Vol.3.Issue.3.2015 (Jul-Sept)

http://www.bomsr.com

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

ISSN:2348-0580

RESEARCH ARTICLE

ON THE BINARY QUADRATIC DIOPHANTINE EQUATION

$$x^2 - 4xy + y^2 + 32x = 0$$

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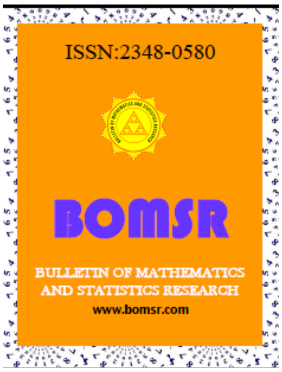

ABSTRACT

The binary quadratic equation $x^2 - 4xy + y^2 + 32x = 0$ representing the hyperbola is studied for its non-zero distinct integer solutions. A few interesting properties among the solutions are presented. Employing the integer solutions of the equation under consideration, integer solutions for special hyperbolas and parabolas are exhibited.

KEYWORDS: Binary quadratic, Integer solutions.

2010 Mathematics Subject Classification : 11D09

INTRODUCTION

The binary quadratic diophantine equations (both homogeneous and non-homogeneous) are rich in variety [1-6]. In [7-18] the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. These results have motivated us to search for infinitely many non-zero integral solutions of an another interesting binary quadratic equation given by $x^2 - 4xy + y^2 + 32x = 0$. The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited.

METHOD OF ANALYSIS:

The binary quadratic equation representing hyperbola is

$$x^2 - 4xy + y^2 + 32x = 0 \tag{1}$$

To start with it is seen that (1) is satisfied by

$(16,6),(16,48),(-32,-128),(48,144),(-128,-512)$.

However, we have solutions for (1) which are illustrated below:

Treating (1) as quadratic in x and solving for x, we have

$$x=\frac{1}{2}\left[4y-32\pm\sqrt{16y^2-256y+32^2-4y^2}\right] \tag{2}$$

To eliminate the square root on the R.H.S of (2), assume

$$\alpha^2=3y^2-64y+256$$

Multiplying both sides of the above equation by 3 and performing a few simplifications, we have

$$Y^2=3\alpha^2+256 \tag{3}$$

where $Y=3y-32$ $\quad(4)$

The smallest positive integer solution to (3) is

$$\alpha_0=16,\ Y_0=32. \tag{5}$$

To find the other solutions of (3), consider the pellian equation

$$Y^2=3\alpha^2+1 \tag{6}$$

where initial positive solution is $\tilde{\alpha}_0=1, \tilde{Y}_0=2.$

The general solution $(\tilde{\alpha}_s, \tilde{Y}_s)$ of (6) is given by

$$\tilde{\alpha}_s=\frac{1}{2\sqrt{3}}g_s$$

$$\tilde{Y}_s=\frac{1}{2}f_s$$

where

$$f_s=\left(2+\sqrt{3}\right)^{s+1}+\left(2-\sqrt{3}\right)^{s+1} \text{ and}$$

$$g_s=\left(2+\sqrt{3}\right)^{s+1}-\left(2-\sqrt{3}\right)^{s+1}.$$

The other values of α and Y satisfying (3) are obtained by applying Brahmagupta lemma between the solutions (α_0, Y_0) and $(\tilde{\alpha}_s, \tilde{Y}_s)$. Then,

$$\left.\begin{aligned}\alpha_{s+1}&=\alpha_0\tilde{Y}_s+Y_0\tilde{\alpha}_s=16\tilde{Y}_s+32\tilde{\alpha}_s=8f_s+\frac{16\sqrt{3}}{3}g_s\\ Y_{s+1}&=Y_0\tilde{Y}_s+3\alpha_0\tilde{\alpha}_s=32\tilde{Y}_s+48\tilde{\alpha}_s=16f_s+8\sqrt{3}g_s\end{aligned}\right\} \tag{7}$$

In view of (4), we have

$$y_{s+1}=\frac{Y_{s+1}+32}{3}, s=0,2,4....$$

$$y_{s+1}=\frac{16f_s+8\sqrt{3}g_s+32}{3}, s=0,2,4..... \tag{8}$$

Substituting (7) and (8) in (2) and taking the positive sign before the squareroot on the R.H.S of (2), we have

$$x_{s+1}=2y_{s+1}-16+\alpha_{s+1}$$

$$x_{s+1}-2y_{s+1}+16=8f_s+\frac{16\sqrt{3}}{3}g_s$$

$$3x_{s+1}-6y_{s+1}+48=24f_s+16\sqrt{3}g_s \tag{9}$$

Then (8) and (9) represent the non-zero distinct integer solutions to (1).

