Vol.3.Issue.4.2015 (Oct-Dec.)

http://www.bomsr.com

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

ISSN:2348-0580

RESEARCH ARTICLE

COMMON FIXED POINT THEOREMS IN GENERALIZED INTUITIONISTIC FUZZY METRIC SPACES WITH PROPERTIES

R.MUTHURAJ[1], M. SORNAVALLI[2], M. JEYARAMAN[3]

[1]PG and Research Department of Mathematics, H.H.The Rajah's College, Pudukkottai – 622 001, India.E-mail : rmr1973@yahoo.co.in.

[2]Department of Mathematics, Velammal College of Engineering & Technology, Madurai – 625 009, India.E-mail: sornavalliv7@gmail.com

[3]PG and Research Department of Mathematics, Raja Dorai Singam Govt. Arts College Sivagangai -630 561, India. Email: jeya.math@gmail.com

M. SORNAVALLI

ABSTRACT

In this paper the notion of generalized intuitionistic fuzzy metric spaces are introduced and some properties are obtained. Two new common fixed point theorems are proved in generalized intuitionistic fuzzy metric spaces under some suitable conditions.

Key words: Fixed point, Generalized intuitionistic fuzzy metric spaces.

Mathematics subject classification: 54H25, 47H10.

1. INTRODUCTION

The concept of fuzzy sets, introduced by Zadeh [13] plays an important role in topology and analysis. Since then, there are many authors to study the fuzzy sets with applications. Especially Kramosil and Michlek [5] put forward a new concept of fuzzy metric space. George and Veeramani [3] revised the notion of fuzzy metric space with the help of continuous t-norm. As a result, many fixed point theorems for various forms of mappings are obtained in fuzzy metric spaces. Dhage [2] introduced the definition of D-metric space and proved many new fixed point theorems in D-metric spaces. Recently, Mustafa and Sims [7] presented a new definition of G-metric space and made great contribution to the development of Dhage theory.

In [12] Guangpeng Sun and Kai yang introduced the notion of Q- fuzzy metric space. In this study we introduce the notion of generalized intuitionistic fuzzy metric space, which can be considered as a generalization of fuzzy metric space. We show some new fixed point theorems in such generalized intuitionistic fuzzy metric spaces. The results presented in this paper improve and extend some known results.

2. GENERALIZED INTUITIONSTIC FUZZY METRIC SPACES

Definition 2.1 : A binary operation $*$: $[0,1] \times [0,1] \rightarrow [0,1]$ is a continuous t-norm if it satisfies the following conditions:

i) $*$ is associative and commutative

ii) $*$ is continuous

iii) $a * 1 = a$ for all $a \epsilon [0,1]$

iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Definition 2.2: A binary operation $\Diamond : [0,1] \times [0,1] \rightarrow [0,1]$ is said to be a continuous t- conorm if it satisfies the following conditions:

i) $\Diamond$ is associative and commutative.

ii) $\Diamond$ is continuous.

iii) $a \Diamond 0 = a$ for all $a \epsilon [0,1]$

iv) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Definition 2.3:[12] A 3-tuple (X, Q, $*$) is called a Q- fuzzy metric space if X is an arbitray (non-empty) set, $*$ is a continuous t-norm, and Q is a fuzzy set on $X^3 \times (0, \infty)$, satisfying the following conditions for each x, y, z, $a \epsilon X$ and $t, s > 0$:

i) $Q(x, x, y, t) > 0$ and $Q(x, x, y, t) \leq Q(x, y, z, t)$ for all $x, y, z \epsilon X$ with $z \neq y$

ii) $Q(x, y, z, t) = 1$ if and only if $x = y = z$

iii) $Q(x, y, z, t) = Q\{P(x, y, z), t\}$, (symmetry) where p is a permutation function,

iv) $Q(x, a, a, t) * Q(a, y, z, s) \leq Q(x, y, z, t+s)$

v) $Q(x, y, z, .) : (0,\infty) \rightarrow [0,1]$ is continuous

A Q – fuzzy metric space is said to be symmetric if $Q(x, y, y, t) = Q(x, x, y, t)$ for all $x, y \epsilon X$.

Definition 2.4: A 5-tuple (X, Q, H, $*$, $\Diamond$) is said to be an generalized intuitionstic fuzzy metric space (for short GIFMS) if X is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and Q, H are fuzzy set on $X^3 \rightarrow (0, \infty)$ satisfying the following conditions. For every x, y, z, $a \epsilon X$ and $t, s > 0$

i) $Q(x, y, z, t) + H(x, y, z, t) \leq 1$

ii) $Q(x, x, y, t) > 0$, for all $x \neq y$

iii) $Q(x, x, y, t) \leq Q(x, y, z, t)$ for $y \neq z$

iv) $Q(x, y, z\ t) = 1$ iff $x = y = z$

v) $Q(x, y, z, t) = Q\{p(x, y, z), t\}$, where p is a permutation function.

