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DUAL SPACES OF GENERALIZED RIESZ SEQUENCE SPACE AND RELATED MATRIX MAPPING

MD. FAZLUR RAHMAN[1], A B M REZAUL KARIM[*2]

[1,2]Department of Mathematics, Eden University College, Dhaka, Bangladesh

A B M REZAUL KARIM

ABSTRACT

In this paper we define the Riesz sequence spaces r^q(u, p) and determine its Kothe-Toeplitz duals. We also establish necessary and sufficient conditions for a matrix A to map $r^q(u,p)$ to l_∞ and $r^q(u,p)$ to c, where l_∞ is the space of all bounded sequences and c is the space of all convergent sequences.

Keywords: Sequence space, Kothe-Toeplitz dual, Matrix transformation.

2000 Mathematics Subject Classification: 46A45; 40C05.

1. INTRODUCTION

Let ω be the space of all sequences real or complex and l_∞, c and c_0 are respectively the space of all bounded sequences, convergent sequences and null sequences.

The main purpose of this paper is to define and investigate the Riesz sequence space $r^q(u,p)$ and prove that it is a complete paranormed space. Later we determine the β-dual (Kothe-Toeplitz dual) of $r^q(u,p)$ and characterize the class of matrices $(r^q(u,p), l_\infty)$ and $(r^q(u,p), c)$.

If (q_n) is a positive sequence of real numbers then for $p = (p_r)$ with $\inf p_r > 0$, we define the Riesz sequence space $r^q(u,p)$ by

$$r^q(u,p) = \left\{x = (x_k)\,\epsilon\,\omega: \sum_{r=0}^{\infty}\left|\frac{1}{Q_{2^r}}\sum_r u_k\, q_k x_k\right|^{p_r} < \infty\right\}$$

where $\sum_r$ *denotes* a sum over the ranges $2^r \leq k < 2^{r+1}$ and

$$Q_{2^r} = \sum_r q_k = q_{2^r} + q_{2^r+1} \ldots\ldots\ldots\ldots\ldots + q_{2^{r+1}-1}.$$

With regard to notation, the dual space of $r^q(u,p)$, that is, the space of all continuous linear functional on $r^q(u,p)$ will be denoted by $[r^q(u,p)]^*$.

In their paper Sheikh and Ganie [10] defined $r^q(u,p)$ in a different norm with the help of [2, 3, 4] and studied completeness and consider some matrix mapping. If $(u_k) = e = (1,1,1, \dots\dots)$ then the sequence space $r^q(u,p)$ of [10] reduces to $r^q(p)$ which introduced by Atlay and Basar [1].

Throughout the paper the following well known inequality (see[7] or [8]) will be frequently used. For any integer $E > 1$ and two complex numbers a and b we have

$$|ab| \le E(|a|^t E^{-t} + |b|^p) \qquad (1)$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{t} = 1$.

To begin with, we show that the space $r^q(u,p)$ is a paranorm space paranormed by

$$g(x) = \left(\sum_{r=0}^{\infty} \left|\frac{1}{Q_{2^r}} \sum_r u_k\, q_k x_k\right|^{p_r}\right)^{1/M} \qquad (2)$$

provided $H = sup_r\ p_r < \infty$ and $M = \max\{1, H\}$.

Clearly

$$g(\theta) = 0 \text{ and } g(x) = g(-x), \text{ where } \theta = (0, 0, 0, \dots\dots\dots\dots\dots\dots..).$$

Since $p_r \le M$, for any $x, y \in r^q(u,p)$ we have

$$\sum_{r=0}^{\infty} \left|\frac{1}{Q_{2^r}} \sum_r u_k q_k (x_k + y_k)\right|^{p_r}$$

$$= \sum_{r=0}^{\infty} \left(\left|\frac{1}{Q_{2^r}} \sum_r u_k\, q_k x_k + \frac{1}{Q_{2^r}} \sum_r u_k\, q_k y_k\right|^{p_r/M}\right)^M$$

$$\le \sum_{r=0}^{\infty} \left(\left|\frac{1}{Q_{2^r}} \sum_r u_k\, q_k x_k\right|^{p_r/M} + \left|\frac{1}{Q_{2^r}} \sum_r u_k\, q_k y_k\right|^{p_r/M}\right)^M$$

and since $M \ge 1$, we see by Minkowski's inequality that g is subadditive.

