Vol.3.Issue.3.2015 (Jul-Sept)

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

http://www.bomsr.com

A Peer Reviewed International Research Journal

ISSN:2348-0580

RESEARCH ARTICLE

AL-TEMEME TRANSFORM METHOD FOR SOLVING SYSTEMS OF LINEAR EQUATIONS

ALI HASSAN MOHAMMED[1], ALAASALLHHADI[2], HASSAN NADEMRASOUL[3]

[1]University of Kufa, Faculty of Education for Girls,Department of Mathematics, Iraq

[2]University of Kufa, Faculty of Education for Girls, Department of Mathematics, Iraq

[3]University of Kufa. Faculty of Computer Science and Math. Department of Mathematics,Iraq

ABSTRACT

Our aim in this paper is to find the solution of Linear systems Ordinary Differential Equations (LSODE) with variable coefficients subjected to some initial conditions by using Al-Tememe Transform ($\mathcal{T}$.T) through generalized the methods that found in [2]

1. INTRODUCTION

Integral transformations are an important role to solve the linear ordinary differential equations (LODE) with constant coefficients and variable coefficients.

We will use Al-Tememe Transform ($\mathcal{T}$.T) to solve systems of linear differential equations of a first-order with variable coefficients. And the method summarized by ta;ing ($\mathcal{T}$.T) to both sides of the equations then we take ($\mathcal{T}^{-1}$. T) to both sides of the equations and by using the given initial conditions we find the constants.

2. Preliminaries

Definition 1: [1]

Let f is defind function at period (a, b)then the integral transformation for$f$ whose it's symbol $F(p)$ is defined as :

$$F(p) = \int_{a}^{b} k(p, x) f(x) dx$$

Where k is a fixed function of two variables, called the kernel of the transformation, and a, b are real numbers or $\mp\infty$, such that the above integral converges.

Definition 2:[3]
The Al-Tememe transformation for the function $f(x)\,; x > 1$ is defined by the following integral :

$$\mathcal{T}[f(x)] = \int_1^{\infty} x^{-p}\, f(x)dx = F(p)$$

Such that this integral is convergent , p is positive constant.
Property 1:[3]
This transformation is characterized by the linear property ,that is
$\mathcal{T}[Af(x) + Bg(x)] = A\mathcal{T}[f(x)] + B\mathcal{T}[g(x)]$ **,**
Where $A\,, B$ are constants ,the functions $f(x)$, $g(x)$ are defined when $x > 1$.
The Al-Tememe transform for some fundamental functions are given in table(1)[3] :

ID	Function ,$\boldsymbol{f(x)}$	$\boldsymbol{F(p) = \int_1^{\infty} x^{-p} f(x) dx} = \boldsymbol{\mathcal{T}[f(x)]}$	Region of convergence
1	$k; k = constant$	$\frac{k}{p-1}$	$\boldsymbol{p} > 1$
2	x^n , $n \in R$	$\frac{1}{p-(n+1)}$	$\boldsymbol{p} > n+1$
3	$ln\,x$	$\frac{1}{(p-1)^2}$	$\boldsymbol{p} > 1$
4	$x^n\, ln\,x\,, n \in R$	$\frac{1}{[p-(n+1)]^2}$	$\boldsymbol{p} > n+1$
5	$sin(a\, ln\, x)$	$\frac{a}{(p-1)^2 + a^2}$	$\boldsymbol{p} > 1$
6	$cos(a\, ln\, x)$	$\frac{p-1}{(p-1)^2 + a^2}$	$\boldsymbol{p} > 1$
7	$sinh(a\, ln\, x)$	$\frac{a}{(p-1)^2 - a^2}$	$\boldsymbol{\lvert p-1 \rvert} > a$
8	$cosh(a\, ln\, x)$	$\frac{p-1}{(p-1)^2 - a^2}$	$\boldsymbol{\lvert p-1 \rvert} > a$

Table (1).