vi) $Q(x, a, a\ t) * Q(a, y, z, s) \leq Q(x, y, z, t+s)$

vii) $Q(x, y, z, .) : (0,\infty) \rightarrow [0,1]$ is continuous

viii) Q is non decreasing function on R^+ $\lim_{t\to\infty} Q(x, y, z, t) = 1$ and

$\lim_{t\to 0} Q(x, y, z, t) = 0$, for all $x, y, z \epsilon X$, $t > 0$

ix) $H(x, x, y, t) < 1$, for all $x \neq y$

x) $H(x, x, y, t) \geq H(x, y, z, t)$ for $y \neq z$

xi) $H(x, y, z\ t) = 0$ iff $x = y = z$

xii) $H(x, y, z, t) = H\{p(x, y, z), t\}$ where p is a permutation function.

xiii) $H(x, a, a\ t) \Diamond H(a, y, z, s) \geq H(x, y, z, t+s)$

xiv) $H(x, y, z, .) : (0,\infty) \rightarrow [0,1]$ is continuous

xv) H is a non- increasing function on R^+ $\lim_{t\to 0} H(x, y, z, t) = 0$ and

$\lim_{t\to 0} H(x, y, z, t) = 1$ for all $x, y, z \epsilon X$, $t > 0$

In this case, the pair (Q, H) is called an generalized intuitionistic fuzzy metric on X.

Example 2.5: Let (X, Q) be a Q- metric space, for all $x, y, z \epsilon X$ and every $t > 0$, consider Q, H to be fuzzy sets on $X^3 \times (0, \infty)$ defined by

$Q(x, y, z, t) = \frac{t}{t+Q(x,y,z,t)}$ and $H(x,y,z,t) = \frac{Q(x,y,z,t)}{t+Q(x,y,z,t)}$ and denote $a * b = ab$ and $a \lozenge b = \min\{a+b, 1\}$. Then $(X, Q, H, *, \lozenge)$ is an generalized intuitionistic fuzzy metric space. Notice that the above example holds even with the t- norm $a * b = \min\{a, b\}$ and t – conorm $a \lozenge b = \max\{a, b\}$.

Remark 2.6 : In an generalized intuitionistic fuzzy metric space Q(x, y, z, .) is non -decreasing and H(x, y, z, .) is non-increasing for all $x, y, z \in X$.

Definition 2.7 : Let $x \in X$, where $(X, Q, H, *, \lozenge)$ is an generalized intuitionistic fuzzy metric space. Then, for $r \in (0, 1)$ and $t > 0$, the set $B_{Q,H}(x, r, t) = \{y \in X : Q(x, y, y, t) > 1- r$ and $H(x, y, y, t) < r\}$ is said to be an open ball with centre x and radius r with respect to t. Note that every open ball $B_{Q,H}(x, r, t)$ is an open set.

Definition 2.8 : Let $(X, Q, H, *, \lozenge)$ be an generalized intuitionstic fuzzy metric space, then

i) a sequence $\{x_n\}$ in X is said to be convergent to x if $\lim_{n\to\infty} Q(x_n, x_n, x, t) = 1$ and $\lim_{n\to\infty} H(x_n, x_n, x, t) = 0$

ii) A sequence $\{x_n\}$ in X is said to be Cauchy sequence if $\lim_{n,m\to\infty} Q(x_n, x_n, x_m, t) = 1$ and $\lim_{n,m\to\infty} H(x_n, x_n, x_m, t) = 0$ that is, for any $\varepsilon > 0$ and for each $t > 0$, there exists $n_0 \in N$ such that $Q(x_n, x_n, x_m, t) > 1- \varepsilon$ and $H(x_n, x_n, x_m, t) < \varepsilon$ for $n, m \geq n_0$.

iii) A generalized intuitionistic fuzzy metric space $(X, Q, H, *, \lozenge)$ is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.9 : Let f and g be two self mappings of a generalized intuitionistic fuzzy metric space $(X, Q, H, *, \lozenge)$ If f and g satisfy the following conditions:

There exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Q(fx_n, u, u, t) = \lim_{n\to\infty} Q(gx_n, u, u, t) = 1$ and $\lim_{n\to\infty} H(fx_n, u, u, t) = \lim_{n\to\infty} H(gx_n, u, u, t) = 0$ for some $u \in X$ and $t > 0$, we say that f and g have the property (E.A).

Definition 2.10 : Let $(X, Q, H, *, \lozenge)$ be a generalized intuitionistic fuzzy metric space. The following conditions are satisfied :

$\lim_{n\to\infty} Q(x_n, y_n, z_n, t_n) = Q(x, y, z, t)$ and $\lim_{n\to\infty} H(x_n, y_n, z_n, t_n) = H(x, y, z, t)$. Whenever $\lim_{n\to\infty} x_n = x$; $\lim_{n\to\infty} y_n = y$; $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} Q(x, y, z, t_n) = Q(x, y, z, t)$, $\lim_{n\to\infty} H(x, y, z, t_n) = H(x, y, z, t)$ then Q, H are called convergent function on $X^3 \times (0, \infty)$.