Finally we have to check the continuity of scalar multiplication. From the definition of $r^q(u,p)$ we have $\inf p_r > 0$. So we may assume that $\inf p_r \equiv \rho > 0$. Now for any complex λ with $\|\lambda\| < 1$, we have

$$g(\lambda x) = \sum_{r=0}^{\infty} \left(\left|\frac{1}{Q_{2^r}} \sum_r u_k\, q_k \lambda x_k\right|^{p_r}\right)^{1/M}$$

$$\le \sup_r \|\lambda\|^{\frac{p_r}{M}} g(x)$$

$$\le \|\lambda\|^{\frac{\rho}{M}}\ g(x) \to 0 \text{ as } \lambda \to 0.$$

It is quite routine to show that $r^q(u,p)$ is a metric space with the metric $d(x,y) = g(x-y)$ provided that $x, y \in r^q(u,p)$, where g is defined by (2); And using a similar method to that in [5] one can show that $r^q(u,p)$ is complete under the metric mentioned above.

2. Duals

If X is a sequence space, then X^+ will denote the generalized Kothe-Toeplitz $(\beta - dual)$ of X.

$X^\beta = X^+ = \{a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \text{ converge for all } x \in X\}$.

Now we are giving the following theorem by which the generalized Kothe-Toeplitz dual will be determined.

Theorem 2.1. If $1 < p_r \le sup_r p_r < \infty$ and $p_r^{-1} + t_r^{-1} = 1,\ r = 0,1,2 \dots\dots\dots.$, then

$$[r^q(u,p)]^\beta = \left\{a = (a_k) : \sum_{r=0}^{\infty} \left|Q_{2^r} \max_r (u_k^{-1} q_k^{-1} a_k)\right|^{t_r} E^{-t_r} < \infty \text{ for some integer } E > 1\right\}$$

Proof. Let $1 < p_r \le sup_r\ p_r < \infty$ and $p_r^{-1} + t_r^{-1} = 1,\ r = 0,1,2 \dots\dots\dots$

Define

$$\mu(t) = \left\{a = (a_k): \sum_{r=0}^{\infty} \left|Q_{2^r} \max_r (u_k^{-1} q_k^{-1} a_k)\right|^{t_r} E^{-t_r} < \infty \; for\ some\ integer\ E > 1\right\} \qquad (3)$$

We want to show that $[r^q(u,p)]^\beta = \mu(t)$.

Let $x \in r^q(u,p)$ and $a \in \mu(t)$. Then using inequality (1), we get

$$\sum_{k=1}^{\infty} a_k x_k = \sum_{r=0}^{\infty} \sum_r (u_k^{-1} q_k^{-1} a_k)(u_k q_k x_k)$$

$$\leq \sum_{r=0}^{\infty} \max_r (u_k^{-1} q_k^{-1} a_k) \sum_r (u_k q_k x_k)$$

$$\leq \sum_{r=0}^{\infty} \left|Q_{2^r} \max_r (u_k^{-1} q_k^{-1} a_k) \frac{1}{Q_{2^r}} \sum_r (u_k q_k x_k)\right|$$

$$\leq E\left(\sum_{r=0}^{\infty} \left|Q_{2^r} \max_r (u_k^{-1} q_k^{-1} a_k)\right|^{t_r} E^{-t_r} + \sum_{r=0}^{\infty} \left|\frac{1}{Q_{2^r}} \sum_r (u_k q_k x_k)\right|^{p_r}\right)$$

$$< \infty$$

which implies that the series $\sum_{k=1}^{\infty} a_k x_k$ is convergent. Therefore,