NOTE:

Substituting (7) and (8) in (2) and taking the negative sign before the squareroot on the R.H.S of (2), another set of x values are obtained and they are given by,

$$x_{s+1} = 2y_{s+1} - 16 - \alpha_{s+1}$$

A few numerical examples are given in the table below:

Table 1: Examples

s	y_{s+1}	x_{s+1}
0	48	144
2	528	1936
4	7216	26896
6	100368	374544
8	1397808	5216656
10	19468816	72658576
12	271165488	1012003344

The recurrence relations for the values of x and y are obtained as follows:

First, we obtain the recurrence relation for y:

Now, (8) is written as

$$3y_{s+1} - 32 = 16f_s + 8\sqrt{3}g_s \tag{10}$$

Replacing s by s+2 in (10) we have,

$$3y_{s+3} - 32 = 208f_s + 120\sqrt{3}g_s \tag{11}$$

Again replacing s by s+4 in (10) we have,

$$3y_{s+5} - 32 = 2896f_s + 1672\sqrt{3}g_s \tag{12}$$

Simplifying the above three equations we obtain the recurrence relation for y as,

$$y_{s+5} - 14y_{s+3} + y_{s+1} = -128, y_1 = 48, y_3 = 528$$

Similarly, we obtain the recurrence relation for x:

Now, (9) is written as,

$$3x_{s+1} - 6y_{s+1} + 48 = 24f_s + 16\sqrt{3}g_s \tag{13}$$

Replacing s by s+2 in (13) we have,

$$3x_{s+3} - 6y_{s+3} + 48 = 360f_s + 208\sqrt{3}g_s \tag{14}$$

Again replacing s by s+4 in (13) we have,

$$3x_{s+5} - 6y_{s+5} + 48 = 5016f_s + 2896\sqrt{3}g_s \tag{15}$$

Simplifying the above three equations we obtain the recurrence relation for x as,

$$x_{s+5} - 14x_{s+3} + x_{s+1} = -64, x_1 = 144, x_3 = 1936$$

A few interesting properties satisfied by the solutions of (1) are presented below:

- x_{s+1} and y_{s+1} are always even.
- $x_{s+1} \equiv 0 \pmod{16}$
- $y_{s+1} \equiv 0 \pmod{16}$
- $x_{3s+1} \equiv 0 \pmod{12}$
- $y_{3s+1} \equiv 0 \pmod{12}$
- $x_{3s+1} \equiv 2 \pmod{10}$

- $y_{3s+1} \equiv 4(\bmod 10)$
- $y_{s+5} = 14y_{s+3} - y_{s+1} - 128$
- $4x_{s+1} = y_{s+1} + y_{s+3}$
- $4x_{s+3} = 15y_{s+3} - y_{s+1} - 128$
- $4x_{s+5} = 8y_{s+5} + 97y_{s+3} - 7y_{s+1} - 1024$
- $15y_{s+3} - y_{s+5} - 4x_{s+1} = 128$
- $56x_{s+1} - 15y_{s+1} - y_{s+5} = 128$
- $y_{s+5} + y_{s+3} - 4x_{s+3} = 0$
- $56x_{s+3} - y_{s+1} - 15y_{s+5} = 128$
- $15y_{s+5} - y_{s+3} - 4x_{s+5} = 128$
- $209y_{s+5} - 56x_{s+5} - y_{s+1} = 1920$
- $4y_{s+3} - x_{s+1} - x_{s+3} = 32$
- $15x_{s+1} - x_{s+3} - 4y_{s+1} = 32$
- $2y_{s+5} + 26y_{s+3} - 7x_{s+1} - x_{s+5} = 256$
- $97x_{s+1} - x_{s+5} - 26y_{s+1} + 2y_{s+5} = 256$
- $7x_{s+3} - x_{s+5} - 2y_{s+3} + 2y_{s+5} = 32$
- $16y_{s+5} - 4x_{s+5} - 15y_{s+3} + y_{s+1} = 0$
- $32x_{s+3} - 4x_{s+5} - 23y_{s+3} + 8y_{s+5} + y_{s+1} = 0$
- Each of the following is a Nasty number:

$$135y_{2s+2} - 9y_{2s+4} - 1152$$

$$36y_{2s+2} - 9x_{2s+2} - 288$$

$$\frac{3}{7}(78y_{2s+6} - 627x_{2s+4} + 1254y_{2s+4} - 10752)$$

$$\frac{3}{7}(1086y_{2s+4} - 45x_{2s+6} + 90y_{2s+6} - 12192)$$

- Each of the following is a cubical integer:

$$2(45y_{3s+3} - 3y_{3s+5} + 135y_{s+1} - 9y_{s+3} - 1792)$$

$$12y_{3s+3} - 3x_{3s+3} + 36y_{s+1} - 9x_{s+1} - 448$$

$$\frac{1}{7}(78y_{3s+7} - 627x_{3s+5} + 1254y_{3s+5} + 234y_{s+5} - 1881x_{s+3} + 3762y_{s+3} - 43456)$$

$$\frac{1}{7}(1086y_{3s+5} - 45x_{3s+7} + 90y_{3s+7} + 3258y_{s+3} - 135x_{s+5} + 270y_{s+5} - 49216)$$

NOTE: Treating (1) as a quadratic in y and following the procedure presented above the other two sets of values of x and y satisfying (1) are given by

Set 1: $x_{s+1} = \frac{1}{3}(X_{s+1}+16)$ where $X = 3x - 16$ and $y_{s+1} = 2x_{s+1} + \alpha_{s+1}$

Set 2: $x_{s+1} = \frac{1}{3}(X_{s+1}+16)$ where $X = 3x - 16$ and $y_{s+1} = 2x_{s+1} - \alpha_{s+1}$

REMARKABLE OBSERVATIONS:

- Each of the following relations represents the hyperbola:

X	Y	HYPERBOLA
$45y_{s+1} - 3y_{s+3} - 448$	$y_{s+3} - 13y_{s+1} + 128$	$X^2 - 12Y^2 = 4096$
$3y_{3s+5} - 45y_{3s+3} + \frac{1}{1024}(45y_{s+1} - 3y_{s+3} - 448)^3 + 448$	$y_{s+3} - 13y_{s+1} + 128$	$X^2 - 108Y^2 = 36864$
$12y_{s+1} - 3x_{s+1} - 112$	$2x_{s+1} - 7y_{s+1} + 64$	$X^2 - 3Y^2 = 256$
$3x_{3s+3} - 12y_{3s+3} + \frac{1}{64}(12y_{s+1} - 3x_{s+1} - 112)^3 + 112$	$2x_{s+1} - 7y_{s+1} + 64$	$X^2 - 27Y^2 = 2304$
$78y_{s+5} - 627x_{s+3} + 1254y_{s+3} - 10864$ And $1086y_{s+3} - 45x_{s+5} + 90y_{s+5} - 12304$	$362x_{s+3} - 724y_{s+3} - 45y_{s+5} + 6272$ And $26x_{s+5} - 627y_{s+3} - 52y_{s+5} + 7104$	$X^2 - 3Y^2 = 12544$
$627x_{3s+5} - 78y_{3s+7} - 1254y_{3s+5} + 10864 + \frac{1}{56^2}(78y_{s+5} - 627x_{s+3} + 1254y_{s+3} - 10864)^3$ And $45x_{3s+7} - 1086x_{3s+7} - 90y_{3s+7} + 12304 + \frac{1}{56^2}(1086y_{s+3} - 45x_{s+5} + 90y_{s+5} - 12304)^3$	$362x_{s+3} - 724y_{s+3} - 45y_{s+5} + 6272$ And $26x_{s+5} - 627y_{s+3} - 52y_{s+5} + 7104$	$X^2 - 27Y^2 = 112896$

- Each of the following relations represents the parabola:

X	Y	PARABOLA
$45y_{2s+2} - 3y_{2s+4} - 384$	$3y_{s+3} - 39y_{s+1} + 384$	$Y^2 = 24X - 3072$
$12y_{2s+2} - 3x_{2s+2} - 96$	$2x_{s+1} - 7y_{s+1} + 64$	$3Y^2 = 8X - 256$
$26y_{2s+6} - 209x_{2s+4} + 418y_{2s+4} - 3584$ And $362y_{2s+4} - 15x_{2s+6} + 30y_{2s+6} - 4064$	$362x_{s+3} - 724y_{s+3} - 45y_{s+5} + 6272$ And $26x_{s+5} - 627y_{s+3} - 52y_{s+5} + 7104$	$3Y^2 = 168X - 12544$

CONCLUSION

In this paper , we have made an attempt to obtain a complete set of non-trivial distinct solutions for the non-homogeneous binary quadratic equation. To conclude , one may search for other choices of solutions to the considered binary equation and further , quadratic equations with multi-variables.