Lemma 2.11:

Let $(X, Q, H, *, \lozenge)$ be a generalized intuitionistic fuzzy metric space. Then Q, H are continuous function on $X^3 \times (0, \infty)$.

Proof:

Since $\lim_{n\to\infty} x_n = x$; $\lim_{n\to\infty} y_n = y$; $\lim_{n\to\infty} z_n = z$.

$\lim_{n\to\infty} Q(x, y, z, t_n) = Q(x, y, z, t)$ and $\lim_{n\to\infty} H(x, y, z, t_n) = H(x, y, z, t)$

There is $n_0 \in N$ such that $|t - t_n| < \varepsilon$ and $|t - t_n| > \delta$ for $n \geq n_0$ and $\varepsilon < t/2$ and $\delta > t/2$

We know that Q(x, y, z, t) is non-decreasing and H(x, y, z, t) is non-increasing with respect to t,

So, we have

$$\begin{aligned}
Q(x_n, y_n, z_n, t) &\geq Q(x_n, y_n, z_n, t- \varepsilon) \\
&\geq Q(x_n, x, x, \tfrac{\varepsilon}{3}) * Q(x, y_n, z_n, t- \tfrac{4\varepsilon}{3}) \\
&\geq Q(x_n, x, x, \tfrac{\varepsilon}{3}) * Q(y_n, y, y, \tfrac{\varepsilon}{3}) * Q(y, x, z_n, t- \tfrac{5\varepsilon}{3}) \\
&\geq Q(x_n, x, x, \tfrac{\varepsilon}{3}) * Q(y_n, y, y, \tfrac{\varepsilon}{3}) * Q(z_n, z, z, \tfrac{\varepsilon}{3}) * Q(z, y, z, t- 2\varepsilon) \text{ and}
\end{aligned}$$

$$H(x_n, y_n, z_n, t) \le H(x_n, y_n, z_n, t-\delta)$$
$$\le H(x_n, x, x, \tfrac{\delta}{3}) \ \lozenge\ H(x, y_n, z_n, t-\tfrac{4\delta}{3})$$
$$\le H(x_n, x, x, \tfrac{\delta}{3}) \ \lozenge\ H(y_n, y, y, \tfrac{\delta}{3}) \ \lozenge\ H(y, x, z_n, t-\tfrac{5\delta}{3})$$
$$\le H(x_n, x, x, \tfrac{\delta}{3}) \ \lozenge\ H(y_n, y, y, \tfrac{\delta}{3}) \ \lozenge\ H(z_n, z, z, \tfrac{\delta}{3}) \ \lozenge\ H(z, y, z, t-2\delta)$$

$$Q(x, y, z, t+2\varepsilon) \ge Q(x, y, z, t_n+\varepsilon)$$
$$\ge Q(x, x_n, x_n, \tfrac{\varepsilon}{3}) * Q(x_n, y, z, t_n+\tfrac{2\varepsilon}{3})$$
$$\ge Q(x, x_n, x_n, \tfrac{\varepsilon}{3}) * Q(y, y_n, y_n, \tfrac{\varepsilon}{3}) * Q(y_n, x_n, z, t_n+\tfrac{\varepsilon}{3})$$
$$\ge Q(x, x_n, x_n, \tfrac{\varepsilon}{3}) * Q(y, y_n, y_n, \tfrac{\varepsilon}{3}) * Q(z, z_n, z_n, \tfrac{\varepsilon}{3}) * Q(z, y, x, t_n) \text{ and}$$

$$H(x, y, z, t+2\delta) \le H(x, y, z, t_n+\delta)$$
$$\le H(x, x_n, x_n, \tfrac{\delta}{3}) \ \lozenge\ H(x_n, y, z, t_n+\tfrac{2\delta}{3})$$
$$\le H(x, x_n, x_n, \tfrac{\delta}{3}) \ \lozenge\ H(y, y_n, y_n, \tfrac{\delta}{3}) \ \lozenge\ H(y_n, x_n, z, t_n+\tfrac{\delta}{3})$$
$$\le H(x, x_n, x_n, \tfrac{\delta}{3}) \ \lozenge\ H(y, y_n, y_n, \tfrac{\delta}{3}) \ \lozenge\ H(z, z_n, z_n, \tfrac{\delta}{3}) \ \lozenge\ H(z, y, x, t_n)$$

Let $n \to \infty$, by continuity of the function Q, H with respect to t, we can get

$Q(x, y, z, t+2\varepsilon) \ge Q(z, y, x, t) \ge Q(z, y, x, t-2\varepsilon)$ and

$H(x, y, z, t+2\delta) \le H(z\, y, x\, t) \le H(z, y, x, t-2\delta)$

Therefore Q, H are continuous function on $X^3 \times (0, \infty)$

Definition 2.12 : Let f and g be self maps on generalized intuitionstic fuzzy metric space (X, Q, H, $*$, $\lozenge$). Then the mappings are said to be weakly compatible if they commute at their coincidence point, that is, fx = gx implies that fgx = gfx.

