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RESEARCH ARTICLE

OSCILLATORY BEHAVIOUR OF THIRD ORDER NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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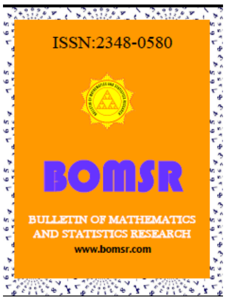

ABSTRACT

Sufficient conditions for oscillation of solutions of third order nonlinear neutral delay differential equations of the form

$$\frac{d}{dt}\left\{\frac{1}{r_1(t)}\frac{d}{dt}\left\{\frac{1}{r(t)}\left\{\frac{d}{dt}\left[y(t)+p(t)y(t-\tau)\right]\right\}\right\}\right\}+f(t)G(y(t-\sigma))=0$$

are obtained, where

$r_1(t), r(t),\ p(t)$, $f(t)$ are real valued continuous functions $r_1(t), r(t),\ p(t)>0$ and $f(t)\geq 0$.

Key words: Oscillation, Third order, Neutral differential equation

Mathematics Subject Classification (2010): 34C10

1. INTRODUCTION

In this paper we consider the nonlinear neutral delay differential equation of third order

$$\frac{d}{dt}\left\{\frac{1}{r_1(t)}\frac{d}{dt}\left\{\frac{1}{r(t)}\left\{\frac{d}{dt}\left[y(t)+p(t)y(t-\tau)\right]\right\}\right\}\right\}+f(t)G(y(t-\sigma))=0 \quad (1.1)$$

where $r_1(t), r(t)\in C([t_0,\infty),(0,\infty))$, $p(t), f(t)\in C([t_0,\infty),[0,\infty))$. We define a function

$$\int_{t_0}^{t} r(t)dt=\int_{t_0}^{t} r_1(t)\,dt \to\infty \ ,\ \ t_0\geq t$$

When *p(t)=0* the above equation reduces to the third order differential equation

$$\frac{d}{dt}\left\{\frac{1}{r_1(t)}\frac{d}{dt}\left\{\frac{1}{r(t)}\left\{\frac{d}{dt}\left[y(t)\right]\right\}\right\}\right\}+f(t)G(y(t-\sigma))=0 \qquad (1.2)$$

The study of behavior of solutions of differential equation (1.2) has been a subject of interest for several researchers. We mention the works of [1, 3, 5 and 7]. Oscillatory behavior of delay differential equations is extensively studied by several authors [6, 8, 13, 14 and 15].

By a solution of equation (1.1) we mean a function $y(t) \in C([T_y, \infty))$ where $T_y \geq t_0$ which satisfies (1.1) on $[T_y, \infty)$. We consider only those solutions of *y(t)* of (1.1) which satisfy $Sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_y$ and assume that (1.1) possesses such solutions.

A solution of equation (1.1) is called oscillatory if it has arbitrary large zeros on $[T_y, \infty)$; otherwise it is called nonoscillatory. Equation (1.1) is said to be oscillatory it all its solutions oscillate. Unless otherwise stated, when we write a functional inequality, it will be assumed to hold for sufficiently large t in our subsequent discussion.

2. MAIN RESULTS

We need the following in our discussion

$(H_1): r_1(t), r(t) \in C'([t_0 \infty), R)$, $r_1(t), r(t) > 0$

$(H_2): f(t) \in C([t_0, \infty), R)$, $f(t) > 0$

$(H_3): p(t) \in C(t_0, \infty), R)$ and $0 \leq p(t) \leq 1$

$(H_4): G(u) \in C(R, R)$ and $uG(u) > 0$, $u \neq 0$

(H_5): There exists $q \in C([t_0, \infty), [0, \infty))$ and $f(t)G(t) \geq q(t)x$

(H_6): $\int_{t_0}^{t} r(s)ds = \infty$ & $\int_{t_0}^{t} r_1(s)ds = \infty$

We set

$$z(t) = y(t) + p(t)y(t-\tau)$$

Lemma 2.1: Let (H_1) and (H_2) hold. Let $y(t)$ be a positive solution of (1.1) and suppose that

$$z(t) > 0,\ z'(t) > 0,\ \left(\frac{1}{r(t)} z'(t)\right)' > 0$$

holds. Then there exists $t_1 \geq t_0$ sufficiently large such that

$$\frac{1}{r(t-\sigma)} z'(t-\sigma) \geq \delta(t-\sigma) \frac{1}{r_1(t)} \left(\frac{1}{r(t)} z'(t)\right)' \quad \text{for} \quad t \geq t_1 \quad (2.1)$$

Let $D_0 = \{(t,s) \in: R^2 : t > s \geq t_0\}$ and $D = \{(t,s) \in R^2 : t \geq s \geq t_0\}$. The function $H \in C(D, R)$ is said to belong to a function of class *F,* if

(i) $H(t,t) = 0$ *for* $t \geq t_0$ and $H(t,s) > 0$ on D_0.

