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PROJECTIVES IN CATEGORIES OF HAUSDORFF U-SPACES

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ABSTRACT

This is the fourth in a series of papers on U-spaces. Here projectiveness in some categories of Hausdorff U-spaces has been introduced and many topological theorems related to projective Hausdorffness have been generalized to U- spaces, as an extension of study of supratopological spaces.

Key Words: Category, projective, Hausdorff, extremally disconnected, , countable basis, compactification, locally compact,.

AMS Mathematics Subject Classification (2010): 54A40, 54D10, 54G12, 54D15

INTRODUCTION

In a previous paper [1] we have introduced U- spaces. There and in [7] and [8] we studied some of their properties. In this paper we use the terminology of [1]. Some study of these spaces was done previously in ([2],[3],[10],[14]) in less general form, and the spaces were called supratopological spaces.

In this paper we have generalized to U- spaces the concepts of projective topological spaces, Stone Čceh compactification, perfect maps, and extremally disconnected spaces. We have also generalized to U- spaces some results on topological spaces occurring in [4] ,[5],[6],[9], [12] and [13]. A few important properties of such U-spaces have been studied. A number of interesting examples have been constructed to prove non-trivialness of such results. For most of the cases of the above generalizations the proof for U- spaces runs parallel to those for topological spaces.

We have constructed 2 examples of proper projective U-spaces which are locally compact but not compact and two examples of proper projective compact U- spaces.

In this paper a U- space will mean a Hausdorff U-space, unless otherwise mentioned.

2. **Projectives in some Categories of Hausdorff U-spaces**

We recall some definitions from [10], and generalize to U- space a few definitions in [5],[6] and [13].

Definition 2.1 A category consists of

(i) A class C of objects A, B, C,.....

(ii) For each pair of objects A, B a set hom(A, B) whose elements are called morphism, with the property that

(a) $\alpha \in$ hom(A, B) and $\beta \in$ hom(B, C) implies there exists $\gamma \in$ hom(A, C) which is written $\gamma = \beta\, \alpha$;

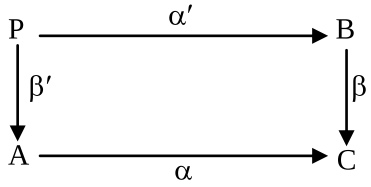
(b) For each A $\in$ C ,there exists $1_A \in$ hom(A, A) such that for each B $\in$ C and for each $\alpha \in$ hom(A, B), $\alpha = \alpha\, 1_A$, and $\alpha = 1_B\, \alpha$,

(c) Let $\alpha \in$ hom(A, B), $\beta \in$ hom(B, C), $\gamma \in$ hom(C, D) then $\gamma(\beta\, \alpha) = (\gamma\, \beta)\alpha$.

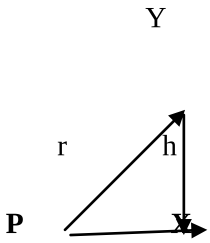
If $\beta\, \alpha = \gamma\, \alpha \Rightarrow \beta = \gamma$ then α is an epimorphism or, epic ;

If $\alpha\, \beta = \alpha\, \gamma \Rightarrow \beta = \gamma$, then α is a monomorphism or, monic.

Definition 2.2 Let A, B, C be three objects and α : A $\rightarrow$ C and β : B $\rightarrow$ C be monics in a category. There are different ways of completing them to a commutative square. A final object in this square is called **Pullback** of α and β .



Definition 2.3 [6](p- 3) A U- space P is **projective**, if for any pair of U– spaces, X, Y and any pair of U- continuous maps h: Y $\rightarrow$ X and f: P $\rightarrow$ X, with h onto, there exists a U- continuous map r: P $\rightarrow$ Y such that hr(p) = f(p) for every p $\in$ P.



Definition 2.4 [13](p-7) A U-continuous function f: X $\rightarrow$ Y where X and Y are arbitrary Hausdorff U-spaces is called U-**perfect** if f is U- closed and the set f^{-1}(y) is compact for each y in Y.

Definition 2.5 [6](p- 482) A U- space X is **extremally disconnected** if the closure of every U- open set is U-open.

Definition 2.6 A U-space Y is an **extension U - space** of another space X if X is U-dense in Y.

The generalization of the construction of the Stone- Čech compactification for a U- completely regular space

Let X be a completely regular T_1 -U-space. Let $\{f_\alpha\}_{\alpha\in A}$ be the collection of all bounded U-continuous real-valued function on X, indexed by some index set A.

For each $\alpha \in$ A, choose $I_\alpha = (-\infty$, sup f_α (X)] regarded as U- subspaces of the usual U- space R. Then define h:X $\rightarrow \prod_{\alpha\in A} I_\alpha$ by the rule h(x)= $(f_\alpha(x))_{\alpha\in A}$. Since X is completely regular T_1 -U-space, for two distinct points x_1, x_2, $\{x_2\}$ is U- closed and $x_1 \notin \{x_2\}$ so there exists f_α such that $f_\alpha(x_1) \neq f_\alpha(x_2)$.Hence $h(x_1) \neq h(x_2)$. Therefore h is one-one. Since f_α : X $\rightarrow I_\alpha$ is U- continuous, it follows from the definition of $\prod_{\alpha\in A} I_\alpha$ that h is U-continuous.

We shall show that h is U-open. Let V_1 be a U-open set of X and $y_0 \in h(V_1)$. Let $x_0 \in V_1$ such that $h(x_0) = y_0$. Since X is completely regular, there exists f_α such that $f_\alpha(x_0) \in (-\infty, \sup f_\alpha(X))$ and $f_\alpha(X - V_1) = \sup f_\alpha(X)$. Let $V_2 = \pi_\alpha^{-1}(-\infty, \sup f_\alpha(X))$. Then V_2 is U- open in $\prod_{\alpha \in A} I_\alpha$, and $W = V_2 \cap h(X)$ is a U- open set of h(X).

