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Some Common Fixed Point Theorems for Pair of Non Commuting Expansive Type Mappings in Hilbert Space

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ABSTRACT

The objective of this paper is to obtain some common unique fixed-point theorems for pair of non commuting expansive type mappings using rational inequality defined on a non-empty closed subset of a Hilbert space.

Mathematics Subject classification: 47H10, 54H25

Keywords: Hilbert space, common fixed point, rational inequality and expansive type mappings.

INTRODUCTION

The Study of properties and applications of fixed points of various types of contractive mapping in Hilbert and Banach spaces were obtained among others by Browder[1]. Browder and Petryshyn [2,3], Hicks and Huffman [4], Huffman [5], Koparde and Waghmode [6]. In this paper we present some common fixed-point theorems for rational inequalities involving expansive type mappings. For the purpose of obtaining the fixed point of the two expansive type mappings. We have constructed a sequence and have shown its convergence to the fixed point. The main results of this paper extend and generalize the theorem 3 of [8], [9] and convert the theorem 1 of [7] for expansive maps.

MAIN RESULTS.

Theorem 2.1- Let S, T be non commuting, surjective self maps of a closed subset C of Hilbert Space H satisfying

$$\|STx-TSy\|^2 \geq \frac{a[\|x-STx\|^2\|x-y\|^2+\|y-TSy\|^2\|x-y\|^2]+b\|x-STx\|^2\|y-TSy\|^2+c\|x-y\|^4}{\|x-STx\|^2+\|y-TSy\|^2+\|x-y\|^2} \text{.....}(A)$$

for each $x, y \in C, x \neq y$, $\;Where\ a,b,c \geq 0\ \ 2a+b+c>3,\ and\ \ c>1$. Then *ST* and *TS* have a common unique fixed point in C.

Proof-- We define a sequence $\{x_n\}$ as follows for n = 0,1,2,3----

$$x_{2n}=STx_{2n+1}, \qquad x_{2n+1}=TSx_{2n+2} ---------(2.1)$$

If $x_{2n}=x_{2n+1}=x_{2n+2}$ for some *n* then we see that x_{2n} is a fixed point of *ST and TS*, therefore we suppose that no two consecutive terms of sequence $\{x_n\}$ are equal.

Now,consider

$$\|x_{2n}-x_{2n+1}\|^2=\|STx_{2n+1}-TSx_{2n+2}\|^2$$

$$\geq \frac{a[\|x_{2n+1}-STx_{2n+1}\|^2\|x_{2n+1}-x_{2n+2}\|^2+\|x_{2n+2}-TSx_{2n+2}\|^2\|x_{2n+1}-x_{2n+2}\|^2]}{\|x_{2n+1}-STx_{2n+1}\|^2+\|x_{2n+2}-TSx_{2n+2}\|^2+\|x_{2n+1}-x_{2n+2}\|^2}$$

$$+\ \frac{b\|x_{2n+1}-STx_{2n+1}\|^2\|x_{2n+2}-TSx_{2n+2}\|^2+c\|x_{2n+1}-x_{2n+2}\|^4}{\|x_{2n+1}-STx_{2n+1}\|^2+\|x_{2n+2}-TSx_{2n+2}\|^2+\|x_{2n+1}-x_{2n+2}\|^2}$$

$$=\frac{a[\|x_{2n+1}-x_{2n}\|^2\|x_{2n+1}-x_{2n+2}\|^2+\|x_{2n+2}-x_{2n+1}\|^2\|x_{2n+1}-x_{2n+2}\|^2]}{\|x_{2n+1}-x_{2n}\|^2+\|x_{2n+2}-x_{2n+1}\|^2+\|x_{2n+1}-x_{2n+2}\|^2}$$

$$+\ \frac{b\|x_{2n+1}-x_{2n}\|^2\|x_{2n+2}-x_{2n+1}\|^2+c\|x_{2n+1}-x_{2n+2}\|^4}{\|x_{2n+1}-x_{2n}\|^2+\|x_{2n+2}-x_{2n+1}\|^2+\|x_{2n+1}-x_{2n+2}\|^2}$$

