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RESEARCH ARTICLE

OPTIMAL CONVEX COMBINATION BOUNDS OF HARMONIC AND QUADRATIC MEANS FOR NEUMAN-SÁNDOR MEAN

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ABSTRACT

In this paper, we present the least value a and the greatest value b such that the double inequality

$$aH(a,b)+(1-a)Q(a,b)<M(a,b)<bH(a,b)+(1-b)Q(a,b)$$

holds for all $a,b>0$ with $a\neq b$. Here $H(a,b)$, $Q(a,b)$ and $M(a,b)$ denote the harmonic, quadratic and Neuman-Sándor means of two positive numbers a and b, respectively.

Keywords: Inequality, Neuman-Sándor mean, harmonic mean, quadratic mean.

1. INTRODUCTION

For $a,b>0$ with $a \neq b$, the Neuman-Sándor mean $M(a,b)$ [1] is defined by

$$M(a,b)=\frac{a-b}{2\sinh^{-1}[(a-b)/(a+b)]}, \qquad (1.1)$$

where $\sinh^{-1}x=\log(x+\sqrt{1+x^2})$ is the inverse hyperbolic sine function.

Recently, the Neuman-Sándor mean has been the subject of intensive research. In particular, many remarkable inequalities for $M(a,b)$ can be found in the literature [1, 2]. Let $H(a,b)=(2ab)/(a+b)$, $G(a,b)=\sqrt{ab}$, $L(a,b)=(a-b)/(\log a-\log b)$, $P(a,b)=(a-b)\big/(4\tan^{-1}\sqrt{a/b}-\pi)$, $A(a,b)=(a+b)/2$, $T(a,b)=(a-b)\big/[2\tan^{-1}(a-b)/(a+b)]$, $Q(a,b)=\sqrt{(a^2+b^2)/2}$ and $C(a,b)=(a^2+b^2)\big/(a+b)$ be the harmonic, geometric, logarithmic, first Seiffert, arithmetic, second Seiffert, quadratic and contra-harmonic mean of a and b, respectively. Then

$$\min\{a,b\} < H(a,b) < G(a,b) < L(a,b) < P(a,b) < A(a,b) < M(a,b) < T(a,b) < Q(a,b) < C(a,b) < \max(a,b) \quad (1.2)$$

hold for all $a,b > 0$ with $a \neq b$.

In [3], Neuman proved that the double inequalities

$$aQ(a,b) + (1-a)A(a,b) < M(a,b) < bQ(a,b) + (1-b)A(a,b) \quad (1.3)$$

and

$$l\,C(a,b) + (1-l)A(a,b) < M(a,b) < mC(a,b) + (1-m)A(a,b) \quad (1.4)$$

hold for all $a,b > 0$ with $a \neq b$ if and only if

$a \le [1-\log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})] = 0.3249\cdots$, $b \ge 1/3$, $l \le [1-\log(1+\sqrt{2})]/\log(1+\sqrt{2})$ and $m \ge 1/6$.

In [4], Li etc showed that the double inequality

$$L_{p_0}(a,b) < M(a,b) < L_2(a,b) \quad (1.5)$$

holds for all $a,b > 0$ with $a \neq b$, where

$L_p(a,b) = [(a^{p+1} - b^{p+1})/(p+1)(a-b)]^{1/p}$ $(p \neq -1, 0)$, $L_0(a,b) = 1/e[(a^a)/b^b]^{1/(a-b)}$ and $L_{-1}(a,b) = (a-b)/(\log a - \log b)$ is the p-th generalized logarithmic mean of a and b, and $p_0 = 1.843\cdots$ is the unique solution of the equation $(p+1)^{1/p} = \log(1+\sqrt{2})$.

In [5], Chu etc proved that the double inequalities

$$a_1L(a,b) + (1-a_1)Q(a,b) < M(a,b) < b_1L(a,b) + (1-b_1)Q(a,b) \quad (1.6)$$

and

$$a_2L(a,b) + (1-a_2)C(a,b) < M(a,b) < b_2L(a,b) + (1-b_2)C(a,b) \quad (1.7)$$

hold for all $a,b > 0$ with $a \neq b$ if and only if

$a_1 \ge 2/5$, $b_1 \le 1 - 1/[\sqrt{2}\log(1+\sqrt{2})] = 0.1977\cdots$, $a_2 \ge 5/8$ and $b_2 \le 1 - 1/[2\log(1+\sqrt{2})] = 0.4327\cdots$.

The main purpose of this paper is to found the least value a and the greatest value b such that the double inequality

$$aH(a,b) + (1-a)Q(a,b) < M(a,b) < bH(a,b) + (1-b)Q(a,b)$$

holds for all $a,b > 0$ with $a \neq b$

2. MAIN RESULTS

THEOREM 2.1. The double inequality

$$aH(a,b) + (1-a)Q(a,b) < M(a,b) < bH(a,b) + (1-b)Q(a,b)$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $a \ge 1/3$ and $b \le 1 - 1/[\sqrt{2}\log(1+\sqrt{2})] = 0.1977\cdots$.