We shall show that $y_0 \in W \subset h(V_1)$. Since $y_0 \in h(V_1) \subseteq h(X)$ and $\pi_\alpha h(x_0) = f_\alpha(x_0)$, $y_0 = h(x_0) = \pi_\alpha^{-1} f_\alpha(x_0) \subseteq V_2$. Therefore $y_0 \in W$.

Let $y \in W$. Then for some $x \in X$, $y = h(x)$ and $\pi_\alpha(y) \in (-\infty, \sup f_\alpha(X))$. Since $\pi_\alpha(y) = \pi_\alpha h(x) = f_\alpha(x)$ and $f_\alpha(X - V_1) = \sup f_\alpha(X)$, so $x \in V_1$, i.e., $y = h(x) \in h(V_1)$.

Therefore $h: X \to \prod_{\alpha \in A} I_\alpha$ is an U-imbedding. Hence ($\overline{h(X)}$, h) is a compactification of X. $\overline{h(X)}$ will be written β(X) and will be called the U-**generalized form of Stone- Čech compactification** of X.

Definition 2.7 [13](p- 8) Let **P** be the category of all paracompact U-spaces and U- perfect maps and **T** be the category of all Tychonoff U-spaces and perfect U-maps. It is to be noted that both of these categories contain **C**, the category of compact U-spaces and U- continuous maps, as a full subcategory. **P** is also a full subcategory of **T**.

Theorem 2.1 The category **P** has pullbacks.

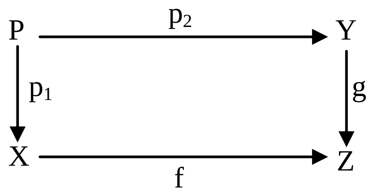
As in ([13].p.8) we have (The proof is similar. The details are given to stress the velidity.)

Proof: Let $f: X \to Z$ and $g: Y \to Z$ be two morphisms in the category **P** (i.e., X, Y, Z be paracompact U-spaces and f, g are perfect U-maps). We have to show the existence of a pullback diagram for f and g.

Let $P = \{(x, y) \in X \times Y: f(x) = g(y)\}$ and p_1 and p_2 be the projection on X and Y respectively. Suppose there exist $p'_1: P' \to X$ and $p'_2: P' \to Y$ such that $fp'_1 = gp'_2$.

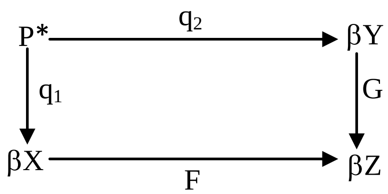
Define $h: P' \to X \times Y$ as follows:

$h(t) = (p'_1(t), p'_2(t))$, $t \in P'$. Since $fp'_1 = gp'_2$, $h(t) \in P$ that is, $h: P' \to P$ such that $p_1 h = p'_1$ and $p_2 h = p'_2$. It is easy to see that the map h is unique. Thus the diagram



is a pullback for f and g. We show that this diagram belongs to **P**, that is, that the maps p_1 and p_2 are U-perfect.

Consider the pullback diagram



for the maps $F: \beta X \to \beta Z$ and $G: \beta Y \to \beta Z$ where F and G are the extensions of the map f and g onto βX and βY respectively (βX, βY and βZ are the generalization of Stone- Čech compactifications and η_X, η_Y and η_Z are reflector maps of X, Y and Z respectively).

We have $F\eta_X = \eta_Z f$, $G\eta_Y = \eta_Z g$ and $P^* = \{(x^*, y^*) \in \beta X \times \beta Y: F(x^*) = G(y^*)\}$. q_1 and q_2 are projections of P^* to βX and βY respectively.

Again, let p_1^*: $\beta P \rightarrow \beta X$, p_2^*: $\beta P \rightarrow \beta Y$ be the extensions of p_1 and p_2 onto βP. Hence $\eta_X p_1 = p_1^* \eta_P$, $\eta_Y p_2 = p_2^* \eta_P$. Since $fp_1 = gp_2$, $\eta_Z fp_1 = \eta_Z gp_2$. Note that $Fp_1^* \eta_P = F\eta_X p_1 = \eta_Z fp_1$ and $Gp_2^* \eta_P = G\eta_Y p_2 = \eta_Z gp_2$.

Therefore, $Fp_1^* \eta_P = Gp_2^* \eta_P$. Since η_P (P) is U- dense in βP

we have $Fp_1^* = G p_2^*$ on βP.

From the definition of pullback there exists a (unique) mapping h: $\beta P \rightarrow P^*$ such that $p_1^* = q_1h$ and $p_2^* = q_2h$.

Again, for the maps $\eta_X p_1$: P $\rightarrow \beta X$ and

$\eta_Y p_2$:P $\rightarrow \beta Y$, we have $F\eta_X p_1 = G\eta_Y p_2$ (this equality is already noted earlier).

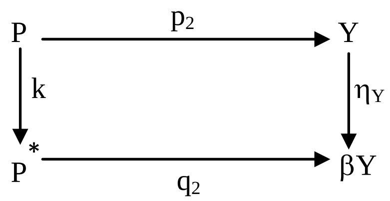
From the definition of pullback once again we get a map k: P $\rightarrow$ P*such that $\eta_X p_1 = q_1k$ and $\eta_Y p_2 = q_2k$. It is easy to see that the map k is as follows:

$k(x, y) = (\eta_X p_1(x, y), \eta_Y p_2(x, y)) = (\eta_X (x), \eta_Y (y)), (x, y) \in P$. k clearly turns out to be a U-homeomorphism into P*. Moreover it is not difficult to notice that $k = h\eta_P$. Now k is a U-homeomorphism of P onto $k(P) \subset P^*$.