From the Al-Tememe definition and the above table, we get:
Theorem1:
If $\mathcal{T}[\,f(x)] = F(p)$ and $a$is constant, then $\mathcal{T}[x^{-a}f(x)] = F(p+a)$.see [3]
Definition 3: [3]
Let $f(x)$ be a function where ($x > 1$) and $\mathcal{T}[f(x)] = F(p), f(x)$ is said to be an inverse for the Al-Tememe transformation and written as $\mathcal{T}^{-1}[F(p)] = f(x)$, where $\mathcal{T}^{-1}$ returns the transformation to the original function.
Property 2:[3]
If $\mathcal{T}^{-1}[F_1(p)\,] = f_1(x)$, $\mathcal{T}^{-1}[F_2(p)\,] = f_2(x)$,…, $\mathcal{T}^{-1}[F_n(p)\,] = f_n(x)$ and $a_1, a_2, \dots, a_n$ are constants then,

$$\mathcal{T}^{-1}[a_1F_1(p) + a_2F_2(p) + \cdots + a_nF_n(p)] = a_1f_1(x) + a_2f_2(x) + \cdots + a_nf_n(x)$$

Theorem 2:[3]

If the function $f(x)$ is defined for $x > 1$ and its derivatives $f^{(1)}(x), f^{(2)}(x), \dots, f^{(n)}(x)$ are exist then:

$$\mathcal{T}\left[x^n f^{(n)}(x)\right] = -f^{(n-1)}(1) - (p-n)f^{(n-2)}(1) - \cdots \\ - (p-n)\big(p-(n-1)\big) \dots ((p-2)f(1) + (p-n)!\,F(p)$$

Definition 4:[4]

A function $f(x)$ is piecewise continuous on an interval $[a, b]$ if the interval can be partitioned by a finite number of points $a = x_0 < x_1 < \cdots < x_n = b$ such that:

1. $f(x)$ is continuous on each subinterval (x_i, x_{i+1}), for $i = 0, 1, 2, \dots, n-1$
2. The function f has jump discontinuity at x_i , thus

$$\left|\lim_{x \to x_i^+} f(x)\right| < \infty , i = 0, 1, 2, \dots, n-1 ; \left|\lim_{x \to x_i^-} f(x)\right| < \infty , \qquad i = 0, 1, 2, \dots, n$$

Note: A function is piecewise continuous on $[0, \infty)$ if it is piecewise continuous in $[0, A]$ for all $A > 0$

Al-Tememe Transform Method for Solving linear Systems of Ordinary Differential Equations:

Let use consider we have alinear system of ordinary differential equation of first order with variable coefficients which we can write it by :

$$\begin{aligned} xy_1' &= a_{11}y_1 + a_{12}y_2 + g_1(x) \\ &\;\;\cdot \\ xy_2' &= a_{21}y_1 + a_{22}y_2 + g_2(x). \end{aligned} \qquad \dots (1)$$

Where a_{11} , a_{12} , a_{21} anda_{22} are constants , y_1' the derivative of function $y_1(x)$ and , y_2' the derivative of function $y_2(x)$,such that $y_1(x)$ and $y_2(x)$ are a continuous function and the ($\mathcal{T}$.T) of $g_1(x)$ and $g_2(x)$ are known. And for solve the system (1) we take ($\mathcal{T}$.T) to both sides of (1) , and after simplification we put $Y_1 = \mathcal{T}(y_1)$, $Y_2 = \mathcal{T}(y_2)$, $G_1 = \mathcal{T}(g_1)$, $G_2 = \mathcal{T}(g_2)$ so,we get:

$$\begin{aligned} (p-1)Y_1 - y_1(1) &= a_{11}Y_1 + a_{12}Y_2 + G_1(p) \\ (p-1)Y_2 - y_2(1) &= a_{21}Y_1 + a_{22}Y_2 + G_2(p). \end{aligned} \qquad \dots (2)$$

So,

$$(p - 1 - a_{11})Y_1 - a_{12}Y_2 = y_1(1) + G_1(p) \qquad \dots (3)$$

$$-a_{21}Y_1 + (p - 1 - a_{22})Y_2 = y_2(1) + G_2(p) \qquad \dots (4)$$

By multiplying eq. (3)by a_{21} and eq. (4) by $(p - 1 - a_{11})$.

and collecting the result terms we have :

$$Y_2 = \frac{h_2(p)}{k_2(p)} \qquad \dots (5)$$

$$\text{By the similar method , we find} Y_1 = \frac{h_1(p)}{k_1(p)} \qquad \dots (6)$$