Proof. Without loss of generality, we assume that $a > b > 0$. Let $x = (a-b)/(a+b) \in (0,1)$ and $l = 1 - 1/[\sqrt{2}\log(1+\sqrt{2})]$. Then

$$\frac{H(a,b)}{A(a,b)} = 1 - x^2, \quad \frac{M(a,b)}{A(a,b)} = \frac{x}{\sinh^{-1}x}, \quad \frac{Q(a,b)}{A(a,b)} = \sqrt{1+x^2}. \quad (2.1)$$

Firstly, we prove that

$$\frac{1}{3}H(a,b)+\frac{2}{3}Q(a,b)< M(a,b) \tag{2.2}$$

Equation (2.1) lead to

$$\frac{\frac{1}{3}H(a,b)+\frac{2}{3}Q(a,b)-M(a,b)}{A(a,b)}=\frac{2\sqrt{1+x^2}}{3}+\frac{1-x^2}{3}-\frac{x}{\log\left(x+\sqrt{1+x^2}\right)}=d(x). \tag{2.3}$$

Because $\log\left(x+\sqrt{1+x^2}\right)=x+\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{[2\cdot 4\cdot 6\cdots(2n)]\cdot(2n+1)}x^{2n+1}$, we have

$\log\left(x+\sqrt{1+x^2}\right)< x$. So that

$$d(x)<\frac{2\sqrt{1+x^2}}{3}+\frac{1-x^2}{3}-1=-\frac{x^4}{3\left(2\sqrt{1+x^2}+2+x^2\right)}<0. \tag{2.4}$$

Therefore the inequality (2.2) follows from (2.3) and (2.4).

Secondly, we prove that

$$\lambda H(a,b)+(1-\lambda)Q(a,b)> M(a,b). \tag{2.5}$$

Equation (2.1) lead to

$$\frac{\lambda H(a,b)+(1-\lambda)Q(a,b)-M(a,b)}{A(a,b)}=\frac{2\sqrt{1+x^2}}{3}+\frac{1-x^2}{3}-\frac{x}{\log\left(x+\sqrt{1+x^2}\right)}$$
$$=\frac{(1-\lambda)\sqrt{1+x^2}+\lambda(1-x^2)}{\log\left(x+\sqrt{1+x^2}\right)}D(x), \tag{2.6}$$

where

$$D(x)=\log\left(x+\sqrt{1+x^2}\right)-\frac{x}{(1-\lambda)\sqrt{1+x^2}+\lambda(1-x^2)}. \tag{2.7}$$

Simple computations yield

$$\lim_{x\to 0^+}D(x)=0, \tag{2.8}$$

$$\lim_{x\to 1^-}D(x)=\log(1+\sqrt{2})-\frac{1}{(1-\lambda)\sqrt{2}}=0, \tag{2.9}$$

and

$$D'(x)=\frac{1}{\sqrt{1+x^2}[(1-\lambda)\sqrt{1+x^2}+\lambda(1-x^2)]^2}F(x), \tag{2.10}$$

where

$$F(x)=\lambda[(2\lambda-3)x^2+1-2\lambda]\sqrt{1+x^2}+\lambda^2x^4+(1-2\lambda-\lambda^2)x^2+\lambda(2\lambda-1). \tag{2.11}$$

Making use of the transform $x=1/2(t-1/t)\quad (t\in(1,1+\sqrt{2}))$ for (2.11), we have

$$F(x)=\frac{1}{16t^4}G(t), \tag{2.12}$$

where

$$G(t) = \lambda^2 t^8 + 2\lambda(2\lambda - 3)t^7 + 4(1 - 2\lambda - 2\lambda^2)t^6 + 2\lambda(7 - 10\lambda)t^5 + 2(23\lambda^2 - 4)t^4 + 2\lambda(7 - 10\lambda)t^3 + 4(1 - 2\lambda - 2\lambda^2)t^2 + 2\lambda(2\lambda - 3)t + \lambda^2, \tag{2.13}$$