Definition 2.13: Let f and g be self maps on generalized intuitionstic fuzzy metric space (X, Q, H, $*$,$\lozenge$). The pair (f,g) is said to be compatible if $\lim_{n\to\infty} Q(fgx_n, gfx_n, gfx_n, t) = 1$ and $\lim_{n\to\infty} H(fgx_n, gfx_n, gfx_n, t) = 0$ Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

3. MAIN THEOREM

We first generalize a classic theorem in generalized intuitionistic fuzzy metric space.

THEOREM 3.1: Let f, g, S and T be self- mappings of a complete symmetric generalized intuitionistic fuzzy metric space (X, Q, H, $*$, $\lozenge$) with $t * t \ge t$ and $t \lozenge t \le 1-t$ if the mappings satisfy the following conditions:

[3.1.1] $f(X) \subseteq S(X)$, $g(X) \subseteq T(X)$

[3.1.2] (f, T) or (g, S) satisfy the property (E.A)

[3.1.3] (f, T) and (g, S) are weakly compatible

[3.1.4] $Q(fx, gy, gz, kt) \ge \{Q(Tx, Sy, Sz, t) * Q(fx, Sy, Sz, t) * Q(Tx, gy, gz, t)\}$ and

[3.1.5] $H(fx, gy, gz, kt) \le \{H(Tx, Sy, Sz, t) \lozenge H(fx, Sy, Sz, t) \lozenge H(Tx, gy, gz, t)\}$,

there exists $k \in (0,1)$ such that for every $x,y,z \in X$ and $t > 0$.

Then f, g, S and T have a unique common fixed point in X.

Proof:

Suppose (f, T) satisfy the property (E.A), hence there exists a sequence $\{x_n\}$ such that

$\lim_{n\to\infty} Q(fx_n, u, u, t) = \lim_{n\to\infty} Q(Tx_n, u, u, t) = 1$ and

$\lim_{n\to\infty} H(fx_n, u, u, t) = \lim_{n\to\infty} H(Tx_n, u, u, t) = 0$ for some $u \in X$ and $t > 0$.

Since $f(x) \subseteq S(x)$, there exists a sequence $\{y_n\}$ such that $fx_n = Sy_n$

$\Rightarrow \lim_{n\to\infty} Q(Sy_n, u, u, t) = 1$ and $\lim_{n\to\infty} H(Sy_n, u, u, t) = 0$.

Therefore,

$Q(fx_n, gy_n, gy_{n+1}, kt) \geq Q(Tx_n, Sy_n, Sy_{n+1}, t) * Q(fx_n, Sy_n, Sy_{n+1}, t) * Q(Tx_n, gy_n, gy_{n+1}, t)$

$\geq \{Q(Tx_n, u, u, ½ t) * Q(u, Sy_n, Sy_{n+1}, ½ t) * Q(fx_n, u, u, ½ t) *$

$Q(u, Sy_n, Sy_{n+1}, ½ t) * Q(Tx_n, gy_n, gy_{n+1}, t)\}$

$H(fx_n, gy_n, gy_{n+1}, kt) \leq H(Tx_n, Sy_n, Sy_{n+1}, t) \lozenge H(fx_n, Sy_n, Sy_{n+1}, t) \lozenge H(Tx_n, gy_n, gy_{n+1}, t)$

$\leq \{H(Tx_n, u, u, ½ t) \lozenge H(u, Sy_n, Sy_{n+1}, ½ t) \lozenge H(fx_n, u, u, ½ t) \lozenge$

$H(u, Sy_n, Sy_{n+1}, ½ t) \lozenge H(Tx_n, gy_n, gy_{n+1}, t)\}$

There exists $\delta > 0$ such that $k + \delta < 1$, for $k \in (0,1)$. On making $n \to \infty$ and by the symmetry generalized intuitionstic fuzzy metric space, we have

$\lim_{n\to\infty} Q(fx_n, gy_n, gy_{n+1}, kt) \geq 1 * 1 * 1 * 1$

$\lim_{n\to\infty} Q(Tx_n, fx_n, fx_n, [1-(k+\delta)] t) * \lim_{n\to\infty} Q(fx_n, gx_n, gy_{n+1}, (k+\delta)] t) \geq 1 * 1 * 1 * 1 * 1$

$\lim_{n\to\infty} Q(fx_n, gy_n, gy_{n+1}, (k+\delta) t)$ and

$\lim_{n\to\infty} H(fx_n, gy_n, gy_{n+1}, kt) \leq 0 \lozenge 0 \lozenge 0 \lozenge 0$

$\lim_{n\to\infty} H(Tx_n, fx_n, fx_n, [1-(k+\delta)] t) \lozenge \lim_{n\to\infty} H(fx_n, gx_n, gy_{n+1}, (k+\delta)] t) \leq 0 \lozenge 0 \lozenge 0 \lozenge 0 \lozenge 0$

