Vol.3.Issue.4.2015 (Oct-Dec)

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

http://www.bomsr.com

RESEARCH ARTICLE

ISSN:2348-0580

OPERATIONS ON DIGRAPHS AND DIGRAPH FOLDING

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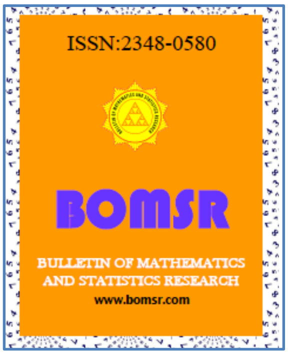


ABSTRACT

In this paper we examined the relation between digraph folding of a given pair of digraphs and digraph folding of new digraphs generated from these given pair of digraphs by some known operations like union, intersection, join, Cartesian product and composition. We first redefined these known operations for digraphs, then we defined some new maps of these digraphs and we called these maps union, intersection, join, Cartesian and composition dimaps .In each case we obtained the necessary and sufficient conditions, if exist ,for a dimap to be digraph folding. Finally we explored the digraph folding, if there exist any, by using the adjacency matrices.

Key words: Digraphs, adjacency matrices , digraph folding ,union,intersection, join, the Cartesian product and the composition of digraphs **.**

1. INTRODUCTION

Graph folding is introduced by E.EL-Kholy and A.AL-Esway [3].The notion of digraph folding is introduced by E.EL-Kholy and H.Ahmed [4].

Definitions (1.1)

1) A digraph D consists of elements, called vertices and a list of ordered pairs of these elements, called arcs. The set of vertices is called the vertex set of D,denoted by V(D), and the list of arcs is called the arc list of D, denoted by A(D).If v and w are vertices of D,then an arc of the form vw is said to be directed from v to w.The digraph with no loops is called simple. Two or more arcs joining the same pair of vertices in the same direction is called multiple arc [5].
2) Let D_1 and D_2 be digraphs and $f: D_1 \rightarrow D_2$ a continuous function. Then f is called a digraph map if, (i)For each vertex $v \in V(D_1)$, f(v) is a vertex in $V(D_2)$. (ii)For each arc $e \in A(D_1)$, dim (f(e)) $\leq$ dim (e) [4].

3) Let D_1 and D_2 be simple digraphs, we call a digraph map $f: D_1 \rightarrow D_2$ a digraph folding iff f maps vertices to vertices and arcs to arcs, i.e., for each $v \in V(D_1)$, $f(v) \in V(D_2)$ and for each e A (D_1), $f(e) \in A(D_2)$ [4] .
   If the digraph contains loops, then the digraph folding must send loops to loops but of the same direction. The set of digraph foldings between digraphs D_1 and D_2 is denoted by Đ(D_1, D_2) and from D into itself by Đ(D).
4) Let D be a diagraph without loops, with n vertices labeled 1,2,3,…,n. The adjacency matrix M(D) is the nxn matrix in which the entry in row i and column j is the number of arcs from vertex i to vertex j [5] .

Proposition (1.2)

Let D be a connected digraph without loops with n vertices. Then a digraph folding of D into itself may be defined, if there is any, as a digraph map f of D to an image f(D) by mapping :

The multiple arc into one of its arcs.

I. (a)The vertex v_i to the vertex v_j if the numbers appearing in the adjacency matrix in the i^{th} and j^{th} rows (or columns) are the same .

(b)The vertex v_i to the vertex v_j if the entries of the i^{th} and j^{th} rows are zeros and if the i^{th} and j^{th} columns are the same, or there exists a row k which has numbers 1 in the i^{th} and j^{th} columns.

II. (a)The arc (v_i,v_k) to the arc (v_j,v_k) if the i^{th} and j^{th} rows (or columns) are the same .

(b) The arc (v_i,v_j) to the arc (v_i ,v_k) if the j^{th} and k^{th} columns (or rows) are the same .

In general the arc (v_i,v_j) will be mapped to the arc (v_k,v_l) if v_i mapped to v_k and v_j mapped to v_l, [4].

(2) Union of digraphs

In the following we redefine the known operation ,union, given for two simple graphs [3] , for digraphs .

Definition (2-1)

Let $D_1=(V_1,A_1)$ and $D_2 =(V_2,A_2)$ be simple digraphs .Then the simple digraph D = (V, A) where $V=V_1UV_2$ and $A=A_1UA_2$ is called the union of digraphs D_1 and D_2 and is denoted by D_1UD_2 .When D_1 and D_2 are vertex disjoint D_1UD_2 is denoted by , D_1+D_2 , and is called the sum of digraphs $D_1$and D_2 .

Definition (2-2)

Let $D_1 =(V_1,A_1)$ and $D_2 =(V_2,A_2)$ be simple digraphs .Let $f: D_1 \rightarrow D_1$ and $g:D_2 \rightarrow D_2$ be digraph maps . By the union dimap of the digraph maps f and g, fUg ,we mean a digraph map from the digraph D_1UD_2 into itself.

$fUg : D_1UD_2 \rightarrow D_1UD_2$ such that $f(v) = g(v)$, for all $v \in V_1 \cap V_2$, $f(e)=g(e)$, for all $e \in A_1 \cap A_2$ defined by

(i) For each $v \in V_1 \cup V_2$, $(fUg)(v) = \begin{cases} f(v), \text{ if } v \in V_1 \\ g(v), \text{ if } v \in V_2 \end{cases}$

(ii) For each e A_1UA_2 , $(fUg)(e) = \begin{cases} f(e), \text{ if } e \in A_1 \\ g(e), \text{ if } e \in A_2 \end{cases}$

Theorem (2-3)

Let $D_1=(V_1,A_1)$ and $D_2=(V_2,A_2)$ be simple connected digraphs.Let $f:D_1 \to D_1$ and $g:D_2 \to D_2$ be digraph maps . Then the union dimap fUg is a digraph folding iff f and g are digraph foldings . In this case $(fUg)(D_1UD_2)=f(D_1) \cup g(D_2)$ The proof is almost as in [2] .