Simple calculations of derivative yield

$$G'(t) = 2[4\lambda^2 t^7 + 7\lambda(2\lambda - 3)t^6 + 12(1 - 2\lambda - 2\lambda^2)t^5 + 5\lambda(7 - 10\lambda)t^4 + 4(23\lambda^2 - 4)t^3 + 3\lambda(7 - 10\lambda)t^2 + 4(1 - 2\lambda - 2\lambda^2)t + \lambda(2\lambda - 3)], \tag{2.14}$$

$$G''(t) = 4[14\lambda^2 t^6 + 21\lambda(2\lambda - 3)t^5 + 30(1 - 2\lambda - 2\lambda^2)t^4 + 10\lambda(7 - 10\lambda)t^3 + 6(23\lambda^2 - 4)t^2 + 3\lambda(7 - 10\lambda)t + 2(1 - 2\lambda - 2\lambda^2)], \tag{2.15}$$

$$G'''(t) = 12[28\lambda^2 t^5 + 35\lambda(2\lambda - 3)t^4 + 40(1 - 2\lambda - 2\lambda^2)t^3 + 10\lambda(7 - 10\lambda)t^2 + 4(23\lambda^2 - 4)t + \lambda(7 - 10\lambda)], \tag{2.16}$$

$$G^{(4)}(t) = 48[35\lambda^2 t^4 + 35\lambda(2\lambda - 3)t^3 + 30(1 - 2\lambda - 2\lambda^2)t^2 + 5\lambda(7 - 10\lambda)t + (23\lambda^2 - 4)], \tag{2.17}$$

$$G^{(5)}(t) = 240[28\lambda^2 t^3 + 21\lambda(2\lambda - 3)t^2 + 12(1 - 2\lambda - 2\lambda^2)t + \lambda(7 - 10\lambda)], \tag{2.18}$$

$$G^{(6)}(t) = 1440[14\lambda^2 t^2 + 7\lambda(2\lambda - 3)t + 2(1 - 2\lambda - 2\lambda^2)], \tag{2.19}$$

and

$$G^{(7)}(t) = 10080\lambda[4\lambda t + (2\lambda - 3)]. \tag{2.20}$$

Notice that $19/100 < \lambda < 1/5$, from (2.13)-(2.20) one has

$$\lim_{t \to 1^+} G(t) = 0, \tag{2.21}$$

$$\lim_{t \to (1+\sqrt{2})^-} G(t) = 16[2(12\sqrt{2} + 17)\lambda^2 - (70\sqrt{2} + 99)\lambda + (12\sqrt{2} + 17)] < -\frac{4}{25}(34\sqrt{2} + 45) < 0, \tag{2.22}$$

$$\lim_{t \to 1^+} G'(t) = 0, \tag{2.23}$$

$$\lim_{t \to (1+\sqrt{2})^-} G'(t) = 32[(47\sqrt{2} + 66)\lambda^2 - (107\sqrt{2} + 151)\lambda + (17\sqrt{2} + 24)] < -\frac{4}{25}(58\sqrt{2} + 81) < 0, \tag{2.24}$$

$$\lim_{t \to 1^+} G''(t) = 144(\frac{2}{9} - \lambda) > 0, \tag{2.25}$$

$$\lim_{t \to (1+\sqrt{2})^-} G''(t) = 32[17(9\sqrt{2} + 13)\lambda^2 - (272\sqrt{2} + 387)\lambda + (39\sqrt{2} + 55)] < -\frac{8}{25}(656\sqrt{2} + 969) < 0, \tag{2.26}$$

$$\lim_{t \to 1^+} G'''(t) = 1296(\frac{2}{9} - \lambda) > 0, \tag{2.27}$$

$$\lim_{t \to (1+\sqrt{2})^-} G'''(t) = 96[13(11\sqrt{2} + 15)\lambda^2 - 38(5\sqrt{2} + 7)\lambda + (23\sqrt{2} + 33)] < -\frac{48}{25}(369\sqrt{2} + 487) < 0, \tag{2.28}$$

$$\lim_{t \to 1^+} G^{(4)}(t) = 96(9\lambda^2 - 65\lambda + 13) > \frac{311904}{10000} > 0, \tag{2.29}$$

$$\lim_{t\to(1+\sqrt{2})^-} G^{(4)}(t) = 96[(300\sqrt{2}+439)\lambda^2 - 5(61\sqrt{2}+88)\lambda + (30\sqrt{2}+43)] < -\frac{24}{25}(1595\sqrt{2}+2304) < 0, \tag{2.30}$$

$$\lim_{t\to 1^+} G^{(5)}(t) = 960(9\lambda^2 - 20\lambda + 3) < -\frac{10560}{25} < 0, \tag{2.31}$$

$$\lim_{t\to 1^+} G^{(6)}(t) = 1440(24\lambda^2 - 25\lambda + 2) < -\frac{12888}{5} < 0, \tag{2.32}$$

and

$$G^{(7)}(t) < 10080\cdot\frac{1}{5}[4\cdot\frac{1}{5}(1+\sqrt{2}) + (2\cdot\frac{1}{5} - 3)] = -\frac{72}{5}\left(231 - 112\sqrt{2}\right) < 0. \tag{2.33}$$