Acknowledgement

The financial support from the UGC, New Delhi (F.MRP-5122/14 (SERO/UGC) dated March 2014) for a part of this work is gratefully acknowledged.

References:

[1]. Banumathy.T.S., (1995)A Modern Introduction to Ancient Indian Mathematics,Wiley Eastern Limited, London.

[2]. Carmichael, R.D., (1950)The Theory of Numbers and Diophantine Analysis, Dover Publications, New York.

[3]. Dickson. L. E., (1952)History of The Theory of Numbers, Vol.II, Chelsia Publicating Co, New York.

[4]. Mordell,L,J., (1969)Diophantine Equations, Acadamic Press,London.

[5]. Nigel,P.Smart., (1999)TheAlgorithm Resolutions of Diaphantine eqations, Cambridge University, Press, London.

[6]. Telang ,S.G., (1996) Number theory, Tata Mc Graw-Hill Publishing Company , NewDelhi.

[7]. Gopalan,M.A., and Parvathy,G., (2010) "Integral Points On The Hyperbola $x^2+4xy+y^2-2x-10y+24=0$,"Antarctica J.Math,Vol 1(2),149-155.

[8]. Gopalan,M.A, Vidhyalakahmi,S, Sumathi.G and Lakshmi.K, Sep (2010) ""Integral Pionts On The Hyperbola $x^2+6xy+y^2+40x+8y+40=0$" , Bessel J.Math. Vol 2(3),159-164 .

[9]. Gopalan,M.A.,Gokila,K.,andVidhyalakahmi,S., (2007) "On the Diophantine Equation $x^2+4xy+y^2-2x+2y-6=0$", ActaCienciaIndica, Vol.XXXIIIM No2, p. 567-570. Telang ,S.G., Number theory, Tata Mc Graw-Hill Publishing Company , NewDelhi(1996).

[10]. Gopalan,M.A.,Vidhyalakahmi,S.,andDevibala,S., (2007). On The Diophantine Equation $3x^2+xy=14$,Acta CienciaIndica,Vol.XXXIII M.No2,P.645-646.

[11]. Gopalan.M.A., and Janaki.G., (2008) "Observations on $x^2-y^2+x+y+xy=2$", ImpactJ.Sci.,Tech, Vol2(3)p.14, 3-148.

[12]. Gopalan.M.A., and., Shanmuganadham,P.,and Vijayashankar,A., (2008)"On Binary Quadratic Equation $x^2-5xy+y^2+8x-20y+15=0$", Acta Ciencia Indica,Vol . XXXIVM. No.4,p.1803-1805c.

[13]. Gopalan,M.A, Vidhyalakahmi,S, Lakshmi.K and Sumathi.G, (2012) " Observation on $3x^2+10xy+4y^2-4x+2y-7=0$" , Diophantus J.Maths.Vol.1(2),123-125.

[14]. Mollion,R.A, (1998) "All Solutions of the Diophatine Equations $X^2-DY^2=n$" Far EastJ,Math.Sci., Speical Volume,Part III,p.257-293.

[15]. Vidhyalakshmi,S, Gopalan,M.A and Lakshmi.K, (2014) " Observation On The Binary Quadratic Equation $3x^2-8xy+3y^2+2x+2y+6=0$", Scholar Journal of Physics, Mathematics and Statistics, Vol.1(2), (Sep-Nov), 41-45.

[16]. Vidhyalakshmi. S, Gopalan,M.A and Lakshmi.K, August(2014) "Integer Solution of the Binary Quadratic Equation $x^2-5xy+y^2+33x=0$", International Journal of Innovative Science Engineering &Technology, Vol.1(6), 450-453.

[17]. Vidhyalakshmi. S, Gopalan, M.A, Kavitha. A and Mary Madona. D, June(2015)"On the Binary Quadratic Diophantine Equation $x^2-3xy+y^2+33x=0$", IJSEAS, Vol.1(4), 222-225.

[18]. Gopalan,M.A, Vidhyalakshmi. S, Presenna. R and Lakshmi. K, June(2015)"On the Binary Quadratic Diophantine Equation $x^2-3xy+y^2+10x=0$", IJESM, Vol.4(2), 68-77.