Example 2.4

Let $D_1=(V_1,A_1)$,where $V_1=\{v_1,v_2,v_3,v_4\}$ and $A_1=\{e_1,e_2,e_3,e_4,e_5\}$.Let $D_2=(V_2,A_2)$, where $V_2=\{v_1,v_2,v_3,v_5\}$ and $A_2=\{e_1,e_2,e_6\}$,see Fig.1 (a).

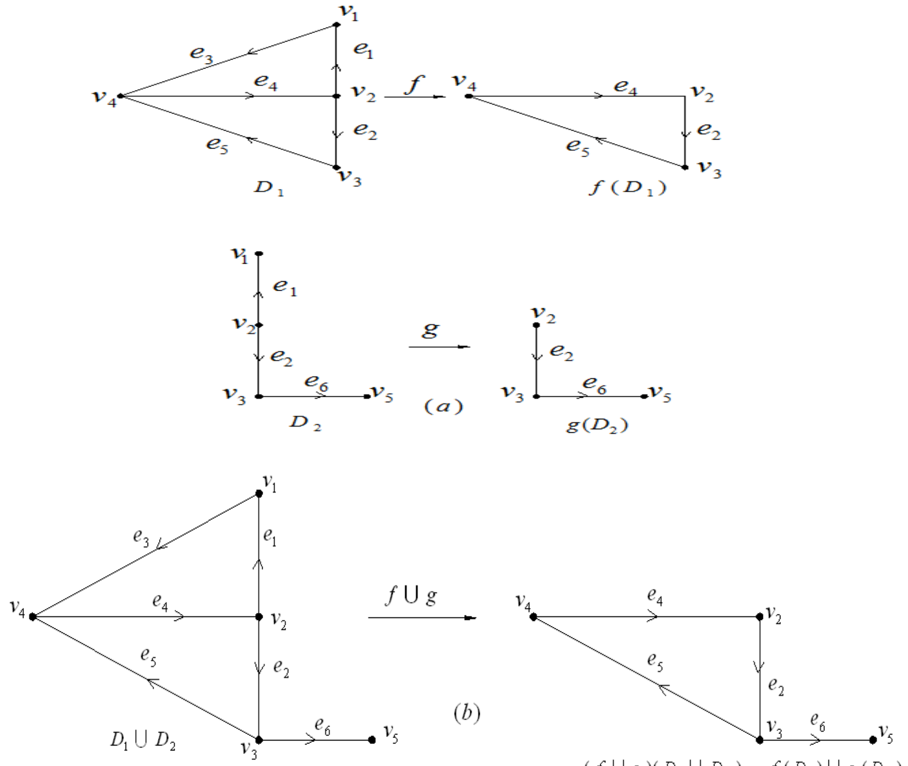


Fig.1

Now let f Đ(D_1) be a digraph folding defined by $f\{v_1\}=\{v_3\}$ and $f\{e_1,e_3\}=\{e_2,e_5\}$, where through this paper the omitted vertices and arcs will be mapped to themselves . Also, let g Đ( )be a digraph folding defined by $g\{v_1\}=\{v_3\}$ and $g\{e_1\}=\{e_2\}$, see Fig.1(a) .The union dimap fUg: $D_1UD_2 \to D_1UD_2$ defined by $(f\,Ug)\{v_1\}=\{v_3\}$ and $(fUg)\{e_1,e_3\}= \{e_2,e_5\}$ is a digraph folding , see Fig.1 (b).The adjacency matrices of D_1 ,D_2 and D_1UD_2 are as follows :

$$M(D_1)=\begin{array}{c} \\ v_1\\ v_2\\ v_3\\ v_4\end{array}\!\!\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4\end{array}\\ \begin{bmatrix}0&0&0&1\\1&0&1&0\\0&0&0&1\\0&1&0&0\end{bmatrix}\end{array},\quad M(D_2)=\begin{array}{c} \\ v_1\\ v_2\\ v_3\\ v_5\end{array}\!\!\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_5\end{array}\\ \begin{bmatrix}0&0&0&0\\1&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\end{array}\text{ and }M(D_1\cup D_2)=\begin{array}{c} \\ v_1\\ v_2\\ v_3\\ v_4\\ v_5\end{array}\!\!\begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5\end{array}\\ \begin{bmatrix}0&0&0&1&0\\1&0&1&0&0\\0&0&0&1&1\\0&1&0&0&0\\0&0&0&0&0\end{bmatrix}\end{array}$$

By using only these adjacency matrices we can define the digraph folding .For example ,by using the adjacency matrix $M(D_1)$ we can easily see that the vertex v_1 will be mapped to the vertex v_3 since the first and third rows of $M(D_1)$ have the same entries .Also the arc $(v_1,v_4)=e_3$ will be mapped to the arc $(v_3,v_4)= e_5$ since the 1st and 3rd rows are the same, finally the arc $(v_2,v_1)=e_1$ will be mapped to the arc

$(v_2,v_3)= e_3$ since the $_1$st and $_3$rd columns are the same. Again by using $M(D_2)$ and $M(D_1UD_2)$ we can describe the digraph folding of both D_2 and D_1UD_2

(3) Intersection of digraphs

Definition (3.1)

Let $D_1 = (V_1,A_1)$ and $D_2 = (V_2,A_2)$ be simple digraphs . Then the simple digraph $D =(V,A)$ where $V = V_1\cap V_2$ and $A = A_1\cap A_2$ is called the intersection of digraphs D_1 and D_2 and is denoted by $D_1\cap D_2$.

Definition ($_{3\cdot 2}$)

Let $D_1=(V_1,A_1)$ and $D_2 =(V_2,A_2)$ be simple digraphs .Let $f : D_1\rightarrow D_1$ and $g:D_2\rightarrow D_2$ be digraph maps. If f and g agree on $V_1\cap V_2$ and $A_1\cap A_2$ then by the intersection dimap of the digraph maps f and g , $f \cap g$,we mean a digraph map $f\cap g: D_1\cap D_2\rightarrow D_1\cap D_2$, where $V_1\cap V_2 \neq \emptyset$ defined by :

(i) For all $v\in V_1\cap V_2$, $(f \cap g)(v) = f(v) = g(v)$

(ii) For all $e\in A_1\cap A_2$,$(f \cap g) (e) = f(e)= g(e)$

Theorem (3-3)

Let $D_1=(V_1,A_1)$ and $D_2=(V_2,A_2)$ be simple connected digraphs. Let $f:D_1\rightarrow D_1$ and $g:D_2\rightarrow D_2$ be digraph maps .Then the intersection dimap $f\cap g$ is a digraph folding iff f and g are digraph foldings . In this case $(f\cap g)(D_1\cap D_2)=f(D_1) \cap g(D_2)$ The proof is easy.