$a \in [r^q(u,p)]^\beta$, that is, $[r^q(u,p)]^\beta \supset \mu(t)$

Conversely, suppose that $\sum_{k=1}^{\infty} a_k x_k$ is convergent for all $x \in r^q(u,p)$, but $a \notin \mu(t)$. Then

$$\sum_{r=0}^{\infty} \left|Q_{2^r} \max_r (u_k^{-1} q_k^{-1} a_k)\right|^{t_r} E^{-t_r} = \infty$$

for every integer $E > 1$.

So, we can define a sequence $0 = n(0) < n(1) < n(2) < \cdots \dots \dots \dots$, such that

$\gamma = 0, 1, 2, \dots \dots \dots \dots \dots$, we have

$M_\gamma = \sum_{r=n(\gamma)}^{n(\gamma+1)-1} \left|Q_{2^r} \max_r (u_k^{-1} q_k^{-1} a_k)\right|^{t_r} (\gamma + 2)^{-t_r/p_r} > 1.$

Now we define a sequence (see [5], [6], [9]) $x = (x_k)$ in the following way:

$x_{N(r)} = Q_{2^r} \left|Q_{2^r} \max_r (u_k^{-1} q_k^{-1} a_k)\right|^{t_r - 1} sgn\ a_{N(r)} (\gamma + 2)^{-t_r} M_\gamma^{-1}$

for $n(\gamma) \leq r \leq n(\gamma + 1) - 1, \gamma = 0, 1, 2, \dots \dots, and\ \ x_k = 0\ for\ k \neq N(r),$

where $N(r)$ is such that $a_{N(r)} = \max_r (u_k^{-1} q_k^{-1} a_k)$, the maximum is taken with respect to k in $[2^r, 2^{r+1})$.

Therefore ,

$$\sum_{r=2^{n(\gamma)}}^{2^{n(\gamma+1)}-1} a_k x_k = \sum_{r=n(\gamma)}^{n(\gamma+1)-1} a_{N(r)} Q_{2^r} \left|Q_{2^r} \max_r (u_k^{-1} q_k^{-1} a_k)\right|^{t_r - 1} (\gamma + 2)^{-t_r} M_\gamma^{-1}$$

$$= \sum_{r=n(\gamma)}^{n(\gamma+1)-1} a_{N(r)} Q_{2^r} \left|Q_{2^r} a_{N(r)}\right|^{t_r - 1} (\gamma + 2)^{-t_r} M_\gamma^{-1}$$

$$= M_\gamma^{-1} (\gamma + 2)^{-1} \sum_{r=n(\gamma)}^{n(\gamma+1)-1} \left|Q_{2^r} a_{N(r)}\right|^{t_r} (\gamma + 2)^{1-t_r}$$

$$= M_\gamma^{-1} (\gamma + 2)^{-1} \sum_{r=n(\gamma)}^{n(\gamma+1)-1} \left|Q_{2^r} \max_r (u_k^{-1} q_k^{-1} a_k)\right|^{t_r} (\gamma + 2)^{-t_r/p_r}$$

$$= M_\gamma^{-1} (\gamma + 2)^{-1} M_\gamma$$

$$= (\gamma + 2)^{-1}$$

diverges. Moreover

$$\sum_{r=n(\gamma)}^{n(\gamma+1)-1}\left|\frac{1}{Q_{2^r}}\sum_{r} u_k q_k x_k\right|^{p_r} = \sum_{r=n(\gamma)}^{n(\gamma+1)-1}\left|\frac{1}{Q_{2^r}} Q_{2^r}\left|Q_{2^r}\max_r (u_k^{-1} q_k^{-1} a_k)\right|^{t_r-1} (\gamma+2)^{-t_r} M_\gamma^{-1}\right|^{p_r}$$