$$\Rightarrow \|x_{2n}-x_{2n+1}\|^2\left[\|x_{2n+1}-x_{2n}\|^2+2\|x_{2n+1}-x_{2n+2}\|^2\right]$$

$$\geq a[\|x_{2n+1}-x_{2n}\|^2\|x_{2n+1}-x_{2n+2}\|^2+\|x_{2n+1}-x_{2n+2}\|^2\|x_{2n+1}-x_{2n+2}\|^2]$$

$$+b\|x_{2n+1}-x_{2n}\|^2\|x_{2n+1}-x_{2n+2}\|^2+c\|x_{2n+1}-x_{2n+2}\|^4$$

$$\Rightarrow \|x_{2n}-x_{2n+1}\|^4+2\|x_{2n}-x_{2n+1}\|^2\|x_{2n+1}-x_{2n+2}\|^2$$

$$\geq \|x_{2n+1}-x_{2n+2}\|^2(2a+b+c)\min\{\|x_{2n}-x_{2n+1}\|^2,\|x_{2n+1}-x_{2n+2}\|^2\}$$

$$\Rightarrow \|x_{2n}-x_{2n+1}\|^4 \geq \|x_{2n+1}-x_{2n+2}\|^2(2a+b+c-2)\min\{\|x_{2n}-x_{2n+1}\|^2,\|x_{2n+1}-x_{2n+2}\|^2\}$$

Case I

$$\Rightarrow \|x_{2n}-x_{2n+1}\|^4 \geq (2a+b+c-2)\|x_{2n+1}-x_{2n+2}\|^4$$

$$\Rightarrow \|x_{2n+1}-x_{2n+2}\| \leq \left[\frac{1}{(2a+b+c-2)}\right]^{1/4}\|x_{2n}-x_{2n+1}\|$$

$$\Rightarrow \|x_{2n+1}-x_{2n+2}\| \leq k_1\|x_{2n}-x_{2n+1}\|[where\ k_1=\left[\frac{1}{(2a+b+c-2)}\right]^{1/4} < 1\ (as\ 2a+b+c>3)]$$

Similarly

$$\|x_{2n+2}-x_{2n+3}\| \leq k_1\|x_{2n+1}-x_{2n+2}\|[where\ k_1=\left[\frac{1}{(2a+b+c-2)}\right]^{1/4} < 1\ (as\ 2a+b+c>3)]$$

Case II

$$\Rightarrow \|x_{2n}-x_{2n+1}\|^4 \geq (2a+b+c-2)\|x_{2n+1}-x_{2n+2}\|^2\|x_{2n}-x_{2n+1}\|^2$$

$$\Rightarrow \|x_{2n+1}-x_{2n+2}\| \leq \left[\frac{1}{(2a+b+c-2)}\right]^{\frac{1}{2}}\|x_{2n}-x_{2n+1}\|$$

$$\Rightarrow \|x_{2n+1}-x_{2n+2}\| \leq k_2\|x_{2n}-x_{2n+1}\|[where\ k_2=\left[\frac{1}{(2a+b+c-2)}\right]^{\frac{1}{2}} < 1\ (as\ 2a+b+c>3)]$$

Similarly

$$\|x_{2n+2}-x_{2n+3}\| \leq k_2\|x_{2n+1}-x_{2n+2}\|[where\ k_2=\left[\frac{1}{(2a+b+c-2)}\right]^{\frac{1}{2}} < 1\ (as\ 2a+b+c>3)]$$

In general

$$\Rightarrow \|x_n-x_{n+1}\| \leq \mathrm{k}\|x_{n-1}-x_n\|[where\ k=\max\{\mathrm{k}_1,\mathrm{k}_2\}\ \text{then}\ k<1]\{\text{for n=1,2,3......}\}$$

$$\Rightarrow \|x_n-x_{n+1}\| \leq \mathrm{k}^n\|x_0-x_1\|$$

Now we shall prove that, $\{x_n\}$ is a Cauchy sequence for the caseI.