Example (3-3)

let $D_1=(V_1,A_1)$,where $V_1=\{ v_1,v_2,v_3,v_4,v_5,v_6 \}$ and $A_1=\{e_1,e_2,e_3,e_4,e_5,e_6,e7,e8\}$.Let $D_2=(V_2,A_2)$, where $V_2=\{ v_1,v_5,v_6,v7\}$ and $A_2=\{ e_5,e8,e9,e_10,e_{11} \}$,see Fig.$_2$ (a).

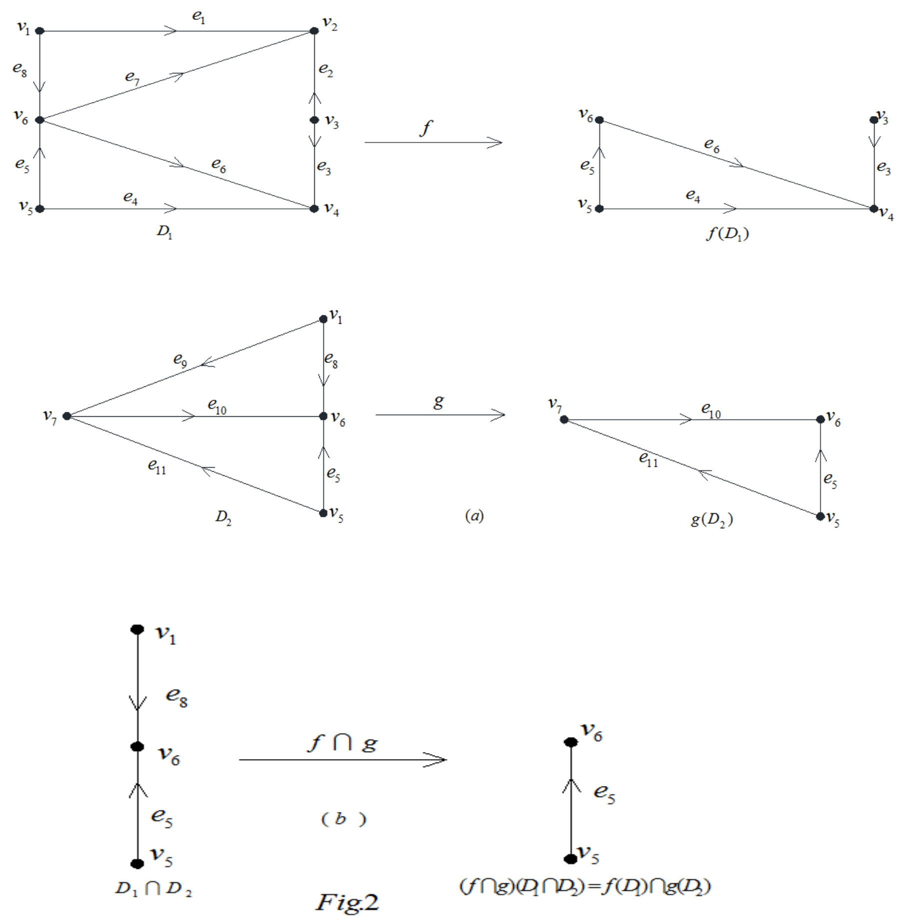


Fig.2

Now let f Đ(D_1) be a digraph folding defined by f{v_1,v_2 }={v_5,v_4}and f{e_1,e_2,e7,e8} = {e_4,e_3,e_6,e_5}. Also , let g Đ(D) be a digraph folding defined by g{v_1}={v_5} and g{e8,e9}={e_5,e_{11}}, see Fig.$_1$(a) .The intersection dimap

f ∩g :D_1∩ $D_2 \rightarrow D_1$∩ D_2 defined by (f ∩ g){v_1}={v_5} and (f ∩g){e8}={e_5} is a digraph folding , see Fig.$_2$(b).The adjacency matrices of D_1 ,D_2 and $D_1 \cap D_2$ are as follows :

$$M(D_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array} , \quad M(D_2) = \begin{array}{c} \\ v_1 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_5 & v_6 & v_7 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$\text{and } M(D_1 \cap D_2) = \begin{array}{c} \\ v_1 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

By using only these adjacency matrices we can define the digraph folding. For example, by using the adjacency matrix M($D_1 \cap D_2$) we can easily see that the vertex v_1 will be mapped to the vertex v_5, since the first and second rows have the same entries. Also, the arc (v_1,v_6)=e8 can be mapped to the arc (v_5,v_6)=e_5, since the first and second rows are the same. Again by using M(D_1) and M(D_2) we can describe the digraph foldings of both D_1 and D_2 .

(4) Join of diagraphs

Definition (4-1)

Let D_1 and D_2 be vertex dis joint diagraphs. Then we define the join digraph, D_1 v D_2 , to be the digraph in which each vertex of D_1 (or D_2) is adjacent to the vertices of D_2 (or D_1) .

Definition (4-2)

Let D_1=(V_1,A_1) ,D_2=(V_2,A_2),D_3=(V_3,A_3) and D_4=(V_4,A_4) be simple digraphs.

Let f : $D_1 \rightarrow D_3$ and g :$D_2 \rightarrow D_4$ be digraph maps. By a join dimap,we mean a digraph map, fvg : D_1v$D_2 \rightarrow D_3$vD_4 defined by

(i) For each vertex v∈V_1 U V_2, (fvg)(v)= $\begin{cases} f(v) \text{ , if v } V_1 \\ g(v) \text{ , if v } V_2 \end{cases}$

(ii) For each arc e =(v_1,v_2) ,v_1∈V_1 and v_2 ∈V_2 , (fvg){e}= {f(v_1),g(v_2)∈ A_3vA_4.