$$= \sum_{r=n(\gamma)}^{n(\gamma+1)-1} \left|Q_{2^r} a_{N(r)}\right|^{(t_r-1)p_r} (\gamma+2)^{-p_r t_r} M_\gamma^{-p_r}$$

$$= \sum_{r=n(\gamma)}^{n(\gamma+1)-1} \left|Q_{2^r} a_{N(r)}\right|^{t_r} (\gamma+2)^{-p_r-t_r} M_\gamma^{-p_r}$$

$$= M_\gamma^{-1}(\gamma+2)^{-2} \sum_{r=n(\gamma)}^{n(\gamma+1)-1} \left|Q_{2^r} a_{N(r)}\right|^{t_r} (\gamma+2)^{2-p_r-t_r} M_\gamma^{1-p_r}$$

$$= M_\gamma^{-1}(\gamma+2)^{-2} \sum_{r=n(\gamma)}^{n(\gamma+1)-1} \left|Q_{2^r} a_{N(r)}\right|^{t_r} (\gamma+2)^{-t_r/p_r} M_\gamma^{1-p_r}(\gamma+2)^{2-p_r-t_r+t_r/p_r}$$

$$= M_\gamma^{-1}(\gamma+2)^{-2} M_\gamma^{1-p_r}(\gamma+2)^{2-p_r-t_r+t_r/p_r}$$

$$= (\gamma+2)^{-2} M_\gamma^{1-p_r}(\gamma+2)^{1-p_r}$$

$$= (\gamma+2)^{-2} M_\gamma^{-p_r t_r}(\gamma+2)^{-p_r t_r}$$

$$= \frac{(\gamma+2)^{-2}}{M_\gamma^{p_r t_r}(\gamma+2)^{p_r t_r}} < (\gamma+2)^{-2} < \infty$$

That is, $x \in r^q(u,p)$, which is a contradiction.

Hence $a \in \mu(t)$, that is, $\mu(t) \supset [r^q(u,p)]^\beta$. Then combining the above two results we get $[r^q(u,p)]^\beta = \mu(t)$.

Theorem 2.2. Let $1 < p_r \le \sup_r p_r < \infty$. Then $[r^q(u,p)]^*$ is isomorphic to $\mu(t)$ which is defined by (3).

Proof. It is easy to check that each $x \in r^q(u,p)$ can be written in the form

$$x = \sum_{k=1}^{\infty} x_k e_k \text{ , where } e_k = (0,0,\dots\dots.0,1,0,\dots\dots\dots..),$$

and the 1 appears at the k-th place. Then for any $f \in [r^q(u,p)]^*$ we have

$$f(x) = \sum_{k=1}^{\infty} x_k f(e_k) = \sum_{k=1}^{\infty} x_k a_k \qquad (4)$$

where $f(e_k) = a_k$. By Theorem 2.1, the convergence of $\sum_{k=1}^{\infty} x_k a_k$ for every x in $r^q(u,p)$ implies that $a \in \mu(t)$.

If $x \in r^q(u,p)$ and if we take $a \in \mu(t)$, then by theorem 2.1 , $\sum_{k=1}^{\infty} x_k a_k$ converges and clearly defines a linear functional on $r^q(u,p)$. Using the same kind of argument as in theorem 2.1 , it is easy to check that

$$\sum_{k=1}^{\infty} x_k a_k \le \sum_{k=1}^{\infty} |x_k a_k| \le E\left(\sum_{r=0}^{\infty}\left|Q_{2^r}\max_r (u_k^{-1} q_k^{-1} a_k)\right|^{t_r} E^{-t_r}+1\right) g(x)$$

when $g(x) \le 1$, where g(x) is defined by (2). Hence $\sum_{k=1}^{\infty} x_k a_k$ defines an element of $[r^q(u,p)]^*$.

Furthermore, it is easy to see that representation (4) is unique. Hence we can define a mapping $T: [r^q(u,p)]^* \to \mu(t)$.