$\lim_{n\to\infty} H(fx_n, gy_n, gy_{n+1}, (k+\delta) t)$

Hence $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Tx_n = u$

Let $(X, Q, H, *, \lozenge)$ is a complete generalized intuitionistic fuzzy metric space, there exists $x_0 \in X$ such that

$Tx_0 = u \Rightarrow Q(fx_0, gy_n, gy_{n+1}, kt) \geq Q(Tx_0, Sy_n, Sy_{n+1}, t) * Q(fx_0, Sy_n, Sy_{n+1}, t) * Q(Tx_0, gy_n, gy_{n+1}, t)$

If $n \to \infty$ we can get $Q(fx_0, u, u, t) \geq 1 * Q(fx_0, u, u, t) * 1$.

By the property of non- decreasing with respect to t,

$Tx_0 = u \Rightarrow H(fx_0, gy_n, gy_{n+1}, k) \leq H(Tx_0, Sy_n, Sy_{n+1}, t) \lozenge H(fx_0, gy_n, gy_{n+1}, t) \lozenge H(Tx_0, Sy_n, Sy_{n+1}, t)$

If $n \to \infty$ we can get $H(fx_0, u, u, t) \leq 0 \lozenge H(fx_0, u, u, t) \lozenge 0$

By the property of non- increasing with respect to t, it is easy to see that $fx_0 = Tx_0 = u$.

As $f(X) \subseteq S(X)$, there exists y_0 such that $fx_0 = Sy_0$. Suppose $Sy_0 \neq gy_0$. Then

$Q(fx_n, gy_0, gy_0, kt) \geq Q(Tx_n, Sy_0, Sy_0, t) * Q(fx_n, Sy_0, Sy_0, t) * Q(Tx_n, gy_0, gy_0, t)$

$\geq Q(Tx_n, u, u, t) * Q(fx_n, u, u, t) * Q(Tx_n, gy_0, gy_0, t)$ and

$H(fx_n, gy_0, gy_0, kt) \leq H(Tx_n, Sy_0, Sy_0, t) \lozenge H(fx_n, Sy_0, Sy_0, t) \lozenge H(Tx_n, gy_0, gy_0, t)$

$\leq H(Tx_n, u, u, t) \lozenge H(fx_n, u, u, t) \lozenge H(Tx_n, gy_0, gy_0, t)$

Letting $n \to \infty$ we have

$Q(u, gy_0, gy_0, kt) \geq Q(u, u, u, t) * Q(u, u, u, t) * Q(u, gy_0, gy_0, t)$ and

$H(u, gy_0, gy_0, kt) \leq H(u, u, u, t) \lozenge H(u, u, u, t) \lozenge H(u, gy_0, gy_0, t)$.

Which is a contradiction. So, $gy_0 = Sy_0 = u$.

Now by (f, T) and (g, S) are weakly compatible. We can get that :

$ffx_0 = fTx_0 = Tfx_0 = TTx_0$ and $ggy_0 = gSy_0 = Sgy_0 = SSy_0$. Suppose $fu \neq u$.

Then $Q(fu, u, u, kt) = Q(fu, gy_0, gy_0, kt)$

$\geq Q(Tu, Sy_0, Sy_0, t) * Q(fu, Sy_0, Sy_0, t) * Q(Tu, gy_0, gy_0, t)$

$\geq Q(Tu, u, u, t) * Q(fu, u, u, t) * Q(Tu, u, u, t)$

$\geq \lim_{n\to\infty} Q(Tu, Tx_n, Tx_n, t) * Q(fu, u, u, t) * Q(Tu, Tx_n, Tx_n, t)$

$\geq \lim_{n\to\infty} Q(Tu, Tx_n, Tx_n, t)$ [3.1.6]

$H(fu, u, u, kt) = H(fu, gy_0, gy_0, kt)$

$\leq H(Tu, Sy_0, Sy_0, t) \lozenge H(fu, Sy_0, Sy_0, t) \lozenge H(Tu, gy_0, gy_0, t)$

$\leq H(Tu, u, u, t) \lozenge H(fu, u, u, t) \lozenge H(Tu, u, u, t)$

$\leq \lim_{n\to\infty}$ H(Tu, Tx_n, Tx_n, t) ◊ H (fu, u, u, t) ◊ H (Tu, Tx_n, Tx_n, t)

$\leq \lim_{n\to\infty}$ H(Tu, Tx_n, Tx_n, t) [3.1.7]

By fu = ffx_0 = fTx_0 = Tfx_0 = TTx_0 = Tu and $t * t \geq t$ and $t \Diamond t \leq 1 - t$,
it is easy to see that [3.1.6] and [3.1.7] yields a contradiction and so fu = u = Tu.
Now following the similar argument, we can get gu = u = Su.
So f, g, S and T have a common fixed point u.