For this for every positive integer *p* we have,

$$\begin{aligned}\|x_n-x_{n+p}\| &= \|x_n-x_{n+1}+x_{n+1}-.........+x_{n+p-1}-x_{n+p}\| \\ &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-x_{n+2}\|+..........+\|x_{n+p-1}-x_{n+p}\| \\ &\leq (k^n+k^{n+1}+k^{n+2}+............+k^{n+p-1})\|x_0-x_1\| \\ &= k^n(1+k+k^2+.....+k^{p-1})\|x_0-x_1\| \\ &\leq \frac{k^n}{1-k}\|x_0-x_1\|\end{aligned}$$

as $n\to\infty, \|x_n-x_{n+p}\|\to 0$,it follows that $\{x_n\}$ is a Cauchy sequence in C

As C is a closed subset of H. So there exist a point *x* in C such that

$\{x_n\}\to x$ as $n\to\infty$,-----------(2.2)

Existence of fixed point: Since *S* and *T* are surjective maps so *ST* and *TS are also* surjective and hence there exist two points y and y' in *C* such that $x = STy\ and\ \ x = TSy'$ *.......................(2.3) .*

Consider,

$$\|x_{2n}-x\|^2 = \|STx_{2n+1}-TSy'\|^2$$

$$\geq \frac{a[\|x_{2n+1}-STx_{2n+1}\|^2+\|y'-TSy'\|^2]\|x_{2n+1}-y'\|^2}{\|x_{2n+1}-STx_{2n+1}\|^2+\|y'-TSy'\|^2+\|x_{2n+1}-y'\|^2}$$

$$+\frac{b\|x_{2n+1}-STx_{2n+1}\|^2\|y'-TSy'\|^2+c\|x_{2n+1}-y'\|^4}{\|x_{2n+1}-STx_{2n+1}\|^2+\|y'-TSy'\|^2+\|x_{2n+1}-y'\|^2}$$

$$= \frac{a[\|x_{2n+1}-x_{2n}\|^2+\|y'-TSy'\|^2]\|x_{2n+1}-y'\|^2+b\|x_{2n+1}-x_{2n}\|^2\|y'-TSy'\|^2+c\|x_{2n+1}-y'\|^4}{\|x_{2n+1}-x_{2n}\|^2+\|y'-x\|^2+\|x_{2n+1}-y'\|^2}$$

As $\{x_{2n}\}, \{x_{2n+1}\}$ **are subsequences of** $\{x_n\}$, **as** $n \to \infty, \{x_{2n}\} \to x, \{x_{2n+1}\} \to x$

Therefore

$$\|x-x\|^2 \geq \frac{a[\|x-x\|^2+\|y'-x\|^2]\|x-y'\|^2+b\|x-x\|^2\|y'-x\|^2+c\|x-y'\|^4}{\|x-x\|^2+\|y'-x\|^2+\|x-y'\|^2}$$

$$0 \geq \frac{1}{2}(a+c)\|y'-x\|^2$$

$$\Rightarrow \|x-y'\|^2 = 0 \ (as\ a+c>1)$$

$$\Rightarrow x = y' \text{.....................}(2.4)$$

In an exactly similar way we can prove that, $x = y$.....................(2.5)

The fact (2.3) along with (2.4 & 2.5) shows that x is a common fixed point of *ST & TS*.