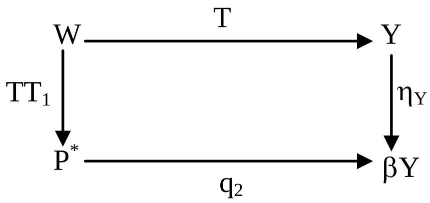
From the property of generalized form of Stone- Čech compactification it follows that

(i) $h(\beta P - \eta_P (P)) \subset \overline{k(P)} - k(P) \subset P^*$.

Now $q_2k = \eta_Y p_2$, that is,



is a commutative diagram. So we consider the pullback diagram for $q_2 : P^* \rightarrow \beta Y$ and $\eta_Y : Y \rightarrow \beta Y$ say



Where W is given by $\{(s, y) \in P^* \times Y: q_2(s) = \eta_Y (y)\}$ and π_1, π_2 are the respective projections to P* and Y. Since $q_2(s) = q_2(x^*, y^*) = y^*$, $q_2(s) = \eta_Y (y)$ implies $y^* = \eta_Y (y)$.

Consequently, $W = \{((x^*, \eta_Y (y)), y) \in P^* \times Y: \eta_Y (y)= y^*\}$

$= \{((x^*, \eta_Y (y)), y) \in (\beta X \times \beta Y) \times Y : F(x^*) = G(\eta_Y (y))\}$.

If $F(x^*) = G(\eta_Y (y))$ then $F(x^*)= G(\eta_Y (y)) = \eta_Z g(y)$. Since f is a U-perfect map, $F(\beta X - \eta_X (x)) \subset \beta Z - \eta_Z (Z)$. As a consequence, $x^* \in \eta_X (x)$, that is, $x^* = \eta_X (x)$ for some $x \in X$. So we have

$W = \{((\eta_X (x), \eta_Y (y)), y) \in (\beta X \times \beta Y) \times Y : F(\eta_X (x)) = G(\eta_Y (y))\}$. Again $\eta_Z g(y) = G(\eta_Y (y)) = F(\eta_X (x)) = \eta_Z f(x)$ and this naturally implies $f(x) = g(y)$.

We then get,

(ii) $W = \{((\eta_X (x), \eta_Y (y)), y) \in (\beta X \times \beta Y) \times Y : f(x) = g (y)\}$

$= \{k(x, y),y): (x, y) \in P \text{ and } p_2(x, y) = y\}$.

Since $\eta_Y p_2 = q_2k$ there exist a unique map j: P $\rightarrow$ W as follows:

$j(x, y) = (k(x, y), p_2(x, y)), (x, y) \in P$.

Easy to see from (ii) that j(P) = W. In fact j is a U-homeomorphism of P and W. Now W is, by construction, a U-closed subset of P*$\times$Y which is paracompact U-space (as P* is a compact U-space and Y is a paracompact U-space). As a result W is paracompact U-space. This makes P paracompact U-space and J a U-isomorphism of P and W in the category **P**. We then obtain that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\quad p_2 \quad} & Y \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ P^* & \xrightarrow[\quad q_2 \quad]{} & \beta Y \end{array}$$

as a pullback diagram. Note that η_Y is a one-one map, that is, η_Y is a U-monomorphism. From the definition of inverse image we see that P = q_2^{-1}(Y) as a sub object of P*.In terms of sets this means that
k(P) = q_2^{-1}(η_Y (Y)). As a result q_2(P*- k(P)) $\subset \beta$ Y - η_Y (Y). We know from (i) that
h(β P - η_P (P)) $\subset \overline{k(P)}$ - k(P) $\subset$P*- k(P), so that p_2*(β P - η_P(P)) = q_2h (β P - η_P (P)) = q_2[h(β P - η_P (P))] $\subset q_2$(P*- k(P)) $\subset \beta$ Y - η_Y (Y).

Hence, by a characterization of Henriksen and Isbell [13], p_2 is a U-perfect map. Similarly, p_1 is a U-perfect map. We generalize the theorems, Lemmas and Corollary of [5] (p- 482- 484)

Theorem 5.2 In any category of U-spaces and maps satisfying conditions
(a) all admissible maps are U-continuous,
(b) if A is an admissible space and {p, q} is a two element space, then A$\times${p, q} and the projection map of this U-space onto A are admissible,
(c) if A is an admissible space and B is a U-closed subspace of A, then B and the inclusion map of B into A are admissible, a projective U-space is extremally disconnected.

Proof: Let X be a projective U-space in such a category. Let G be any U-open subset of X; we must prove $\overline{G}$ is U-open. In X$\times${p, q} consider the U-closed set Y = ((X - G) $\times$ {p})$\cup$ ($\overline{G}$ $\times${q}), and its inclusion map i. Let π be the projection of X$\times${p, q} onto X. Our hypothesis on the category implies that π o i is an admissible map of Y onto X and that the identity ϕ is an admissible map of X into X. Since X is projective U- space, there is an admissible map ψ of X into Y such that ϕ = π o i o ψ . Because π o i is one -to-one on G$\times${q} it is clear that ψ (x) = $\langle x, q \rangle$ for x$\in$G; from the continuity of ψ follows
ψ (x) = $\langle x, q \rangle$ for x$\in \overline{G}$ .Similarly, for x$\notin \overline{G}$, ψ (x) = $\langle x, p \rangle$.
Thus we have proved $\overline{G} = \psi^{-1}(\overline{G}$ $\times${q}). Since ψ is U-continuous and $\overline{G}$ $\times${q} is U-open in Y, $\overline{G}$ is U-open in X as required.

Theorem 2.3 In an extremally disconnected U-space no sequence is convergent unless it is ultimately constant.