(iii) If e=(u_1,v_1)∈A_1 , then (fvg){e}=(fvg){(u_1,v_1)}={f(u_1),f(v_1)}, Also if e =(u_2,v_2) ∈A_2 ,then (fvg){e}=(fvg){(u_2,v_2)}={ g(u_2),g(v_2)}

Note that If f{u_1}= f{v_1},then the image of the join dimap (fvg){e} will be a vertex of D_3 v D_4 and thus is not a digraph folding .

Theorem (4-3)

Let D_1 , D_2 , D_3 and D_4 be digraphs, let f:$D_1 \rightarrow D_3$ and g :$D_2 \rightarrow D_4$ be digraph maps .Then Đ(D_1vD_2,D_3vD_4) is a digraph folding iff f and g are digraph foldings .

Proof :

Suppose f and g are digraph foldings .Then (fvg){ V_1UV_2 }= {f(V_1)Ug(V_2)}.But f(V_1)∈ D_3) , g(V_2) ∈V(D_4) .Thus {f(V_1)Ug(V_2)} ∈ V(D_3vD_4) , i.e., fvg maps vertices to vertices .Now,let e∈A(D_1vD_2).Then either e∈A(D_1) or e∈A(D_2) or e is an arc joining a vertex of D_1 (or D_2) to a vertex of D_2 (or D_1).In the first two cases and since each of f and g is a digraph folding ,(fvg){e} A(D_3vD_4).Now, if e=(v_1,v_2) , v_1∈ D_1 and v_2∈D_2 .Then(fvg){e}=(fvg){(v_1,v_2)}={f(v_1),g(v_2) ={(v_3,v_4)} A(D_3vD_4). Thus fvg maps arcs to arcs and hence the join digraph map is a digraph folding. The converse is guaranteed by the definition of the join digraph.

Example (4-4)

Let $D_1=(V_1,A_1)$, where $V_1=\{v_1,v_2,v_3,v_4\}$ and $A_1=\{e_1,e_2,e_3,e_4\}$ and $D_2=(V_2,A_2)$, where $V_2=\{v_5,v_6,v7\}$ and $A_2=\{e_5,e_6\}$, see Fig.3 (a) .

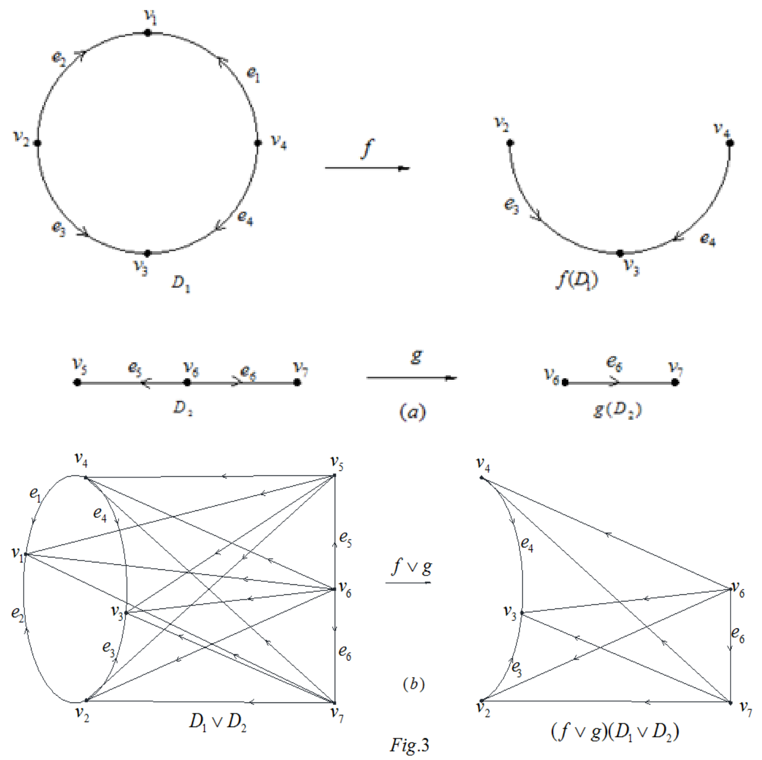


Fig.3

Let f Đ(D_1) be defined by $f\{v_1\}=\{v_3\}$ and $f\{e_1,e_2\}=\{e_4,e_3\}$ .Also , let g Đ(D_2) be defined by $g\{v_5\}=\{v7\}$ and $g\{e_5\}=\{e_6\}$.The join dimap

$fvg:D_1 v\ D_2 \rightarrow D_1 v\ D_2$ is defined by $(fvg)\{v_1,v_5\}=\{v_3,v7\}$and $(fvg)\{e_1\}=(fvg)\{(v_4,v_1)\}=\{(v_4,v_3)\}=\{e_4\}$,also,$(fvg)\{e_5\}=(fvg)\{(v_6,v_5)\}=\{(v_6,v7)\}=\{e_6\}$and $(fvg)\{(v_5,v_1)\}=\{(v7,v_3)\}$,and so on, see Fig.3 (b) .The adjacency matrices of D_1 , D_2 and D_1vD_2 are as follows :

$$M(D_1)=\begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} , \quad M(D_2)=\begin{array}{c c} & \begin{array}{ccc} v_5 & v_6 & v_7 \end{array} \\ \begin{array}{c} v_5 \\ v_6 \\ v_7 \end{array} & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

And $$M(D_1 \vee D_2)=\begin{array}{c c} & \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

By using these adjacency matrices we can describe the digraph foldings. The adjacency matrix $M(D_1)$ suggests that the vertex v_1 can be mapped to the vertex v_3 since the first and third columns of $M(D_1)$ have the same entries . Also the arc $(v_4,v_1)= e_1$ can be mapped to the arc$(v_4,v_3)=e_4$ and the arc$(v_2,v_1)=e_2$ can be mapped to the arc$(v_2,v_3)=e_3$ since the 1st and $_3$rd columns are the same.Again by using $M(D_2)$and $M(D_1vD_2)$we can describe the digraph foldings of both D_2 and D_1vD_2 .