Uniqueness:

Let *z* be another common fixed point of *ST & TS,* that is

$$STz = z \ and \ \ TSz = z$$

$$\|x-z\|^2 = \|STx - TSz\|^2$$

$$\geq \frac{a\|x-STx\|^2\|x-z\|^2+\|z-TSz\|^2[a\|x-z\|^2+b\|x-STx\|^2]+c\|x-z\|^4}{\|x-STx\|^2+\|z-TSz\|^2+\|x-z\|^2}$$

$$\Rightarrow \|x-z\|^2 \geq c\|x-z\|^2$$

$$\Rightarrow (1-c)\|x-z\|^2 \geq 0$$

$$\Rightarrow \|x-z\|^2 = 0 \ (as \ \ \mathrm{c} > 1)$$

$$\Rightarrow x = z$$

This completes the proof of the theorem 2.1.

Theorem 2.2- Let S, T be non commuting surjective self maps of a closed subset C of Hilbert Space H satisfying

$$\|STx-TSy\|^2 \geq \frac{[a\|x-STx\|^2\|x-y\|^2+b\|y-TSy\|^2\|x-y\|^2+c\|x-STx\|^2\|y-TSy\|^2]}{\|x-STx\|^2+\|y-TSy\|^2} \text{..........}(B)$$

for each *x,y* $\in$ *C and* $\|x-STx\|^2+\|y-TSy\|^2 \neq 0$

$Where\ a,c \geq 0, b > 0\ and\ a+b+c > 2$ Then *S*T and *TS have* a common fixed point in C.

Proof: Proof of this theorem is similar to the proof of theorem 2.1.

Theorem 2.3 Let *S, T* be non commuting, surjective self maps of a closed subset *C* of Hilbert Space *H* satisfying

$$\|STx-TSy\|^2 \geq \frac{a\|x-STx\|^2\left[1+\|y-TSy\|^2\right]}{1+\|x-y\|^2}+b\left[\|x-STx\|^2+\|y-TSy\|^2\right]+c\|x-y\|^2 \ldots.(C)$$

for each $x,y \in C, x \neq y$, Where $a,b \geq 0,\ \ 0<c<1$ *satisfy* $a+2b+c>1$

Then *ST* and *TS* have a common unique fixed point in C.

.

Proof-- We define a sequence $\{x_n\}$ as follows for n = 0,1,2,3----

$$x_{2n}=STx_{2n+1}, \qquad x_{2n+1}=TSx_{2n+2}$$

If $x_{2n}=x_{2n+1}=x_{2n+2}$ for some *n* then we see that x_{2n} is a fixed point of *ST and TS*, therefore we suppose that no two consecutive terms of sequence $\{x_n\}$ are equal.

Now consider

$$\|x_{2n}-x_{2n+1}\|^2=\|STx_{2n+1}-TSx_{2n+2}\|^2$$

$$\geq \frac{a\|x_{2n+1}-STx_{2n+1}\|^2\left[1+\|x_{2n+2}-TSx_{2n+2}\|^2\right]}{1+\|x_{2n+1}-x_{2n+2}\|^2}+b\left[\|x_{2n+1}-STx_{2n+1}\|^2+\|x_{2n+2}-TSx_{2n+2}\|^2\right]$$

$$+c\|x_{2n+1}-x_{2n+2}\|^2$$

$$=\frac{a\|x_{2n+1}-x_{2n}\|^2\left[1+\|x_{2n+2}-x_{2n+1}\|^2\right]}{1+\|x_{2n+1}-x_{2n+2}\|^2}+b\left[\|x_{2n+1}-x_{2n}\|^2+\|x_{2n+2}-x_{2n+1}\|^2\right]$$

$$+c\|x_{2n+1}-x_{2n+2}\|^2$$

$$=(a+b)\|x_{2n}-x_{2n+1}\|^2+(b+c)\|x_{2n+1}-x_{2n+2}\|^2$$

$$\Rightarrow [1-(a+b)]\|x_{2n}-x_{2n+1}\|^2 \geq (b+c)\|x_{2n+1}-x_{2n+2}\|^2$$

$$\Rightarrow \|x_{2n+1}-x_{2n+2}\|^2 \leq \frac{[1-(a+b)]}{(b+c)}\|x_{2n}-x_{2n+1}\|^2$$

$$\Rightarrow \|x_{2n+1}-x_{2n+2}\| \leq \mathrm{k}\|x_{2n}-x_{2n+1}\| [where\ k=\left[\frac{[1-(a+b)]}{(b+c)}\right]^{1/2} < 1]$$