Proof: Suppose that the sequence {x_n} converges to p in the extremally disconnected U-space X. Assume this sequence is not ultimately constant, we shall deduce a contradiction.
First we construct inductively a disjoint sequence {U_i} of U-open sets in X such that each U_i contains a member $x_{n(i)}$ of the given sequence, where {n(i)} is an increasing sequence of integers. Let n(1) be an index for which $x_{n(1)} \neq$ p, and choose a U-open set U_1 such that $x_{n(1)} \in U_1$ but p$\notin \overline{U_1}$. Suppose we have chosen disjoint U-open sets U_1,U_2,U_3,……..,U_k and increasing integers n_1,n_2,n_3,……..,n_k such that $x_{n(i)} \in U_i$ and p$\notin \overline{U_i}$ for i = 1,2,3,……..,k. Then V = X - ($\overline{U_1} \cup \overline{U_2} \cup \overline{U_3} \cup$………. $\cup \overline{U_k}$) is an U-open neighborhood of p, so $x_q \in$V for all sufficiently large q. By a suitable choice of $n_{(k+1)}$ we shall have $n_{(k+1)} > n_k$, $x_{n(k+1)} \in$ V but $x_{n(k+1)} \neq$ p since the original sequence is not ultimately constant. Choose an U-open set W such that

$x_{n(k+1)} \in W$ but $p \notin \overline{W}$, and let $U_{k+1} = W \cap V$. This completes the inductive construction.

Let $G = \cup U_{2q}$. Since X is extremally disconnected U-space, $\overline{G}$ is an U-open set, and $p \in \overline{G}$ being the limit of $\{x_{n(2q)}\}$. Thus $\overline{G}$ is a neighborhood of p, so $x_r \in \overline{G}$ for all large r; in particular, $x_{n(s)} \in \overline{G}$ for some odd integer s. Since U_s is a neighborhood of $x_{n(s)}$, $U_s \cap G$ is not empty, contrary to the definition of G and disjoint ness of the U's.

Definition 2.8 A U-space is said to have a **countable basis at x** if there is a countable collection B of neighborhoods of x such that each neighborhood of x contains at least one of the elements of B.

A U-space that has a countable basis at each of its points is said to **satisfy the first countability axiom**, or to be **first-countable**.

Corollary 2.1 In a category of U-spaces in which all Hausdorff U-spaces satisfy the first axiom of countability and properties

(a) all admissible maps are U-continuous,

(b) if A is an admissible space and {p, q} is a two-element space, then $A \times \{p, q\}$ and the projection map of this space onto A are admissible,

(c) if A is an admissible space and B is a U-closed subspace of A, then B and the inclusion map of B into A are admissible hold, then every projective Hausdorff U-space is discrete topological Hausdorff U-spaces.

Lemma 2.1 Let A and E be U-spaces. Suppose f is a U-continuous map of E onto A such that $f(E_o) \neq A$ for any proper closed subset E_o of E.

Then, for any U-open set $G \subset E$, $f(G) \subset \overline{A - f(E-G)}$.

Proof: There is nothing to prove if G is empty. Suppose otherwise, let a be any point of f(G), and let N be any U-open neighborhood of a.

The lemma will follow if we prove that $N \cap (A - f(E-G))$ is not empty. Because $G \cap f^{-1}(N)$ is a nonempty U-open subset of E, $f(E - (G \cap f^{-1}(N))) \neq A$. Take $x \in A - f(E - (G \cap f^{-1}(N)))$; clearly $x \in A - f(E - G)$. Since f is onto, x = f(y) where evidently $y \in (G \cap f^{-1}(N))$. Therefore $x = f(y) \in f(f^{-1}(N)) = N$,

so $x \in N \cap (A - f(E - G))$, and the latter set is not empty.

Lemma 2.2 In an extremally disconnected U-space, if U_1 and U_2 are disjoint U-open sets, then $\overline{U_1}$ and $\overline{U_2}$ are also disjoint.

Proof: First, $\overline{U_1}$ and U_2 are disjoint because U_2 is U-open; then $\overline{U_1}$ and $\overline{U_2}$ are disjoint because $\overline{U_1}$ is U-open.

Lemma 2.3 Let A be an extremally disconnected Hausdorff compact U-space, and let E be a compact U-space. Suppose f is a U-continuous map of E onto A such that $f(E_o) \neq A$ for any proper U-closed subset E_o of E.

Then f is a U-homeomorphism.

Proof: We need only show that f is one-to one. Suppose, on the contrary, that x_1 and x_2 are distinct points of E for which $f(x_1) = f(x_2)$. Let G_1 and G_2 be disjoint U-open neighborhoods of x_1 and x_2 respectively. Both the sets $E - G_1$ and $E - G_2$ are compact, so $A - f(E - G_1)$ and $A - f(E - G_2)$ are U-open.

The latter sets are disjoint because $E = (E - G_1) \cup (E - G_2)$. By the Lemma- 2.2, $\overline{A - f(E - G_1)}$ and $\overline{A - f(E - G_2)}$ are disjoint. On the other hand, it follows from Lemma- 2.1 that $f(x_1) = f(x_2)$ is a point common to these sets. This contradiction establishes Lemma- 2.3.

Lemma 2. 4 [5](p- 484) Let A and D be compact Hausdorff U-spaces, and let f map D continuously onto A. Then D contains a compact U-subspace E such that f(E) = A but $f(E_o) \neq A$ for any proper U-closed subset E_o of E.

Proof: This is a well known consequence of Zorn's Lemma.

Theorem- 2.4 In the category of compact U-spaces and U-continuous maps, the projective U-spaces are precisely the extremally disconnected U-spaces.

Proof: To prove that all projective U-spaces in the category are extremally disconnected U-space, we have only to verify the conditions of Theorem-2.2. We turn to the opposite inclusion.
Let A be an extremally disconnected compact U-space, let B and C be compact U-spaces, let f be a U-continuous map of B onto C, and let ϕ be a U-continuous map of A into C. We must prove that there exists a U-continuous map ψ of A into B such that ϕ = fψ .
In the space $A\times B$ consider D = {(a, b) $|\phi(a)$ =f(b)}. This set is clearly closed and therefore compact U-space. Since f is onto, the projection π_1 of $A\times B$ onto A carries D onto A. By Lemma- 2.4 there is a U-closed subset E of D such that π_1(E) = A but $\pi_1(E_o) \neq$ A for any proper U- closed subset E_o of E. Let ρ be the restriction of π_1 to E. Lemma-2.3 asserts that ρ is a U-homomorphism. Let $\psi = \pi_2 \rho^{-1}$, where π_2 is the projection of $A\times B$ into B; this is the required map. Suppose $a\in A$; since $\rho^{-1}(a) \in D$, $f(\pi_2(\rho^{-1}(a))) = \phi(\pi_1(\rho^{-1}(a))) = \phi(a)$.
Thus $\phi = f\pi_2 \rho^{-1} = f\psi$; this completes the proof.