Uniquness:

Let v ≠ u be another common fixed point of f, g, S and T. Then,

Q(v, u, u, kt) = Q (fv, gu, gu, kt)
≥ Q (Tv, Su, Su, t) ∗ Q (fv, Su, Su, t) ∗ Q (Tv, gu, gu, t)
= Q (v, u, u, t) ∗ Q (v, u, u, t) ∗ Q (v, u, u,t)

H(v, u, u, kt) = H(fv, gu, gu, kt)
≤ H(Tv, Su, Su, t) ◊ H (fv, Su, Su, t) ◊ H(Tv, gu, gu, t)
= H(v, u, u, t) ◊ H (v, u, u, t) ◊ H (v, u, u,t)

By $t * t \geq t$ and $t \Diamond t \leq 1 - t$, we can get Q (v, u, u, kt) ≥ Q (v, u, u, t) and
H (v, u, u, kt) ≤ H(v, u, u, t) is a contradiction thus v = u.
Hence f, g, S and T have a unique common fixed point in X.

Theorem 3. 2 : Let f, g, S and T be self mappings of a complete generalized intuitionistic fuzzy metric space (X, Q, H, ∗, ◊) with $t * t > t$ and $t \Diamond t < 1 - t$ if the mappings satisfy the following conditions:

[3.2.1] f(x) ⊆ T(x) , g(x) ⊆ S(x)

[3.2.2] Suppose (f, S) satisfy the property (E.A)

[3.3.3] (f, S) and (g, T) are weakly compatible

[3.3.4] $Q(fx, gy, gz, t) \geq \phi\left[\min\begin{pmatrix} Q(Sx,Ty,Tz,\ t), Q(fx,Ty,Tz,t), \\ Q(Sx,gy,gz,t), Q(fx,Sx,Sx,t)\end{pmatrix}\right]$ and

$H(fx, gy, gz, t) \leq \Psi\left[\max\begin{pmatrix} H(Sx,Ty,Tz,t), H(fx,Ty,Tz,t), \\ H(Sx,gy,gz,t), H(fx,Sx,Sx,t)\end{pmatrix}\right]$

for all x, y, z ϵ X and t > 0 where ϕ, Ψ : [0,1] → [0,1] is a continuous and increasing function with ϕ(s) > s and Ψ(s) < s for 0 < S < 1 and $\varphi(1) = 1$, $\Psi(0) = 0$.
Then f, g, S and T have a unique common fixed point in X.

Proof:

Let (f, S) satisfy the property (E.A). By the definition of (E.A) we can get

$\lim_{n\to\infty} Q(fx_{n,}\ u,\ u, t) = \lim_{n\to\infty} Q(Sx_{n,}\ u,\ u,\ t) = 1$ and

$\lim_{n\to\infty} H(fx_{n,}, u,\ u,\ t) = \lim_{n\to\infty} H(Sx_{n,},\ u,\ u,\ t) = 0$ for some u ϵ X and every t > 0.

Because generalized intuitionistic fuzzy metric space is complete and f(x) ⊆ T(x), there exists a sequence $\{y_n\}$ such that $f(x_n) = T(y_n)$, which implies $\lim_{n\to\infty} Q(Ty_{n,}u,u,t) = 1$ and $\lim_{n\to\infty} H(Ty_{n,}u,u,t) = 0$. Now

$Q(fx_n, gy_n, gy_{n+1}, t) \geq \phi\left[\min\begin{pmatrix} Q(Sx_n,Ty_n,Ty_{n+1},t), Q(fx_n,Ty_n,Ty_{n+1},t) \\ Q(Sx_n,gy_n,gy_{n+1},t), Q(fx_n,\ Sx_n,Sx_n,t)\end{pmatrix}\right]$

$H(fx_n, gy_n, gy_{n+1}, t) \leq \Psi\left[\max\begin{pmatrix} H(Sx_n,Ty_n,Ty_{n+1},t), H(fx_n,Ty_n,Ty_{n+1},t) \\ H(Sx_n,gy_n,gy_{n+1},t), H(fx_n,\ Sx_n,Sx_n,t)\end{pmatrix}\right]$

By the definition of generalized intuitionistic fuzzy metric space, we can get

Q(x, y, z, t) ≥ Q (x u, u, 1/3 t) ∗ Q (u, y, z, 2/3 t)
≥ Q (x, u, u,1/3 t) ∗ Q(y, u, u, 1/3 t) ∗ Q (z, u, u, 1/3 t) and

H(x, y, z, t) ≤ H (x u, u, 1/3 t) ◊ H (u, y, z, 2/3 t)

$\leq H(x, u, u, 1/3\, t) \; \Diamond \; H(y, u, u, 1/3\, t) \; \Diamond \; H(z, u, u, 1/3\, t)$

