R}^m, v^* \in \mathbb{R}^k$ and $w_i^* \in \mathbb{R}^n$, for each i = 1, 2, ..., k such that $(x^*, y^*, \mu^*, v^*, w^*)$ satisfies (2)–(6). This, in turn, implies that $(x^*, y^*, \mu^*, v^*, w^*)$ is a feasible solution for (D). From the weak duality theorem, for any feasible solution (x, y, μ, v, w) to (D), we have $F(x^*) \not\le v,$ that is, there does not exist any feasible solution (x, y, μ, v, w) to (D) for which $v^* < v$. Hence we conclude that $(x^*, y^*, \mu^*, v^*, w^*)$ is a weakly efficient solution to (D) and the objective functions of (MFP) and (D) are equal at these points in view of (3). This completes the proof.

Theorem 3.3 (Strict Converse Duality) Assume that x^* and $(u^*, y^*, \mu^*, v^*, w^*)$ be weakly efficient solutions for (MFP) and (D), respectively with $v_i^* = \dfrac{f_i(x^*) + x^{*T} w_i^*}{g_i(x^*)}$ for all *i* = 1,2, ...,*k*. Assume that

$$A(\cdot) = \sum_{i=1}^{k} y_i^* [f_i(\cdot) + (\cdot)^T w_i^* - v_i^* g_i(\cdot)] + \sum_{j=1}^{m} \mu_j^* h_j(\cdot)$$

is strictly (H_p,*r*)-invex at u^*. Then $x^* = u^*$, that is, u^* is weakly efficient solution for (MFP).

Proof. Suppose on the contrary that $x^* \neq u^*$. From the strict (H_p,*r*)-invexity of $A(\cdot)$ at u^*, we have

$$\frac{1}{r} e^{rA(x^*)} > \frac{1}{r} e^{rA(u)} \left[1 + r \frac{\nabla A(u)^T}{e^u} H_p'(x^*, u; 0^+) \right].$$

Using the fundamental property of exponential functions, the above inequality together with (8), implies that

$$A(x^*) > A(u^*). \tag{20}$$

From (9), (10), and (12), we get

$$A(u^*) = \sum_{i=1}^{k} y_i^* [f_i(u^*) + u^{*T} w_i^* - v_i^* g_i(u^*)] + \sum_{j=1}^{m} \mu_j^* h_j(u^*) \geq 0. \tag{21}$$

Since

$$v_i^* = \frac{f_i(x^*) + x^{*T} w_i^*}{g_i(x^*)}, \quad \text{for all } i = 1, 2, ..., k$$

that is, $f_i(x^*) + x^{*T} w_i^* - v_i^* g_i(x^*) = 0,$ for all *i* = 1, 2, ..., *k*. (22)

By the feasibility of x^* and (12), we have

$$\sum_{j=1}^{m} \mu_j^* h_j(x^*) \leq 0. \tag{23}$$

Therefore, from (12), (22) and (23), we have

$$A(x^*) = \sum_{i=1}^{k} y_i^* [f_i(x^*) + x^{*T} w_i^* - v_i^* g_i(x^*)] + \sum_{j=1}^{m} \mu_j^* h_j(x^*) \leq 0. \tag{24}$$

Hence from (20) and (24), we obtain

$$A(u^*) < 0,$$

which contradicts (21). Hence $x^* = u^*$.

Now we establish sufficient optimality conditions for $x^* \in X_0$ to be a weakly efficient solution of (MFP) under (H_p,*r*)-invexity.

Theorem 3.4 Let x^* be a feasible solution of (MFP) and that there exist $y^* \in \mathbb{R}^k$, $\mu^* \in \mathbb{R}^m$, $v^* \in \mathbb{R}^k$ *and* $w_i^* \in \mathbb{R}^n$ for each *i* = 1, 2, ...,*k* such that (2) – (6) are satisfied . If $S(\cdot) = \sum_{i=1}^{k} y_i^* [f_i(\cdot) + (\cdot)^T w_i^* - v_i^* g_i(\cdot)] + \sum_{j=1}^{n} \mu_j^* h_j(\cdot)$ is (H_p, *r*)-invex at x^* then x^* will be a weakly efficient solution of (MFP).

Proof: Suppose x^* is not a weakly efficient solution of (MFP). Then there exists some $x \in X_0$ such that

$$F(x) < F(x^*),$$

that is, for all *i* = 1, 2,..., *k*

$$\frac{f_i(x) + s(x \mid D_i)}{g_i(x)} < \frac{f_i(x^*) + s(x^* \mid D_i)}{g_i(x^*)}.$$

Now using the facts that $s(x \mid D_i) \geq x^T w_i^*$, for all *i* = 1,2,.., *k*,

$$\frac{f_i(x) + x^T w_i^*}{g_i(x)} \leq \frac{f_i(x) + s(x \mid D_i)}{g_i(x)} < v_i^*$$

$$\Rightarrow \qquad f_i(x) + x^T w_i^* - v_i^* g_i(x) < 0.$$

Using the feasibility of *x*, and from (6) we get

$$\sum_{i=1}^{k} y_i^* [f_i(x) + x^T w_i^* - v_i^* g_i(x)] + \sum_{j=1}^{m} \mu_j^* h_j(x) < 0$$

$$\Rightarrow \qquad S(x) < 0 \qquad (25)$$

From the (H_p, *r*)-invexity of $S(\cdot)$ at x^*, we have

$$\frac{1}{r} e^{rS(x)} \geq \frac{1}{r} e^{rS(x^*)} \left[1 + \frac{r \nabla S(x^*)^T}{e^{x^*}} H_p'(x, x^*, 0+) \right].$$

Using the fundamental property of exponential functions, the above inequality together with (2), implies that

$$S(x) \geq S(x^*) = 0,$$

which contradicts (25). This completes the proof.

5. Conclusion

In this paper, we have used the concept of ($H_{p,}$,*r*)-invex functions related to nondifferentiable multiobjective fractional programming problems in which each component of the objective function contains a term involving the support function of a compact convex set and established sufficient optimality conditions and duality results for weakly efficient solutions of nondifferentiable fractional programming problems under the ($H_{p,}$,*r*)-invexity assumptions.

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