Definition 2.9 A map is said to be **U-proper** if and only if it is U-continuous and the inverse image of every compact U- space is compact.

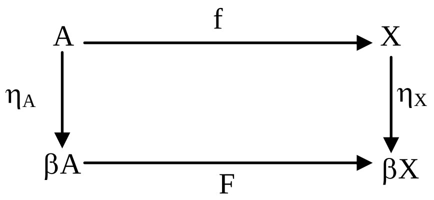
Example- 2.1 (Proper extremally disconected compact U- space). Let X = {a, b, c, d}, U = {X, Φ ,{a, b},{a, c},{a, b, c},{d},{a, b, d},{a, c, d}}.Since {a, b}$\cap${a, c}={a}$\notin$ U. U is a U-structure.
Then (X, U) is a proper U-space.
Here $\overline{\{a,b\}} = \overline{\{a,c\}} = \overline{\{a,b,c\}}$ = {a, b, c}, $\overline{\{d\}}$ = {d}, $\overline{\{a,b,d\}}$ = X, $\overline{\{a,c,d\}}$ = X .
Hence X is a proper extremally disconnected and compact U-space.

Example - 2.2 (a proper projective compact U-space)
Let X = {a, b, c, d} and U = {X, Φ , {a, b},{a, c}, {a, d},{b, c}, {b, d},{c, d},{b, c, d}, {a, c, d}, {a, b, d}, {a, b, c}}. Then X is a proper U-space which is clearly, Hausdorff, compact and extremally disconnected U-space. Thus X is a proper projective compact U-space.

Example - 2.3 Let X =N be U-space, n_o is a fixed element of N and
let U ={{ N, Φ }$\cup${{n$\in$ N $|$n $\leq n_o$}, {n$\in$N $|$n > n_o}, {n$\in$ N $|$n < n_o + 3}, {n$\in$ N $|$n $\geq n_o$ + 3}, $|$ $n_o \in$ N }}, and their unions.
Now { n$\in$ N $|$n < n_o + 3}$\cap${n$\in$ N $|$n > n_o } = { n_o+ 1, n_o + 2}$\notin$ U.
Thus U is a U-structure but not a topology, and so, **(X,U) is a proper U-space.**
(i) **X is clearly compact.**
(ii) **X is Hausdorff.** For, if $n_1, n_2 \in$ N and $n_1 \neq n_2$, say $n_1 < n_2$, then $n_1 \in U_1$ = {1,2,3,.........,n_1}, $n_2 \in U_2$ = {n$\in$ N $|$n > n_1 } and $U_1 \cap U_2 = \Phi$.
(iii) **X is extremally disconnected** U- space**,** since, for each U-open set G of X, $\overline{G}$ = G is U-open.
Hence by Theorem 2.4, X is a proper projective compact U-space.

Definition 2.11 If $A\subset X$, a **U-retraction** of X onto A is a U-continuous map r: X $\rightarrow$A such that r$|$ A is the identity map of A. If such a map r exists, we say that A is a U-**retract** of X.
We now generalize the theorems of ([14], p- 11-12)

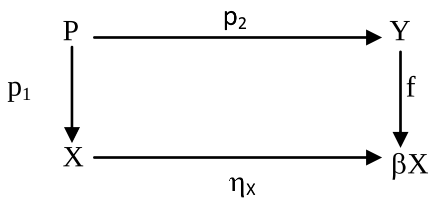
Theorem 2.5 Let X be any extremally disconnected object from the category **P**. Any perfect U-mapping f: A$\rightarrow$X of another object A onto X is a U-retraction.
Proof: We have f: A$\rightarrow$X onto. Then we can draw the following diagram



Where F is the unique U-continuous extension of f onto β A taking values in β X. Since f is a surjection, F is also onto. But β X is extremally disconnected U-space and F is an onto map. Since β X is projective U-space in the category **C**. F is a U-retraction, that is there exists a mapping g: $\beta X \to \beta A$ such that Fg = $1_{\beta X}$ = the identity map on β X. Since f is a perfect U-map, F(β A – η_A(A)) = β X – η_X(X). Therefore, g(η_X (X)) $\subset \eta_A$ (A). Put h = $\eta_A^{-1} g \eta_X$: X $\to$ A. Now fh(x) = f $\eta_A^{-1} g \eta_X$ (x). But F (g η_X (x)) = η_X (x) and g(η_X (x)) $\in \eta_A$ (A), that is, g(η_X (x)) = η_A (a) for some a $\in$ A. Therefore, η_X (x)= F(η_A (a))= η_X f(a). So, a = η_A^{-1}(η_A (a)) = $\eta_A^{-1} g \eta_X$ (x) and x = f(a) and hence, f(a) = f $\eta_A^{-1} g \eta_X$ (x) = x. Consequently fh(x) = x for each x $\in$ X, that is, fh = 1_x.Naturally f is a U-retraction.

Theorem 2.6 The category **P** has projectives that is any paracompact U-space is the perfect U-image of a projective U-space object. In fact, for every object X there is a projective U-space objects P and an onto U-perfect mapping p_1: P $\to$ X such that p_1 maps no proper U-closed subspace of P onto X. For any other such object P′ and p'_1:P′ $\to$ X there is an U-isomorphism e: P $\to$ P′ such that $p_1 = p_1'e$.