Thus

$$\lim_{n\to\infty} Q(fx_n, gy_n, gy_{n+1}, t) \geq \lim_{n\to\infty} \phi\left[\min\begin{pmatrix} Q(Sx_n, Ty_n, Ty_{n+1}, t), & Q(fx_n, Ty_n, Ty_{n+1}, t) \\ Q(Sx_n, gy_n, gy_{n+1}, t), & Q(fx_n, Sx_n, Sx_n, t) \end{pmatrix}\right]$$

$$\geq \lim_{n\to\infty} \phi\left[\min\begin{pmatrix} 1*1*1, & Q(Sx_n, gy_n, gy_{n+1}, t) \\ 1*1*1, & 1*1*1 \end{pmatrix}\right] \text{ and}$$

$$\lim_{n\to\infty} H(fx_n, gy_n, gy_{n+1}, t) \leq \lim_{n\to\infty} \Psi\left[\max\begin{pmatrix} H(Sx_n, Ty_n, Ty_{n+1}, t) & H(fx_n, Ty_n, Ty_{n+1}, t) \\ H(Sx_n, gy_n, gy_{n+1}, t) & H(fx_n, Sx_n, Sx_n, t) \end{pmatrix}\right]$$

$$\leq \lim_{n\to\infty} \Psi\left[\max\begin{pmatrix} 0 \Diamond 0 \Diamond 0 & H(Sx_n, gy_n, gy_{n+1}, t) \\ 0 \Diamond 0 \Diamond 0 & 0 \Diamond 0 \Diamond 0 \end{pmatrix}\right]$$

If $gy_n \neq u$ then $\lim_{n\to\infty} Q(fx_n, gy_n, gy_{n+1}, t) \geq \lim_{n\to\infty} \phi\,[Q(Sx_n, gy_n, gy_{n+1}, t)]$

$$> \lim_{n\to\infty} Q(Sx_n, gy_n, gy_{n+1}, t)$$

$$\lim_{n\to\infty} H(fx_n, gy_n, gy_{n+1}, t) \leq \lim_{n\to\infty} \Psi\,[H(Sx_n, gy_n, gy_{n+1}, t)]$$

$< \lim_{n\to\infty} H(Sx_n, gy_n, gy_{n+1}, t)$ is a contradiction by the above lemma.

Therefore $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = u$

Let $(X, Q, H, *, \Diamond)$ is a complete generalized intuitionistic fuzzy metric space.

There exists $x_0 \in X$ such that $Sx_0 = u$,

$$\text{Hence } Q(fx_0, gy_n, gy_{n+1}, t) \geq \phi\left[\min\begin{pmatrix} Q(Sx_0, Ty_n, Ty_{n+1}, t), & Q(fx_0, Ty_n, Ty_{n+1}, t) \\ Q(Sx_0, gy_n, gy_{n+1}, t), & Q(fx_0, Sx_0, Sx_0, t) \end{pmatrix}\right]$$

$$H(fx_0, gy_n, gy_{n+1}, t) \leq \Psi\left[\max\begin{pmatrix} H(Sx_0, Ty_n, Ty_{n+1}, t), & H(fx_0, Ty_n, Ty_{n+1}, t) \\ H(Sx_0, gy_n, gy_{n+1}, t), & H(fx_0, Sx_0, Sx_0, t) \end{pmatrix}\right]$$

On making $n \to \infty$, $Q(fx_0, u, u, t) \geq \phi\left[\min\begin{pmatrix} Q(u,u,u,t), & Q(u,u,u,t) \\ Q(fx_0, u, u, t), & Q(fx_0, u, u, t) \end{pmatrix}\right]$ and

$$H(fx_0, u, u, t) \leq \Psi\left[\max\begin{pmatrix} H(u,u,u,t), & H(u,u,u,t) \\ H(fx_0, u, u, t), & H(fx_0, u, u, t) \end{pmatrix}\right]$$

which can imply $fx_0 = u$, with $\varphi(s) > s$ and $\Psi(s) < s$ for $0 < s < 1$

As $f(x) \subseteq T(x)$, there exists y_0 such that $fx_0 = Ty_0$.

Suppose $Ty_0 \neq g\, y_0$. Now

$$Q(fx_n, gy_0, gy_0, t) \geq \phi\left[\min\begin{pmatrix} Q(Sx_n, Ty_0, Ty_0, t), & Q(fx_n, Ty_0, Ty_0, t) \\ Q(Sx_n, gy_n, gy_0, t), & Q(fx_n, Sx_n, Sx_n, t) \end{pmatrix}\right]$$

$$H(fx_n, gy_0, gy_0, t) \leq \Psi\left[\max\begin{pmatrix} H(Sx_n, Ty_0, Ty_0, t), & H(fx_n, Ty_0, Ty_0, t) \\ H(Sx_n, gy_n, gy_0, t), & H(fx_n, Sx_n, Sx_n, t) \end{pmatrix}\right]$$

If $n \to \infty$, $Q(u, gy_0, gy_0, t) \geq \phi\left[\min\begin{pmatrix} Q(u,u,u,t), & Q(u, gy_0, gy_0, t) \\ Q(u,u,u,t), & Q(u,u,u,t) \end{pmatrix}\right]$ and

$H(u, gy_0, gy_0, t) \leq \Psi\left[\max\begin{pmatrix} H(u,u,u,t), & H(u, gy_0, gy_0, t) \\ H(u,u,u,t), & H(u,u,u,t) \end{pmatrix}\right]$ by the continuity of Q, H and ϕ, Ψ

Hence $Q(u, gy_0, gy_0, t) \geq \phi\,(Q(u, gy_0, gy_0, t)) > Q(u, gy_0, gy_0, t)$ and

$H(u, gy_0, gy_0, t) \leq \Psi\,(H(u, gy_0, gy_0, t)) < H(u, gy_0, gy_0, t)$ is a contradiction.