(5) The Cartesian product of diagraph

Definition (5-1)

The Cartesian product D_1xD_2 of two simple diagraphs is a simple diagraph with vertex set $V(D_1xD_2)=V_1xV_2$ and arc set $A(D_1xD_2)=[(A_1xV_2)U(V_1xA_2)]$

such that two vertices (u_1,u_2) and (v_1,v_2) are adjacent in D_1xD_2 iff ,either

I. $u_1 = v_1$ and u_2 is adjacent to v_2 in D_2 , or
II. u_1 is adjacent to v_1 in D_1 , $u_2 = v_2$. Definition (5-2)

Let D_1,D_2,D_3 and D_4 be simple digraphs .Let $f: D_1 \rightarrow D_3$ and $g: D_2 \rightarrow D_4$ be diagraph maps .Then by the Cartesian product dimap fxg : $D_1xD_2 \rightarrow D_3 \times D_4$ we mean a dimap defined as follows :

I. If $v=(v_1,v_2) \in V_1xV_2$, $v_1 \in V_1$, $v_2 \in V_2$,then $(fxg)(v)=(fxg)(v_1,v_2)=(f(v_1),g(v_2))$ V_3xV_4
II. If the arc $e=\{(\{v_1\}_i , \{v_2\}_j),(\{v_1\}_i ,\{v_2\}_k)\}$, where $\{v_1\}_i \in V(D_1)$ and $\{v_2\}_j$,$\{v_2\}_k \in V(D_2)$, then $(fxg)\{e\}=(fxg)\{(\{v_1\}_i ,\{v_2\}_j) ,(\{v_1\}_i,\{v_2\}k)\} =\{(\{v_1\}_i, g\{v_2\}_j),(\{v_1\}_i, g\{v_2\}_k)\}$.
III. If the arc $e=\{(\{v_1\}_i,\{v_2\}_j),(\{v_1\}_k ,\{v_2\}_j)\}$, where $\{v_1\}_i ,\{v_1\}_k \in V(D_1)$ and $\{v_2\}_j \in V(D_2)$, then $(fxg)\{e\}=(fxg)\{(\{v_1\}_i ,\{v_2\}_j),(\{v_1\}_k , \{v_2\}_j)\}=\{(f\{v_1\}_i,\{v_2\}_j),(f\{v_1\}_k ,\{v_2\}_j)\}$. Note that if $g\{v_2\}j=g\{v_2\}k$or $f\{v_1\}i=f\{v_1\}k$,the image of the arc will be a vertex

Theorem (5-3)

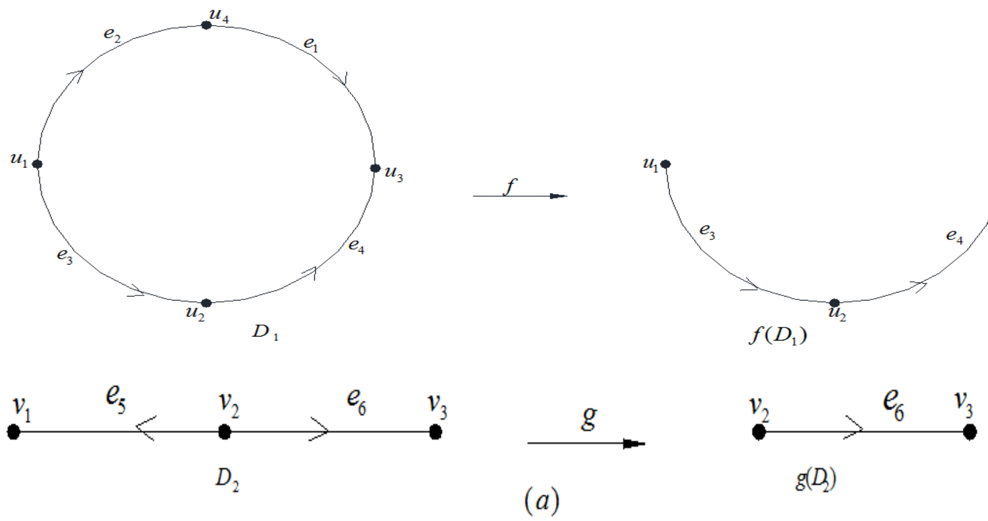
Let D_1 , D_2 , D_3 and D_4 be digraphs, let $f:D_1 \rightarrow D_3$ and $g :D_2 \rightarrow D_4$ be digraph maps . Then $(fxg) \in Đ(D_1xD_2,D_3xD_4)$ is a digraph folding iff f $Đ(D_1,D_3)$ and g $Đ(D_2,D_4)$ are digraph foldings.In this case $(fxg)(D_1xD_2)= f(D_1) \times g(D_2)$.

Proof:

Suppose f and g are digraph foldings. Then for each vertex $(v_1,v_2) \in V(D_1xD_2)=V_1xV_2$,$(fxg)\{(v_1,v_2)\}=\{(f(v_1),g(v_2))\}=(v_3,v_4) \in V(D_1xD_2)=V_3 xV_4$, i.e., fxg maps vertices to vertices . Now , let e $A(D_1 xD_2)$, then if $e=\{(\{v_1\}i,\{v_2\}j),(\{v_1\}k,\{v_2\}j)\}$, where $\{v_1\}i$ is adjacent to $\{v_1\}k$ in $D_1$and $\{v_2\}j$ then $(fxg)\{e\}=(fxg)\{(\{v_1\}i,\{v_2\}j),(\{v_1\}k,\{v_2\}j)\}=\{(f\{v_1\}i,\{v_2\}j),(f\{v_1\}k,\{v_2\}j)\}$, since $\{v_1\}i$ is adjacent to $\{v_1\}k$ and f is a digraph folding,Then $f\{v_1\}i\neq f\{v_1\}k$.Thus $(fxg)\{e\}$ $A(D_3xD_4)$.By the same procedure,if $e=\{(\{v_1\}i ,\{v_2\}j),(\{v_1\}i,\{v_2\}k)\}$, where $\{v_1\}i$ $V(D_1)$ and $\{v_2\}j$ is adjacent to$\{v_2\}k$in D_2,then $(fxg)\{e\} \in A(D_3xD_4)$ i.e., fxg maps arcs to arcs and hence the Cartesian product dimap is a digraph foldings .To prove the converse suppose that (fxg) is a digraph folding and f,or g ,is not a digraph folding. In this case f, or g , will maps an arc to a vertex ,say $f\{(u_1,v_1)\}=\{u_3\} \in V(D_3)$.Then $(fxg)\{(\{u_1\},\{v_2\}j),(\{v_1\},\{v_2\}j)\}=\{(f\{u_1\},\{v_2\}j),(f\{v_1\},\{v_2\}j)\}=\{(\{u_3\},\{v_2\}j),(\{u_3\},\{v_2\}j)\} \in V(D_3xD_4)$.This contradicts the assumption and thus each of f and g must be a digraph folding .