Proof: Let X be any object of **P**. Look at β X, the Stone - Čech compactification of X. There exists an extremally disconnected compact U-space Y and a U-continuous onto map f: Y $\to$ β X such that f(S) $\neq$ β X for any proper U-closed subspace S of Y. Consider the pull- back diagram



for the morphisms η_X : X $\to$ β X and f: Y $\to$ β X,

where P = {(x, y) $\in X \times Y$: η_X (x) = f(y)} and p_1 and p_2 are projections to X and Y respectively. We do not claim that this is a pullback in **P**. Clearly, η_X p_1 = fp_2. Since η_X is a U-monomorphism, p_2 is U-monomorphism. Since f is onto, p_1 is onto. Again, P is a U-closed subset of $X \times Y$ and the latter is paracompact U-space P is, hence, paracompact U-space. p_1 is also U-closed so that p_1 becomes a perfect U-map. fp_2 = η_X p_1 $\Rightarrow$ fp_2 (P) = η_X (X). Let W = p_2(P). Since f is a U-closed map, $f\overline{(p_2(P))} = f(\overline{W}) = \beta X.$ Observe that $\overline{W}$ is a U-closed subset of Y and f($\overline{W}$) = β X. From the choice of Y it follows that $\overline{W}$ = Y, that is, W = p_2(P) is dense in Y. Y is extremally disconnected U-space rendering W extremally disconnected U-space. Now it is not very difficult to see that p_2 is a U-perfect map onto W. Since P is paracompact U-space and p_2 is a U-perfect map onto W, W is a paracompact U- space.

By Theorem-2.5, p_2 is a U-retraction. Since p_2 is a U-monomorphism and a U-retraction also, it is an U-isomorphism, that is p_2 is a U-homeomorphism of P and W. Thus P is an extremally disconnected paracompact U-space. So P is projective U-space due to “In the category **P**, the projective objects are precisely the extremally disconnected paracompact U-spaces”. Since p_1 is a U-perfect map of P onto X, X is a U-perfect image of a projection object. Let Q be a proper U-closed subset of P. Then p_2(Q) is a proper U-closed subset of p_2(P)= W. Write p_2(Q) = W(F) where F is a U-closed subset of Y. Since p_2(Q) is a proper U-closed subset of W, F is a proper U-closed subset of Y.

If $p_1(Q)= X$, then $\eta_X(X) = \eta_X p_1(Q)= fp_2(Q)=f(W(F)) \subset f(F)$. Since f is a U-closed map of X onto βX, f(F) is a U-closed and hence equals βX. This is a contradiction. Consequently P enjoys the property that no proper U-closed subspace of P is mapped onto X by p_1.

If possible let P' be a projective paracompact U-space with a U-perfect map p_1': $P' \to X$ such that $p_1'(P') = X$ and if Q is any proper U-closed subspace of P' then $p_1'(Q) \neq X$. Then there exist a morphism $e : P \to P'$ and a morphism $e': P' \to P$ such that $p_1 = p_1'e$ and $p_1' = p_1e'$. Then $p_1(P) = X = p_1'(P') \Rightarrow p_1'e(P) = X = p_1e'(P')$. Naturally, e and e' are onto; we shall show that $ee' = 1_p$, that is, e is a U-co-retraction. If $ee' \neq 1_p$, there exists a proper U-closed subset S of P such that $d^{-1}(S) \cup S = P$ where $d = e'e$.

Obviously, $d(d^{-1}(S)) \subset S$ whence $p_1d(d^{-1}(S)) \subset p_1(S)$. But $p_1d = p_1e'e = p_1'e = p_1$, hence $p_1(S) \supset p_1d(d^{-1}(S)) = p_1(d^{-1}(S))$; so that $p_1(S) = p_1(P) = X$, a contradiction as S is a proper U-closed subset of P. We thus conclude that e is a U-co-retraction. Already e is a U-retraction; hence e is a U-isomorphism, that is, e is a U-homeomorphism of P onto P'.

Theorem 2.7 [6](p- 7) Let P be a compact Hausdorff U-space. Then P is projective if and only if for every compact Hausdorff U-space W and U- continuous $g: W \to P$, onto, there exists a U-continuous $s: P \to W$ such that $gos(p) = p$.

Proof: Assume that P is projective U-space and let s be a lifting of the identity map on P.

Conversely, assume that P is projective U- space and let X and Y be U-spaces and $h: Y \to X$ and $f: P \to X$, U-continuous map with h onto. Then there exists a U-continuous map $r: P \to Y$ such that $h\mathbf{o}r(p) = f(p)$ for every $p \in P$.

Let $W = \{(p, y) \in P \times Y: f(p) = h(y)\}$ and define $g : W \to P$ by $g(p, y) = p$ and $q: W \to Y$ by $q(p, y) = y$. If $s: P \to W$ is as above then $r = q\mathbf{o}s$ is a lefting of f.

Theorem 2.8 [11](p- 70) If P is a U-retract of P' and P' is projective, then P is projective.

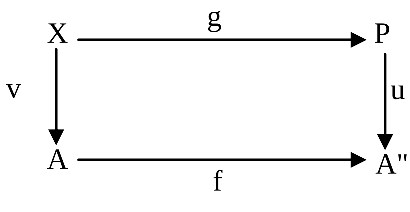
Proof: Let $P \to P' \to P = 1_P$. If $A \to A''$ is an U-epimorphism and $P \to A''$ is any morphism, then using projectivity of P' we have $P \to A'' = P \to P' \to P \to A'' = P \to P' \to A \to A''$ for some morphism $P' \to A$. This establishes U-projectivity of P.

Theorem 2.9 [11] (p-70) If P is projective U-space in A, then every U-epimorphism $A \to P$ is a U-retraction. Conversely if P has the property that every U-epimorphism $A \to P$ is a U-retraction, and if A either has projective or is abelian, then P is projective U- space.

Proof: If P is projective U-space, then given a U-epimorphism $A \to P$ there is a morphism $P \to A$ such that $P \to A \to P$ is 1_P. In other words $P \to A$ is a U-retraction.