Similariy

$$\|x_{2n+2}-x_{2n+3}\| \leq \mathrm{k}\|x_{2n+1}-x_{2n+2}\| [where\ k=\left[\frac{[1-(a+b)]}{(b+c)}\right]^{1/2} < 1]$$

In general

$$\Rightarrow \|x_n-x_{n+1}\| \leq \mathrm{k}\|x_{n-1}-x_n\|$$

$$\Rightarrow \|x_n-x_{n+1}\| \leq \mathrm{k}^n\|x_0-x_1\| \ldots\ldots\ldots\ldots\ldots\ldots(2.6)$$

We can prove that, $\{x_n\}$ is a Cauchy sequence (using(2.6)) (as proved in theorem 2.1)As C is a closed subset of H. So there exist a point *x* in C such that

$\{x_n\} \to x$ as $n \to \infty$,-----------(2.7)

Existence of fixed point: Since *S* and *T* are surjective maps so *ST* and *TS are also* surjective and hence there exist two points y and y' in *C* such that $x = STy \; and \;\; x = TSy'$ ....*(2.8)*

$$\|x_{2n} - x\|^2 = \|STx_{2n+1} - TSy'\|^2$$

$$\geq \frac{a\|x_{2n+1} - STx_{2n+1}\|^2\left[1 + \|y' - TSy'\|^2\right]}{1 + \|x_{2n+1} - y'\|^2} + b\left[\|x_{2n+1} - STx_{2n+1}\|^2 + \|y' - TSy'\|^2\right] + c\|x_{2n+1} - y'\|^2$$

$$= \frac{a\|x_{2n+1} - x_{2n}\|^2\left[1 + \|y' - TSy'\|^2\right]}{1 + \|x_{2n+1} - y'\|^2} + b\left[\|x_{2n+1} - x_{2n}\|^2 + \|y' - TSy'\|^2\right] + c\|x_{2n+1} - y'\|^2$$

As $\{x_{2n}\}, \{x_{2n+1}\}$ are subsequences of $\{x_n\}$,as $n \to \infty, \{x_{2n}\} \to x, \{x_{2n+1}\} \to x$

Therefore

$$\|x - x\|^2 \geq \frac{a\|x - x\|^2\left[1 + \|y' - x\|^2\right]}{1 + \|x - y'\|^2} + b\left[\|x - x\|^2 + \|y' - x\|^2\right] + c\|x - y'\|^2 \; (as \; x = Ty')$$

$$0 \geq (b + c)\|x - y'\|^2$$

$$\Rightarrow \|x - y'\|^2 = 0 \;\; [as \;\; (b + c) > 0]$$

$$\Rightarrow x = y' -----------------(2.9)$$

In an exactly similar way we can prove that,

$$x = y ----------------------(2.10)$$

The fact (2.8) along with (2.9 & 2.10) shows that *x* is a common fixed point of *ST &TS*.

Uniqueness-

Let *z* be another common fixed point of *S & T,* that is

$$STz = z \; and \;\; TSz = z$$

$$\|x - z\|^2 = \|STx - TSz\|^2$$

$$\geq \frac{a\|x - STx\|^2\left[1 + \|z - TSz\|^2\right]}{1 + \|x - z\|^2} + b\left[\|x - STx\|^2 + \|z - TSz\|^2\right] + c\|x - z\|^2$$

$$\Rightarrow (1 - c)\|x - z\|^2 \geq 0$$

$$\Rightarrow \|x - z\|^2 = 0 \;\; (as \;\; 0 < \; c < 1)$$

$$\Rightarrow x = z$$

This completes the proof of the theorem 2.3

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