Examples (5-4)

(a) Let $D_1=(V_1,A_1)$, where $V_1= \{u_1, u_2, u_3, u_4\}$, $A=\{e_1,e_2,e_3,e_4\}$and $D_2=(V_2,A_2)$, where $V_2 = \{v_1,v_2,v_3\}$, $A= \{e_5,e_6\}$, see Fig .4 (a).



(a)

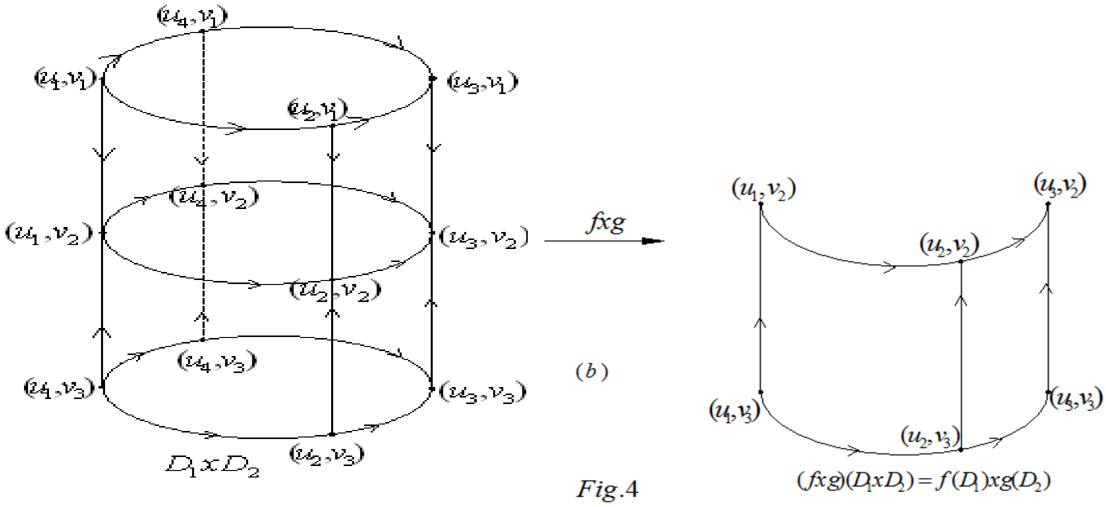


Fig.4

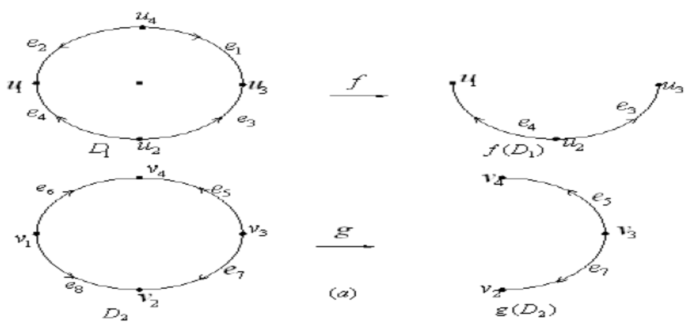
Let f Đ( ) defined by f{u_4}={u_2} and f{e_1,e_2}={e_4,e_3}.Also , let g Đ( ) defined by g{v_1} = {v_3}and g{e_5} = {e_6}.Then the Cartesian product dimap fxg : $D_1xD_2 \to D_3xD_4$ is defined as follows : (fxg){(u_4,v_1),(u_4,v_2),(u_4,v_3),(u_1,v_1),(u_2,v_1),(u_3,v_1)}={(u_2,v_1),(u_2,v_2),(u_2,v_3),(u_1,v_3),(u_2,v_3),(u_3,v_3)} .Also, (fxg){((u_4,v_1),(u_3,v_1)),((u_3,v_1),(u_3,v_2))}={((u_2,v_3),(u_3,v_3)),((u_3,v_3),(u_3,v_2))} and so on, see Fig.4(b).The adjacency matrices of D_1 , D_2 and D_1xD_2 are as follows:

$$M(D_1)=\begin{array}{c} \\ u_1\\ u_2\\ u_3\\ u_4\end{array}\begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4\end{array}\\ \begin{bmatrix}0&1&0&1\\0&0&1&0\\0&0&0&0\\0&0&1&0\end{bmatrix}\end{array} \quad , \quad M(D_2)=\begin{array}{c} \\ v_1\\ v_2\\ v_3\end{array}\begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3\end{array}\\ \begin{bmatrix}0&1&0\\0&0&0\\0&1&0\end{bmatrix}\end{array} \quad \textit{and}$$

$$M(D_1xD_2)=\begin{array}{c} \\ (u_1,v_1)\\ (u_2,v_1)\\ (u_3,v_1)\\ (u_4,v_1)\\ (u_1,v_2)\\ (u_2,v_2)\\ (u_3,v_2)\\ (u_4,v_2)\\ (u_1,v_3)\\ (u_2,v_3)\\ (u_3,v_3)\\ (u_4,v_3)\end{array}\begin{array}{c} \begin{array}{cccccccccccc} (u_1,v_1) & (u_2,v_1) & (u_3,v_1) & (u_4,v_1) & (u_1,v_2) & (u_2,v_2) & (u_3,v_2) & (u_4,v_2) & (u_1,v_3) & (u_2,v_3) & (u_3,v_3) & (u_4,v_3)\end{array}\\ \begin{bmatrix}0&1&0&1&1&0&0&0&0&0&0&0\\0&0&1&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0&0&0\\0&0&1&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&1&0&1\\0&0&0&0&0&1&0&0&0&0&1&0\\0&0&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&1&0&0&1&0\end{bmatrix}\end{array}$$

Once again we can describe the digraph foldings by using M(D_1),M(D_2) and M(D_1XD_2).For example, from M(D_1XD_2) we can see that the vertex (u_4,v_1) can be mapped to the vertex (u_2,v_1) since the second and fourth columns are the same . Also, the arc ((u_1,v_1),(u_4,v_1)) will be mapped to the arc ((u_1,v_3),(u_2,v_3)) since the vertex (u_4,v_1) is mapped to the vertex (u_2,v_3) and the vertex (u_1,v_1) is mapped to the vertex (u_1,v_3), and so on, see Fig.4 .