From (2.33) we clearly see $G^{(6)}(t)$ is strictly decreasing in $(1, 1+\sqrt{2})$. $G^{(4)}(t)$ is strictly decreasing in $(1, 1+\sqrt{2})$ resulting from (2.31) and (2.32) together with the monotonicity of $G^{(6)}(t)$ too. It follows from (2.29) and (2.30) together with the monotonicity of $G^{(4)}(t)$ that there exists $t_0 \in (1, 1+\sqrt{2})$ such that $G^{(4)}(t) > 0$ for $t \in (1, t_0)$ and $G^{(4)}(t) < 0$ for $t \in (t_0, 1+\sqrt{2})$, hence $G'''(t)$ is strictly increasing in $(1, t_0)$ and strictly decreasing in $(t_0, 1+\sqrt{2})$. From (2.27) and (2.28) together with the monotonicity of $G'''(t)$ we know that there exists $t_1 \in (1, 1+\sqrt{2})$ such that $G'''(t) > 0$ for $t \in (1, t_1)$ and $G'''(t) < 0$ for $t \in (t_1, 1+\sqrt{2})$. By (2.21)-(2.26) the same reasoning applied to $G''(t)$, $G'(t)$ and $G(t)$ we can derive that there exists $t_2 \in (1, 1+\sqrt{2})$ such that $G(t) > 0$ for $t \in (1, t_2)$ and $G(t) < 0$ for $t \in (t_2, 1+\sqrt{2})$.

We write that $x_0 = 1/2(t_2 - 1/t_2)$. From
$x \in (0, x_0) \Leftrightarrow t \in (1, t_2)$, $x \in (x_0, 1) \Leftrightarrow t \in (t_2, 1+\sqrt{2})$ and (2.12) together with the signs of $G(t)$ we affirm that $F(x) > 0$ for $x \in (0, x_0)$ and $F(x) < 0$ for $x \in (x_0, 1)$. This fact and (2.10) imply that $D'(x) > 0$ for $x \in (0, x_0)$ and $D'(x) < 0$ for $x \in (x_0, 1)$, thus $D(x)$ is strictly increasing in $(0, x_0)$ and strictly decreasing in $(x_0, 1)$. From (2.8) and (2.9) together with the monotonicity of $D(x)$ we elicit that

$$D(x) > 0. \tag{2.34}$$

Therefore the inequality (2.5) follows from (2.6) and (2.34).

Finally we prove that $1/3H(a,b) + 2/3Q(a,b)$ is the best possible lower convex combination bound and $\lambda H(a,b) + (1-\lambda)Q(a,b)$ is the best possible upper convex combination bound of the harmonic and the quadratic means for the Nueman-Sándor mean.

Equation (2.1) lead to

$$\frac{Q(a,b) - M(a,b)}{Q(a,b) - H(a,b)} = \frac{\sqrt{1+x^2} - \dfrac{x}{\sinh^{-1}x}}{\sqrt{1+x^2} - (1-x^2)} = B(x). \tag{2.35}$$

From (2.35) one has

$$\lim_{x \to 0^+} B(x) = \frac{1}{3}, \tag{2.36}$$

and

$$\lim_{x \to 1^-} B(x) = \lambda . \tag{2.37}$$

If $\alpha < 1/3$, then (2.35) and (2.36) lead to the conclusion that there exists $0 < \delta_1 < 1$ such that $\alpha H(a,b) + (1-\alpha)Q(a,b) > M(a,b)$ for all $a,b > 0$ with $(a-b)/(a+b) \in (0,\delta_1)$.

If $\beta > \lambda$, then (2.35) and (2.37) lead to the conclusion that there exists $0 < \delta_2 < 1$ such that $M(a,b) > \beta H(a,b) + (1-\beta)Q(a,b)$ for all $a,b > 0$ with $(a-b)/(a+b) \in (1-\delta_2, 1)$.

3. REFERENCES

[1]. E. Neuman and J. Sándor, 2003, On the Schwab-Borchardt mean, *Math. Pannon.* 14, 2, 253-266.

[2]. E. Neuman and J. Sándor, 2006, On the Schwab-Borchardt mean II, *Math. Pannon.* 17, 1, 49-59

[3]. Y.-M. Li, B.-Y. Long and Y.-M. Chu, 2012, Sharp bounds for the Neuman-Sándor mean in terms of generalized logarithmic mean, *J. Math. Inequal*. 6, 4, 567-577.

[4]. E. Neuman, 2012, A note on a certain bivariate mean, *J. Math. Inequal.* 6, 4, 637-643.

[5]. Y.-M. Chu, T.-H. Zhao and B.-Y. Liu, 2014, Optimal bound for Neuman-S_andor mean in terms of the convex combination of logarithmic and quadratic or contra-harmonic means, *J. Math. Inequal*. 8, 2, 201-217.