Conversely, suppose that every U-epimorphism $A \to P$ is a U-retraction.

If A has projective then we may take A projective and then it follows from Theorem 5.8. On the other hand, if A is abelian, then, given an U-epimorphosm $f: A \to A''$ and a morphism $u: P \to A''$, we can form the pullback diagram

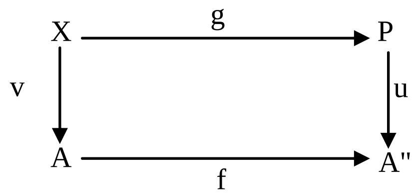


we know that g is an U-epimorphism. Then by assumption we can find $h: P \to X$ such that $gh = 1_P$. Then we have $fvh = ugh = u$. This proves that P is projective U-space.

Theorem 5.10 [13](p- 12) In the category **P**, the projective U-space objects are precisely the extremally disconnected paracompact U-spaces.

Proof: If P is projective U-space, then given a U-epimorphism $A \to P$ there is a morphism $P \to A$ such that $P \to A \to P$ is 1_P. In other words $P \to A$ is a U-retraction.

Conversely, suppose that every U-epimorphism A $\rightarrow$P is a U-retraction. If A has projective then we may take A projective U-space and then it follows from Theorem 5.8. On the other hand, if A is abelian, then, given an U-epimorphosm f: A $\rightarrow$A" and a morphism u:P$\rightarrow$A", we can form the pullback diagram



we know that g is an U-epimorphism. Then by assumption we can find h: P $\rightarrow$X such that gh = 1_P. Then we have fvh = ugh = u. This proves that P is projective U-space. Therefore the projective U-space objects of P are the objects for which perfect U-maps onto them are U-retraction.
Hence the theorem follows from theorems 2.5, 2.8 and 2.9.
Let X be any extremally disconnected U-space object from the category P. By theorem- 2.5 we can prove that any U-perfect mapping f: A $\longrightarrow$X of another object A onto X is a U-retraction.

By theorem- 2.8 'If P is a U-retract of P' and P' is projective U-space, then P is projective U-space' And theorem- 2.9 "If P is projective U-space in A, then every U-epimorphism A$\longrightarrow$P is a U-retraction. Conversely if P has the property that every U-epimorphism A$\longrightarrow$P is a U-retraction, and if A has projective U- space, then P is projective U- space." P is projective U-space.
Hence the theorem is proved.

Examples of proper projective U-spaces which are locally compact but not compact.

Example- 5.4 Let X = R ,U ={X, Φ, (-∞, $\frac{1}{2}$),[0,1), [$\frac{1}{2}$,1), [1,2),..,[n, n + 1), …., and their unions}.

(i) Then (X, U) is a U-space but not a topological space.

Since (-∞, $\frac{1}{2}$) $\cap$ [0, 1) = [0, $\frac{1}{2}$) $\notin$ U.

(ii) **X is not compact**, since C = {(-∞, $\frac{1}{2}$),[0, 1), [1, 2),…..,[n, n + 1),….,} is U-open cover of X but it has no finite sub cover.

(iii) **X is locally compact**. For let $x_o \in$X. If $x_o < \frac{1}{2}$, then (-∞, $\frac{1}{2}$) is a neighborhood of x_o whose closure is (-∞,1), which is compact U-space, since every U-open cover of (-∞, $\frac{1}{2}$) must contain either X or both (-∞, $\frac{1}{2}$) and [$\frac{1}{2}$,1) and each such cover is clearly finite.

If x$\geq \frac{1}{2}$, x$\in$[n, n + 1) for some n$\in${0}$\cup$N. Then $\overline{[n, n+1)}$ = [n, n + 1) which is obviously compact, since [n, n + 1) is U-closed.

(iv) All the U-open sets except (-∞, $\frac{1}{2}$) and [0, 1) are both U-open and U-closed & so the U-closure of any union of these is U-open. Also, $\overline{\left(-\infty, \frac{1}{2}\right)}$ = (-∞, 1), $\overline{[0,1)}$= (-∞, 1).

Hence the closure of every U-open set is U-open.

Thus X is extremally disconnected U-space, and so, X is projective U-space.

Example 2.5 Let X = Z ,U ={X, Φ ,{n$\in$ Z$\mid$ - ∞< n$\leq$1}, {0,1,2}, {3,4,5},{6,7,8} and their unions}. X is a proper U-space.

For {n$\in$ Z $\mid$ - ∞< n$\leq$1}$\cap${0,1,2} = {0, 1}$\notin$ U.

(i) **X is not compact.** For the U-open cover

{{n$\in$ Z $\mid$ - ∞< n$\leq$1},{0,1,2}, {3,4,5},{6,7,8},…………….} has no finite sub cover .

(ii) However, **X is locally compact**. To sec this, let $x_0 \in$X. If $x_0 \leq$1,

the {n$\in$ Z$\mid$ - ∞< n$\leq$1} is a U-open neighborhood of x_0 and its closure is

{n$\in$ Z $\mid$ - ∞ < n$\leq$2} which is clearly compact. If x_0 >1, then for x_0 = 2, {0, 1, 2} is a U-open neighborhood of x_0 and its closure is {n$\in$ Z $\mid$ - ∞< n $\leq$ 2} which again is U-compact, and for x_0 = n >2, x$\in${3r, 3r + 1, 3r + 2} for some positive r, and this set is a U-open neighborhood of x_0. Also, it is its own closure. Clearly it is compact.
Thus X is locally compact U-space.

(iii) The sets {3r, 3r + 1, 3r + 2} are both U-open and U-closed for each r $\geq$1, $\overline{\{n \in Z \mid -\infty < n \leq 1\}}$ = {n$\in$ Z $\mid$ - ∞< n $\leq$ 2}= {n$\in$ Z $\mid$ - ∞ < n $\leq$ 1}$\cup${0, 1, 2}

(iv) which is U-open. Also, $\overline{\{0,1,2\}}$ = {n$\in$ Z $\mid$ - ∞ < n $\leq$ 2} is U-open, as before.