(b) Let D_1=(V_1,A_1), where V_1={u_1,u_2,u_3,u_4} , A_1={e_1,e_2,e_3,e_4} and D_2=(V_2,A_2) , where V_2 = {v_1,v_2,v_3,v_4}, A_2 = {e_5,e_6,e7,e8}, see Fig.5(a)



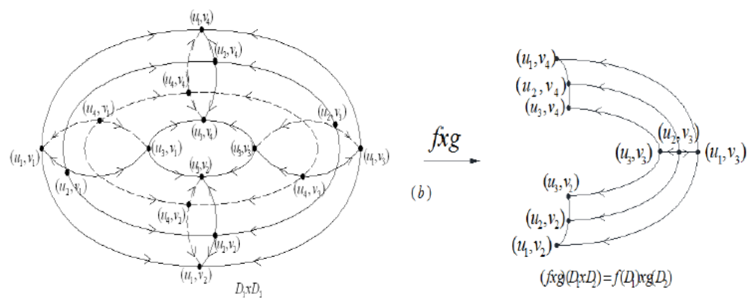


Figure 5

Let f $Đ(D_1)$ be defined by $f\{u_4\}=\{u_2\}$, $f\{e_1,e_2\} = \{e_3,e_4\}$ and g $Đ(D_2)$ be defined by $g\{v_1\} = \{v_3\}$, $g\{e_6, e8\}= \{e_5,e7\}$ .Then the cartesian product dimap $h = f \times g : D_1xD_2 \rightarrow D_1 \times D_2$ is defined as follows: $h\{(u_4,v_2), (u_3,v_1)\} = \{(u_2,v_2), (u_3,v_3)\}$, and so on . Also , $h\{((u_4,v_2),(u_1,v_2)),((u_3,v_1),(u_3,v_2))\}= \{((u_2,v_2),(u_1,v_2)),((u_3,v_3),(u_3,v_2))\}$, and so on, see Fig.$_5$(b) .The adjacency matrices of D_1,D_2 and $D_1 \times D_2$ are as follows :

$$M(D_1) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}\begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \quad , \quad M(D_2) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \text{ and}$$

$$M(D_1xD_2) = \begin{array}{c|cccccccccccccccc} & (u_1,v_1) & (u_2,v_1) & (u_3,v_1) & (u_4,v_1) & (u_1,v_2) & (u_2,v_2) & (u_3,v_2) & (u_4,v_2) & (u_1,v_3) & (u_2,v_3) & (u_3,v_3) & (u_4,v_3) & (u_1,v_4) & (u_2,v_4) & (u_3,v_4) & (u_4,v_4) \\ \hline (u_1,v_1) & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ (u_2,v_1) & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ (u_3,v_1) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ (u_4,v_1) & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ (u_1,v_2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (u_2,v_2) & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (u_3,v_2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (u_4,v_2) & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (u_1,v_3) & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ (u_2,v_3) & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ (u_3,v_3) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ (u_4,v_3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ (u_1,v_4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (u_2,v_4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ (u_3,v_4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (u_4,v_4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{array}$$

Once again we can describe the digraph foldings by using $M(D_1)$ and $M(D_1 xD_2)$ .For example, from $M(D_1 \times D_2)$ we can see that the vertex (u_4,v_2) can be mapped to the vertex (u_2,v_2) since the $_6$th and 8th rows have the same entries . And the vertex (u_3,v_1) can be mapped to the vertex (u_3,v_3) since the $_3$rd and $_{11}$th rows are the same .Also, the arcs $((u_4,v_2),(u_1,v_2))$and $((u_4,v_2),(u_3,v_2))$ can be mapped to the arcs $((u_2,v_2),(u_1,v_2))$ and $((u_2,v_2),(u_3,v_2))$,respectively, since the $_6$th and 8th rows are the same. Finally the arcs $((u_3,v_1),(u_3,v_2))$ and $((u_3,v_1),(u_3,v_4))$ can be mapped to the arcs $((u_3,v_3),(u_3,v_2))$ and $((u_3,v_3),(u_3,v_4))$,respectively, since the $_3$th and 11^{th} rows are the same . And so on , see Fig.$_5$(b) .

($_6$) The composition of digraphs

Definition (6-1)

The composition $D_1[D_2]$of two simple diagraphs is a simple diagraphs with $V(D_1[D_2])=V_1xV_2$. The vertices $u=(u_1,u_2)$ and $v=(v_1,v_2)$ are adjacent if either u_1 is adjacent to v_1 and $u_2=v_2$ or $u_1 = v_1$ and u_2 is adjacent to v_2 .

Definition (6-2)

Let D_1,D_2,D_3 and D_4 be simple diagraphs.Let $f: D_1 \rightarrow D_3$ and $g:D_2 \rightarrow D_4$ be diagraph maps .By the composition dimap $f[g]: D_1[D_2]\ D_3[D_4]$ we mean a map defined as follows

I. If $v=(v_1,v_2) \in V(D_1[D_2])=V_1xV_2$,then $f[g]\{(v_1,v_2)\}=\{(f(v_1),g(v_2))\} \in V(D_3[D_4])$

II. Let $e=\{(\{v_1\}i ,\{v_2\}j),(\{v_1\}k,\{v_2\}l)\}$.If $\{v_1\}i=\{v_1\}k$ and $\{v_2\}j$ is adjacent to $\{v_2\}l$,

then $f[g]\{e\}=\{(\{v_1\}i ,g\{v_2\}j),(\{v_1\}i ,g\{v_2\}l)\}$.Also, if $\{v_2\}j=\{v_2\}l$ and $\{v_1\}i$ is adjacent to $\{v_1\}k$, then $f[g]\{e\}=\{(f\{v_1\}i ,\{v_2\}j),(f\{v_1\}k ,\{v_2\}j)\}$.