By $T(f) = (a_1, a_2, \dots\dots\dots\dots..)$, where a_k appears in representation (4). It is evident that T is linear and bijective. Hence $[r^q(u,p)]^*$ is isomorphic to $\mu(t)$.

3. Matrix Transformations

Let $A = (a_{nk})$ be an infinite matrix of complex numbers $(a_{nk})_{n,k=1,2,\dots\dots\dots}$ and U, V be two subsets of the spaces of complex sequences. We say that the matrix A defines a matrix transformation from U *into* V and denote it by $A \in (U,V)$, if for every sequence $x = (x_k) \in U$ the sequence $A(x) = A_n(x)$ is in V, where

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk}\, x_k \qquad (5)$$

provided the series on the right is convergent.

In this section we characterize the class of matrices $(r^q(u,p), l_\infty)$ and $(r^q(u,p), c)$, where l_∞ and c are respectively the Banach spaces of all bounded and convergent sequence $x = (x_k)$ endowed with the norm $\|x\| = sup_k |x_k|$.

Theorem 3.1. Let $1 < p_r \le sup_r\, p_r < \infty$. Then $A \in (r^q(u,p), l_\infty)$ if and only if there exists an integer $E > 1$ such that

$$U(E) = \sup_n \sum_{r=0}^{\infty} \left| Q_{2^r} \max_r \left(u_k^{-1} q_k^{-1} a_k\right)\right|^{t_r} E^{-t_r} < \infty$$

and $p_r^{-1} + t_r^{-1} = 1$, $r = 0, 1, 2, 3, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Proof. Sufficiency. Suppose there exists an integer $E > 1$ such that $U(E) < \infty$. Then by inequality (1), we have

$$\sum_{k=1}^{\infty} a_{nk}\, x_k = \sum_{r=0}^{\infty} \sum_r a_{nk}\, x_k$$

$$= \sum_{r=0}^{\infty} \sum_r u_k^{-1}\, q_k^{-1} a_{nk}\, u_k q_k\, x_k$$

$$\le \sum_{r=0}^{\infty} \max_r \left(u_k^{-1}\, q_k^{-1} a_{nk}\right) \sum_r u_k q_k\, x_k$$

$$\le \sum_{r=0}^{\infty} \left| Q_{2^r} \max_r \left(u_k^{-1}\, q_k^{-1} a_{nk}\right) \frac{1}{Q_{2^r}} \sum_r u_k q_k x_k \right|$$

$$\le E\left(\sum_{r=0}^{\infty} \left| Q_{2^r} \max_r \left(u_k^{-1}\, q_k^{-1} a_{nk}\right)\right|^{t_r} E^{-t_r} + \sum_{r=0}^{\infty} \left| \frac{1}{Q_{2^r}} \sum_r u_k q_k x_k \right|^{p_r} \right)$$

$< \infty.$

Therefore, $A \in (r^q(u,p), l_\infty)$.

Necessity. Suppose that $A \in (r^q(u,p), l_\infty)$, but

$$\sup_n \sum_{r=0}^{\infty} \left| Q_{2^r} \max_r \left(u_k^{-1}\, q_k^{-1} a_{nk}\right)\right|^{t_r} E^{-t_r} = \infty$$

for every integer $E > 1$. Then $\sum_{k=1}^{\infty} a_{nk}\, x_k$ converges for every *n* and $x \in r^q(u,p)$, whence

$(a_{nk})_{n,k=1,2,\ldots\ldots\ldots\ldots} \in [r^q(u,p)]^\beta$

for every *n.*

By theorem 2.1 , it follows that each A_n defined by

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk}\, x_k$$

is an element of $[r^q(u,p)]^*$. Since $r^q(u,p)$ is complete and since $\sup_n |A_n(x)| < \infty$ on $r^q(u,p)$, by the uniform boundedness principle there exists a number L independent of *n* and *x*, and a number $\delta < 1$, such that

$$|A_n(x)| \le L \qquad (6)$$

For every *n* and $x \in S[\theta, \delta]$, where $S\,[\theta, \delta]$ is the closed sphere in $r^q(u,p)$ with center at the origin θ and radius δ.