Hence X is extremally disconnected U-space.
Therefore X is projective U-space.

3. Cover of compact Hausdorff U-space

We now generalize definitions and results of [6] (p- 7 - 8). The proofs in [6] carry over to U-spaces as we shall see below.

Definition 3.1 Let X be a compact Hausdorff U-space. A pair (C, f) is called a **U-cover of X**, provided that C is a compact Housdorff U-space and f: C $\to$ X is a U-continuous map that is onto X.

Definition 3.2 Let X and C be compact Housdorff U-spaces and f: C $\to$X a U-continuous map that is onto X. A pair (C, f) is called a **U-essential cover of X** if it is a U-cover and whenever Y is a compact, Hausdorff U-space, h: Y $\to$C is U-continuous and f(h(y)) = X, then necessarily h(Y) = C.

Definition 3.3 Let X and C be compact Housdorff U-space and f: C $\to$X a U-continuous map that is onto X. A pair (C, f) is called a **U-rigid cover of X** if it is a U-cover and the only U-continuous map h: C $\to$C satisfying f(h(c)) = f(c) for every c$\in$C is the identity map.

Theorem 3.1 Let X be a compact Hausdorff U-space and let (C, f) be a U- essential cover of X. Then (C, f) is a U-rigid cover of X.

Proof: Let h: C $\to$C satisfy f(h(c)) = f(c) for every c$\in$C. Let C_1 = h(C) which is a compact U-subset of C that still maps onto X. The inclusion map of i: $C_1 \to$C satisfies, f(i(C_1)) = X and hence must be onto C. Thus h(C) = C.
Next, we claim that if G$\subseteq$ C is any non- empty U-open set, then G$\cap$ h^{-1}(G) is non- empty. For assume to the contrary, and let F = C \ G. Then F is compact U-space and given any c$\in$G there exist y$\in h^{-1}$(G) with h(y) = c. Hence, y$\in$F and f(c) = f(h(y)) = f(y). Thus f(F) = X, again contradicting the essentiality of C. Thus, for every U-open set G, we have that G$\cap$ h^{-1}(G) is non-empty.
Now fix any c$\in$C and for every neighborhood G of c pick $x_G \in$ G$\cap$ h^{-1}(G). We have that the net {x_G} converges to c. Hence, by continuity, {h(x_G)} converges to h(c). But since h(x_G)$\in$ G for every G, we also have that {h(x_G)} converges to c. Thus, h(c) = c and since c was arbitrary, C is U-rigid cover of X.

Theorem 3.2 Let (C, f) be a U-cover of X with C a projective U-space. Then (C, f) is a U-essential cover if and only if (C, f) is a U-rigid cover.

Proof: We already have that a U-essential cover is always a U-rigid cover. So assume that (C, f) is a
U-rigid cover. Let h: Y $\to$ C with f(h(Y))= X. Since C is projective, then there exists a map s: C $\to$ Y with (foh)os = f. We have hos : C $\to$C and f(hos(c))= f(c) and so by rigidity, hos(c)= c for every c$\in$C. In particular, h must be onto and so C is U-essential cover.

Theorem 3.3 Let (Y, f) be a U-cover of X and let C $\subset$ Y be a minimal, compact U-subset of Y that maps onto X. Then (C, f) is a U-rigid, essential cover of X.

Proof: First, we prove U-essential. Given any compact Hausdorff U-space Z and h: Z $\rightarrow$C such that f(h(Z)) = X, we have that h(Z) $\subseteq$ C is compact U-space and hence h(Z) = C by minimality.

Since (C, f) is a U-essential cover of X, by the above results it is also a U-rigid cover.

REFERENCE

[1]. Akhter N., Das S. K. and Majumdar S.(2014), On Hausdorff and Compact U-spaces, Annals Pure and Applied Mathematics, Vol. 5, No. 2, 168-182.

[2]. Andrijevic D.(1996), On b- open sets, Mat. Vesnik, 48, 59-64.

[3]. Devi R, Sampathkumar S and Caldas M (2008), On supra α open sets and Sα - continuous functions, General Mathematics, Vol. 16, Nr. 2, 77-84

[4]. Dieudonne (1994), Algebraic topology and differential geometry.

[5]. Gleason A. M (1958), 'Projective topological spaces', Illinois J. Math. 2, 482 - 489.

[6]. Hadwin D. and Paulsen V. I.(2007) ,Injectivity and Projectivity in analysis and topology.

[7]. Majumdar S., Akher N. and Das S. K, Paracompact U- spaces (2014), Journal of Physical Sciences,Vol.19,ISSN: 2350-0352, 26 December 2014.

[8]. S. Majumdar and M. Mitra (31 Dec 2012), Anti- Hausdorff spaces, Journal of Physical Sciences Vol.16, 2012, 117- 123. ISSN: 0972 - 8791.

[9]. Majumdar S. and Akhter N.(2009), Topology (Bengali), Adhuna Prakashan.

[10]. Mashhour A. et al.(1983), On supratopological spaces, Indian J. pure appl, Math, 14(4), 502-510.

[11]. Mitchell B.(1965), Theory of categories, (Academic Press, New York).

[12]. Munkres J. R.(2006), Topology, Prentice Hall of India Private Limited, New Delhi-110001.

[13]. Raha A. B. (1983), Projectives in some categories of Hausdorff spaces, J. Austral. Math. Soc.(Series A) 34 ,7-15.

[14]. Sayed O. R. and Noiri T (2010), On supra b- open sets and supra b- continuity on topological spaces, European Journal of pure and applied Mathematics, Vol. 3, No. 2, 295-302.

[15]. Henriksen M. and Isbell J.R., Some properties of compactifications, Duke Math. J. 25 (1958) 88-106.