Theorem (6-3)

Let D_1, D_2, D_3 and D_4 be digraphs. let $f:D_1 \to D_3$and $g:D_2 \to D_4$ be digraph maps .Then the composition dimap $Đ(D_1[D_2], D_3[D_4])$ is a digraph folding if f $Đ(D_1,D_3)$ and $g \in Đ(D_2,D_4)$ are digraph foldings .

Proof:

Let f and g be digraph folding , then

I. For each vertex $v=(v_1,v_2)$ $V(D_1[D_2])=V_1 x V_2$, $f[g]\{(v_1,v_2)\}=\{(f(v_1),g(v_2))\}$.But $f(v_1)$ $V(D_3)$ and $g(v_2)$ $V(D_4)$,then $\{(f(v_1),g(v_2))\}$ $V(D_3[D_4]$, i.e., f[g] maps vertices to vertices .

II. Let $e=\{(\{v_1\}i ,\{v_2\}j),(\{v_1\}k ,\{v_2\}l)\}$ and suppose $\{v_1\}i$ is adjacent to $\{v_1\}k$, then there exists an arc $\{(\{v_1\}i,\{v_1\}k)\} \in A_1$, since f is a digraph folding and $\{(\{v_1\}i,\{v_1\}k)\} \in A_1$, then $f[g]\{e\} \in A(D_3[D_4])$. Now, if $\{v_1\}i=\{v_1\}k$ and $\{v_2\}j$ is adjacent to $\{v_2\}l$,then $f[g]\{e\}=\{(\{v_1\}i ,g\{ v_2\}j),(\{v_1\}i ,g\{v_2\}l)\}$, since$\{v_2\}j$ is adjacent to $\{v_2\}l$, then there exists an arc $\{(\{v_2\}j \ ,\{v_2\}l)\} \in A_2$ such that $\{(g\{v_2\}j,g\{v_2\}l)\}$ A_3 ,i.e., $g\{v_2\}j \neq g\{v_2\}l$ and hence $f[g]\{e\}$ $A(D_3[D_4])$, i.e., f[g] maps arcs to arcs.

The converse is not true since if f[g] is a digraph folding and f ,or g ,is not a digraph folding .In this case f, or g, maps an arc to a vertex , say $f (u_1,v_1) = (u_3,u_3)$, $u_3 \in V(D_3)$.Then $f[g]\{(u_1,\{v_2\}i),(v_1,\{v_2\}j)\}=\{(f(u_1),g\{v_2\}i),(f(v_1),g\{v_2\}j)\}=\{(u_3 , g\{v_2\}i),(u_3 , g\{v_2\}j)\}$ which is an arc of $D_3[D_4]$.

Example (6-4)

Let D_1 , D_2 ,f and g be the digraphs and digraph foldings given in Example (5.4). The adjacency matrix of $D_1[D_2]$ is as follows :

$$M(D_1[D_2]) = \begin{array}{c} \\ (u_1,v_1) \\ (u_2,v_1) \\ (u_3,v_1) \\ (u_4,v_1) \\ (u_1,v_2) \\ (u_2,v_2) \\ (u_3,v_2) \\ (u_4,v_2) \\ (u_1,v_3) \\ (u_2,v_3) \\ (u_3,v_3) \\ (u_4,v_3) \end{array} \begin{array}{c} \begin{array}{cccccccccccc} (u_1,v_1) & (u_2,v_1) & (u_3,v_1) & (u_4,v_1) & (u_1,v_2) & (u_2,v_2) & (u_3,v_2) & (u_4,v_2) & (u_1,v_3) & (u_2,v_3) & (u_3,v_3) & (u_4,v_3) \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Now a digraph folding $f[g]: D_1[D_2] \to D_1[D_2]$ can be defined as follows: $f[g]\{(u_4,v_1),(u_4,v_2),(u_4,v_3),(u_1,v_1),(u_2,v_1),(u_3,v_1)\} = \{(u_2,v_1),(u_2,v_2),(u_2,v_3),(u_1,v_3),(u_2,v_3),(u_3,v_3)\}$. Also ,$f[g]\{((u_4,v_1),(u_3,v_1)),((u_3,v_2),(u_3,v_1)),((u_2,v_2),(u_3,v_1))\}=\{((u_2,v_1),(u_3,v_3)),((u_3,v_2),(u_3,v_3)),((u_2,v_2),(u_3,v_3))\}$, and so on, see Fig.6 .

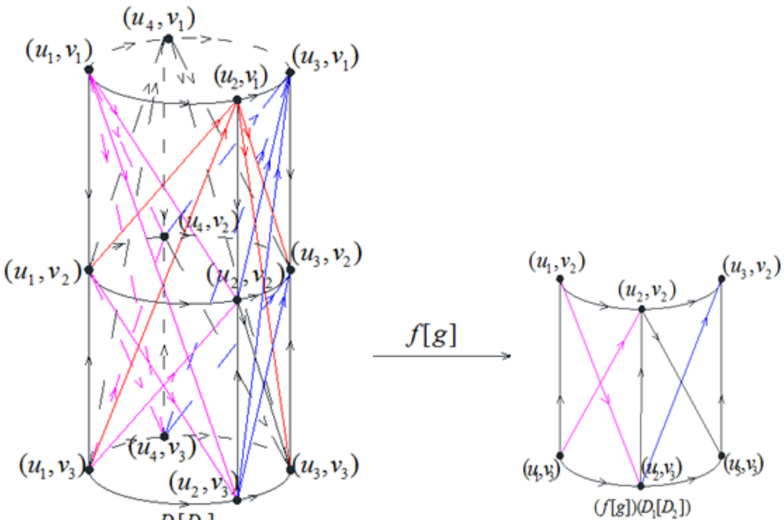


Fig.6

We can describe the digraph foldings by using $M(D_1)$, $M(D_2)$ and $M(D_1[D_2])$. For example , from $M(D_1[D_2])$ we can see that the vertex (u_4,v_1) can be mapped to the vertex (u_2,v_1) since the second and fourth rows have the same entries. Also, the arc $((u_1,v_1),(u_4,v_1))$ can be mapped to the arc $((u_1,v_1),(u_2,v_1))$ since the second and fourth rows are the same. Also the vertex (u_1,v_1) can be mapped to the vertex (u_1,v_3) since $_1$st and 9th rows have the same entries , and so on .

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