Where h_1 , h_2 , k_1 and$k_1$are polynomials of p , such that the degree of h_1 is less than the degree of k_1 and the degree of h_2 is less than the degree of k_2 .By taking the inverse of Al-Tememe transformation ($\mathcal{T}^{-1}$. T) to both sides of equations (5) and (6) we get:

$$y_1 = \mathcal{T}^{-1}\left[\frac{h_1(p)}{k_1(p)}\right]$$
$$y_2 = \mathcal{T}^{-1}\left[\frac{h_2(p)}{k_2(p)}\right] \quad \dots(7)$$

Equations (7) represents the general solution of system (1) which we can be written it as follows:

$$y_1 = A_1 g_1(x) + A_2 g_2(x) + \cdots + A_m g_m(x)$$
$$y_2 = B_1 g_1(x) + B_2 g_2(x) + \cdots + B_m g_m(x) \quad \dots(8)$$

Where as $g_1, g_2, \dots, g_m$ are function of x, and$A_1, A_2, \dots, A_m$are constants, which is number equals to the degree of $k_1(p)$also $B_1, B_2, \dots, B_m$are constants which is ,number equals to the degree of$k_2(p)$ .To find the values of constants of$A_1, A_2, \dots, A_m$ and $B_1, B_2, \dots, B_m$ we use the initial conditions $y_1(1)$and$y_2(1)$ in system:

But the conditions $y_1(1)$ and $y_2(1)$ are not enough to find out the above constants, thus we find $y_1'(1), \dots, y_1^{(m-1)}(1)$ and $y_2'(1), \dots, y_2^{(m-1)}(1)$ by using system (1) so we get the number of equations equal to $2m$ of initial conditions. Also we use the equation (8) for finding derivatives and by using the initial conditions , we get $2m$ equations.which is formed linear system this linear system can be solved to obtain the values of $A_1, A_2, \dots, A_m$ and$B_1, B_2, \dots, B_m$.

Example 1: For solving the system:

$$xy_1' = y_1 + 4y_2 + 2$$
$$xy_2' = y_1 - 2y_2 + x^{-1} \qquad ; y_1(1) = y_2(1) = 0$$

We take Al-Tememe transform to both sides of above system we get :

$$(p-1)Y_1 - y_1(1) = Y_1 + 4Y_2 + \frac{2}{p-1} \quad \dots(10)$$

$$(p-1)Y_2 - y_2(1) = Y_1 - 2Y_2 + \frac{1}{p} \quad \dots(11)$$

By multiplying eq. (10) by 1 and eq. (11) by $(p-2)$,we get

$$(p-2)Y_1 - 4Y_2 = \frac{2}{p-1} \quad \dots(12)$$

$$-(p-2)Y_1 + (p-2)(p+1)Y_2 = \frac{p-2}{p} \quad \dots(13)$$

From equations (12) and (13) we get :

$$Y_2 = \frac{h_1(p)}{p(p-1)(p-3)(p+2)} \; ; \; h_1(p) = p^2 - p + 2$$

And

$$Y_1 = \frac{h_2(p)}{p(p-1)(p-3)(p+2)} \; ; \; h_2(p) = 2p^2 + 6p - 4$$

Therefore,(after using $\mathcal{T}^{-1}$.T)we get:

$$y_2 = A_2 x^{-1} + B_2 + C_2 x^2 + D_2 x^{-3}$$

Also so , $y_1 = A_1 x^{-1} + B_1 + C_1 x^2 + D_1 x^{-3}$

The conditions $y_1(1)$ and $y_2(1)$ are not enough to get the above constants from the above two equations so we will substitute the two conditions and their derivatives in the differential equations (linear system) to get :

$$y_1'(1) = 2 \,, y_2'(1) = 1 \,, y_1''(1) = 4 \,, y_2''(1) = -2 \,, y_1'''(1) = -12 \,, y_2'''(1) = 14$$

By substituting these initial conditions in the derivatives of the general solution of $y_1(x)\,, y_2(x)$ we get:

$A_1 + B_1 + C_1 + D_1 = 0$

$-A_1 + 2C_1 - 3D_1 = 2$

$2A_1 + 2C_1 + 12D_1 = 4$

$-6A_1 - 60D_1 = -12$

By solving these equations we get:

$A_1 = {}^{-2}/_{3}\ \ , B_1 = -{}^{2}/_{3}\ \ , C_1 = {}^{16}/_{15}\ \ , D_1 = {}^{4}/_{15}$

$\Rightarrow \quad y_1 = {}^{-2}/_{3}\, x^{-1} - {}^{2}/_{3} + {}^{16}/_{15}\, x^{2} + {}^{4}/_{15}\, x^{-3}$