Now choose an integer $G > 1$, such that

$$G\delta^M > L$$

Since

$$\sup_n \sum_{r=0}^{\infty} \left| Q_{2^r} \max_r \; (u_k^{-1} q_k^{-1} a_{nk}) \right|^{t_r} G^{-t_r} = \infty,$$

there exists an integer $m_0 > 1$ such that

$$R = \sum_{r=0}^{m_0} \left| Q_{2^r} \max_r \; (u_k^{-1} q_k^{-1} a_{nk}) \right|^{t_r} G^{-t_r} > 1. \tag{7}$$

Define a sequence $x = (x_k)$ as follows:

$$x_k = 0 \; if \;\; k \geq 2^{m_0+1}$$

$$x_{N(r)} = Q_{2^r} \delta^{M/p_r} \, sgn \, a_{nN(r)} \left| Q_{2^r} \, a_{nN(r)} \right|^{t_r - 1} R^{-1} \, G^{-t_r/p_r}$$

$$x_k = 0 \; if \;\; k \geq 2^{m_0+1}$$

$$x_k = 0 \; if \;\; k \neq N(r) \, for \;\; o \leq r \leq m_0,$$

where $N(r)$ is the smallest integer such that

$$a_{nN(r)} = \max_r \; (u_k^{-1} q_k^{-1} a_{nk}).$$

Then one can easily show that,

$g(x) \leq \delta \;\; but \;\; |A_n(x)| > L,$ which contradicts (6). This complete the proof of the theorem.

Theorem 3.2. Let $1 < p_r \leq sup_r \, p_r < \infty$. Then $A \in (r^q(u,p), c)$ if and only if

(i) there exists an integer $E > 1$, such that

$$U(E) = \sup_n \sum_{r=0}^{\infty} \left| Q_{2^r} \max_r \; (u_k^{-1} q_k^{-1} a_{nk}) \right|^{t_r} E^{-t_r} < \infty,$$

(ii) $a_{nk} \to \alpha_k \;\; (n \to \infty, \;\; k \, is \, fixed)$.

Proof. Necessity. Suppose $A \in (r^q(u,p), c)$. Then $A_n(x)$ exists for each $n \geq 1$, and $\lim_{n \to \infty} A_n(x)$ exists for every $x \in r^q(u,p)$. Therefore, by an argument similar to that in theorem 2.1, we have condition (i). Condition (ii) is obtained by taking $x = e_k \in r^q(u,p)$, where e_k is a sequence with 1 at the k-th place and zeros elsewhere.

Sufficiency. The condition of the theorem imply that

$$\sum_{r=0}^{\infty} \left| Q_{2^r} \max_r \; (u_k^{-1} q_k^{-1} a_k) \right|^{t_r} E^{-t_r} \leq U(E) < \infty \tag{8}$$

By (8), it is easy to check that $\sum_{k=1}^{\infty} a_k \, x_k$ is absolutely convergent for each $x \in r^q(u,p)$. For each $x \in r^q(u,p)$ and $\epsilon > 0$, we can choose an integer $m_0 \geq 1$ such that

$$g_{m_0}(x) = \sum_{r=m_0}^{\infty} \left| \frac{1}{Q_{2^r}} \sum_r u_k q_k x_k \right|^{p_r} < \epsilon^M.$$