So $Ty_0 = gy_0$

Now by (f, T) and (g, S) are weakly compatible, we can get,

$ffx_0 = fSx_0 = Sfx_0 = SSx_0$ and $ggy_0 = gTy_0 = Tgy_0 = TTy_0$

$$\text{Then, } \lim_{n\to\infty} Q(fu, gy_n, gy_{n+1}, t) \geq \phi\left[\min\begin{pmatrix} Q(Su, Ty_n, Ty_{n+1}, t), & Q(fu, Ty_n, Ty_{n+1}, t) \\ Q(Su, gy_n, gy_{n+1}, t) & Q(fu, Su, Su, t) \end{pmatrix}\right]$$

$$\lim_{n\to\infty} H(fx_n, gy_n, gy_{n+1}, t) \leq \Psi\left[\max\begin{pmatrix} H(Su, Ty_n, Ty_{n+1}, t), & H(fu, Ty_n, Ty_{n+1}, t) \\ H(Su, gy_n, gy_{n+1}, t), & H(fu, Su, Su, t) \end{pmatrix}\right]$$

$\Rightarrow$ $fu = u = Su$.

Similarly we can get Tu = gu = u

Uniqueness:

Let v be another common fixed point of f, g, S and T. Then

Q(v, u, u, t) = Q(fv, gu, gu, t)

$$\geq \phi\left[\min\begin{pmatrix} Q(Sv, Tu, Tu, t), Q(fv, Tu, Tu, t) \\ Q(Sv, gu, gu, t), \ Q(fv, Sv, Sv, t) \end{pmatrix}\right]$$

H(v, u, u, t) = H(fv, gu, gu, t)

$$\leq \Psi\left[\max\begin{pmatrix} H(Sv, Tu, Tu, t), H(fv, Tu, Tu, t) \\ H(Sv, gu, gu, t), H(fv, Sv, Sv, t) \end{pmatrix}\right]$$

It implies v= u. Hence f, g, S and T have a unique common fixed point in X.

REFERENCES

[1]. Attanssov, K. “Intuitionistic fuzzy sets, VII ITKR’s session, sofia, june 1983 (Deposed in central Science- Technical Library of Bulg. Academy of Science, 1697/84) (in Bulgarian)

[2]. Dhage, B.C., Generalized metric spaces and mappings with fixed point, Bull. Calcutta Math. Soc., 84(4), 1992, 329-336

[3]. George.A and Veeramani.P , “ On Some results in fuzzy metric spaces”, Fuzzy sets and Systems,64(1994), 395 - 399.

[4]. Hu. X-Qi, Luo. Q “Coupled coincidence point theorem for contractions in generalized fuzzy metric spaces”, Fixed point theory Appl, 2012, 196 (2012).

[5]. Kramosil. O and Michalek. J, “ Fuzzy metric and statistical metric spaces”, Ky-bernetics,11(1975) 330 -334.

[6]. Mohiuddine, SA, Sevli, H, “ Stability of Pexiderized quadratic functional equation in intuitionstic fuzzy normed space” journal of computer Appl. Math, 2011, 2137-2146.\

[7]. Mustafa, Z. and B. Sims, “A new approach to generalized metric space”, J. Nonlinear Convex Analysis, 7, 2006, 289-297.

[8]. Mustafa. Z, Obiedat.H, Awawdeh.F, “ Some Fixed point theorem for mapping on Complete G-metric spaces, Fixed point theory Appl, 2008, Article ID 189870.

[9]. Park, J.H. “Intuitionstic fuzzy metric spaces,” Chaos Solitons Fractals” 2004, 22, 1039-1046.

[10]. Rao, K.P.R, Altun. I, Bindu S.H, “ Common Coupled fixed point theorem in generalized fuzzy metric spaces”, Adv. Fuzzy Syst. 2011, Article ID 986748.

[11]. Saadati, R, Park. J. H, “ On the intuitionstic fuzzy topological spaces”, Chos Solitons Fractals, 27, 331-344.

[12]. Sun, G. Yang, K. “Generalized fuzzy metric spaces with Properties” Res.J. Appl. Sci.2, 2010, 673-678.

[13]. Zadeh L.A., “Fuzzy sets”, Inform. and Control, 8 (1965), 338- 353