By the same method we find :

$A_2 = {}^{1}/_{3}\ \ , B_2 = {}^{-1}/_{3}\ \ , C_2 = {}^{4}/_{15}\ \ , D_2 = {}^{-4}/_{15}$

$\Rightarrow \quad y_2 = {}^{1}/_{3}\, x^{-1} - {}^{1}/_{3} + {}^{4}/_{15}\, x^{2} - {}^{4}/_{15}\, x^{-3}$

Example 2 : for solving the system:

$$xy_1' = 2y_1 - y_2 + x$$
$$xy_2' = 3y_1 - 2y_2 + \ln x \qquad ; y_1(1) = y_2(1) = 1$$

By using Al-Tememe transform to both sides of above system we get :

$$(p-1)Y_1 - y_1(1) = 2Y_1 - Y_2 + \frac{1}{p-2} \qquad \dots (14)$$

$$(p-1)Y_2 - y_2(1) = 3Y_1 - 2Y_2 + \frac{1}{(p-1)^2} \qquad \dots (15)$$

Therefore

$$(p-3)Y_1 + Y_2 = \frac{p-1}{p-2} \qquad \dots (16)$$

$$(p+1)Y_2 - 3Y_1 = \frac{p^2 - 2p + 2}{(p-1)^2} \qquad \dots (17)$$

By multiply eq. (16) by 3 and eq. (17) by $(p-3)$we get:

$$3(p-3)Y_1 + \quad 3Y_2 \quad = \frac{3(p-1)}{p-2} \qquad \dots (18)$$

$$-3(p-3)Y_1 + (p-3)(p+1)Y_2 = \frac{(p-3)(p^2 - 2p + 2)}{(p-1)^2} \qquad \dots (19)$$

From equations (18) and (19) we get:

$$Y_2 = \frac{h_1(p)}{p(p-1)^2(p-2)^2}\ ; h_1(p) = p^4 - 4p^3 + 9p^2 - 13p + 9$$

And

$$Y_1 = \frac{h_2(p)}{p(p-1)^2(p-2)^2}\ ; h_2(p) = p^4 - 3p^3 + 4p^2 - 4p + 3$$

Therefore,

$$y_1 = A_1 x^{-1} + B_1 + C_1 \ln x + D_1 x + E_1 x \ln x$$
$$y_2 = A_2 x^{-1} + B_2 + C_2 \ln x + D_2 x + E_2 x \ln x$$

The conditions $y_1(1)$ and $y_2(1)$ are not enough to get out the constants from above equations so, we will substitute the two conditions and their derivatives in the differential equations (linear system) to get :-

$$y_1'(1) = 2\ , y_2'(1) = 1\ , y_1''(1) = 2\ , y_2''(1) = 4\ ,\ y_1'''(1) = -4\ ,\ y_2'''(1) = -11\ , y_1^{(4)}(1) = 15\ , y_2^{(4)}(1) = 45$$

By substituting these initial conditions in the general solution of $y_1(x)\,, y_2(x)$ and its derivatives we get:

$A_1 = 3/4\,, B_1 = 0\ , C_1 = 1\ , D_1 = 1/4\,,\ E_1 = 3/2$

$\Longrightarrow\ y_1 = 3/4\, x^{-1} + ln\, x + 1/4\, x + 3/2\, x\, ln\, x$

By the same method we find :

$A_2 = 9/4\ \ , B_2 = -1\ , C_2 = 2\ , D_2 = -1/4\,,\ E_2 = 3/2$

$\Longrightarrow\ y_2 = 9/4\, x^{-1} + 2ln\, x - 1/4\, x + 3/2\, x\, ln\, x - 1$

REFERENCES

[1]. Gabriel Nagy , “ Ordinary Differential Equations ” Mathematics Department, Michigan State University,East Lansing, MI, 48824.October 14, 2014.

[2]. Herbert Kreyszig , Edward J. Norminton , “ Advanced Engineering Mathematics ” Ohio State University Columbus, Ohio,Tenth edition, 2011.

[3]. Mohammed, A.H. ,AtheraNemaKathem, “ Solving Euler’s Equation by Using New Transformation”, Karbala university magazine for completely sciences ,volume (6), number (4. (2008).

[4]. William F. Trench , “ Elementary Differential Equations ” Trinity University, 2013.