Then by the proof of theorem 2.1 and by inequality (1) we have

$$\sum_{k=2^{m_0}}^{\infty} (a_{nk} - \alpha_k) \, x_k / (g_{m_0}(x))^{1/M} = \sum_{r=m_0}^{\infty} \sum_r (a_{nk} - \alpha_k) x_k / (g_{m_0}(x))^{1/M}$$

$$= \sum_{r=m_0}^{\infty} \sum_r u_k^{-1} \, q_k^{-1} (a_{nk} - \alpha_k) \, u_k q_k x_k / (g_{m_0}(x))^{1/M}$$

$$\leq \sum_{r=m_0}^{\infty} \max_r \; (u_k^{-1} \, q_k^{-1} \, (a_{nk} - \alpha_k)) \sum_r \; u_k q_k x_k / (g_{m_0}(x))^{1/M}$$

$$= \sum_{r=m_0}^{\infty} Q_{2^r} \max_r \left(u_k^{-1} q_k^{-1}(a_{nk} - \alpha_k)\right) \frac{1}{Q_{2^r}} \sum_r u_k q_k x_k / (g_{m_0}(x))^{1/M}$$

$$\leq \sum_{r=m_0}^{\infty} \left| Q_{2^r} \max_r \left(u_k^{-1} q_k^{-1}(a_{nk} - \alpha_k)\right) \frac{1}{Q_{2^r}} \sum_r u_k q_k x_k / (g_{m_0}(x))^{1/M} \right|$$

$$\leq \sum_{r=m_0}^{\infty} E\left[\left| Q_{2^r} \max_r \left(u_k^{-1} q_k^{-1}(a_{nk} - \alpha_k)\right) \right|^{t_r} E^{-t_r} + \left| \frac{1}{Q_{2^r}} \sum_r u_k q_k x_k / (g_{m_0}(x))^{1/M} \right|^{p_r} \right]$$

$$= E\left[\sum_{r=m_0}^{\infty} \left| Q_{2^r} \max_r \left(u_k^{-1} q_k^{-1}(a_{nk} - \alpha_k)\right) \right|^{t_r} E^{-t_r} + \sum_{r=m_0}^{\infty} \left| \frac{1}{Q_{2^r}} \sum_r u_k q_k x_k \right|^{p_r} / (g_{m_0}(x))^{p_r/M} \right]$$

$$= E\left[\sum_{r=m_0}^{\infty} \left| Q_{2^r} \max_r \left(u_k^{-1} q_k^{-1}(a_{nk} - \alpha_k)\right) \right|^{t_r} E^{-t_r} + g_{m_0}(x) / (g_{m_0}(x))^{p_r/M} \right]$$

$$\leq E\left[\sum_{r=m_0}^{\infty} \left| Q_{2^r} \max_r \left(u_k^{-1} q_k^{-1}(a_{nk} - \alpha_k)\right) \right|^{t_r} E^{-t_r} + 1 \right]$$

$< E(2U(E) + 1)$

That is ,

$$\sum_{k=2^{m_0}}^{\infty} (a_{nk} - \alpha_k) x_k < E(2U(E) + 1) \left(g_{m_0}(x)\right)^{\frac{1}{M}} = E(2U(E) + 1)\epsilon$$

Where

$$\sum_{r=m_0}^{\infty} \left| Q_{2^r} \max_r \left(u_k^{-1} q_k^{-1}(a_{nk} - \alpha_k)\right) \right|^{t_r} E^{-t_r} \leq 2U(E) < \infty.$$

It follows immediately that

$$\lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} x_k = \sum_{k=1}^{\infty} \alpha_k x_k$$

This shows that $A \in (r^q(u,p), c)$, which proves the theorem.

Corollary 3.3. Let $1 < p_r \leq \sup_r p_r < \infty$ and $p_r^{-1} + t_r^{-1} = 1,\ r = 0, 1, 2, 3, \dots \dots \dots \dots$

Then $A \in (r^q(u,p), c_0)$ if and only if

(i) there exists an integer $E > 1$, such that

$U(E) = \sup_n \sum_{r=0}^{\infty} \left| Q_{2^r} \max_r \left(u_k^{-1} q_k^{-1} a_{nk}\right) \right|^{t_r} E^{-t_r} < \infty,$

and

$(ii)\, a_{nk} \to 0.\ (n \to \infty, k \text{ is fixed})$

where c_0 is the space of all null sequences.

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