(ii) *H* has continuous partial derivatives $H'(t,s)$ on D_0 with respect to the second variable.

Theorem 2.2: Assume that $(H_1) - (H_4)$ hold. Let $\rho(t)$ be a positive real valued continuous differentiable function and let $H : D \to R$ be real valued continuous functions such that *H* belongs to the class *F* where

$$\underset{t\to\infty}{Lim\sup} \frac{1}{H(t,T_2)} \int_{T_2}^{t} \left[H(t,s)\rho(s)q(s)[1 - p(s-\sigma)] - \frac{k(s)}{H(t,s)} \phi^2(t,s) \right] ds = \infty \qquad (2.2)$$

where $k(s) = \dfrac{1}{4} \dfrac{1}{\delta(s-\sigma) r(s-\sigma) \rho(s)}$, $\qquad \phi(s) = H'(t,s)\rho(s) + H(t,s)\rho'(s)$

Then (1.1) is oscillatory.

Proof: Suppose to the contrary and let *y(t)* be a nonoscillatory solution of the equation (1.1). Without loss of generality we may assume that *y(t)* is eventually positive, since the case when y(t) is negative can be dealt similarly.

We set

$z(t) = y(t) + p(t)y(t-\tau)$

Hence from (H_5) equation (1.1) becomes

$$\frac{d}{dt}\left\{\frac{1}{r_1(t)}\frac{d}{dt}\left\{\frac{1}{r(t)}\left\{\frac{d}{dt}\left[y(t)+p(t)y(t-\tau)\right]\right\}\right\}\right\} \le -q(t)y(t-\sigma) \tag{2.3}$$

Further we obtain

$$\begin{aligned} y(t) &= z(t) - p(t)y(t-\tau) \\ y(t-\sigma) &= z(t-\sigma) - p(t-\sigma)y(t-\sigma-\tau) \\ &\ge z(t-\sigma) - p(t-\sigma)y(t-\sigma) \\ &\ge \left[1-p(t-\sigma)\right]z(t-\sigma) \end{aligned} \tag{2.4}$$

Hence from (2.3) and (2.4) we have

$$\frac{d}{dt}\left\{\frac{1}{r_1(t)}\frac{d}{dt}\left\{\frac{1}{r(t)}\left\{\frac{d}{dt}\left[y(t)+p(t)y(t-\tau)\right]\right\}\right\}\right\} \le -q(t)\left[1-p(t-\sigma)\right]z(t-\sigma) \tag{2.5}$$

Define

$$\omega(t) = \rho(t)\frac{\frac{1}{r_1(t)}\left(\frac{1}{r(t)}z'(t)\right)'}{z(t-\sigma)} \tag{2.6}$$

$$\omega'(t) = \rho'(t)\left[\frac{\frac{1}{r_1(t)}\left(\frac{1}{r(t)}z'(t)\right)'}{z(t-\sigma)}\right] + \rho(t)\left[\frac{\frac{1}{r_1(t)}\left(\frac{1}{r(t)}z'(t)\right)'}{z(t-\sigma)}\right]'$$

$$\omega'(t) = \rho'(t)\left[\frac{\frac{1}{r_1(t)}\left(\frac{1}{r(t)}z'(t)\right)'}{z(t-\sigma)}\right] + \rho(t)\left[\frac{z(t-\sigma)\left(\frac{1}{r_1(t)}\left(\frac{1}{r(t)}z'(t)\right)'\right)' - \frac{1}{r_1(t)}\left(\frac{1}{r(t)}z'(t)\right)'z'(t-\sigma)}{z^2(t-\sigma)}\right]$$

$$\omega'(t) = \rho'(t)\left[\frac{\frac{1}{r_1(t)}\left(\frac{1}{r(t)}z'(t)\right)'}{z(t-\sigma)}\right] + \rho(t)\left[\frac{z(t-\sigma)\left(\frac{1}{r_1(t)}\left(\frac{1}{r(t)}z'(t)\right)'\right)'}{z(t-\sigma)z(t-\sigma)}\right] - \rho(t)\left[\frac{\frac{1}{r_1(t)}\left(\frac{1}{r(t)}z'(t)\right)'z'(t-\sigma)}{z(t-\sigma)z(t-\sigma)}\right]$$

$$\omega'(t)=\rho'(t)\left[\frac{\frac{1}{r_1(t)}\left(\frac{1}{r(t)}z'(t)\right)'}{z(t-\sigma)}\right]+\rho(t)\left[\frac{\left(\frac{1}{r_1(t)}\left(\frac{1}{r(t)}z'(t)\right)'\right)'}{z(t-\sigma)}\right]-\rho(t)\left[\frac{\frac{1}{r_1(t)}\left(\frac{1}{r(t)}z'(t)\right)'z'(t-\sigma)}{z(t-\sigma)z(t-\sigma)}\right]$$

$$\omega'(t)\le\frac{\rho'(t)}{\rho(t)}\omega(t)-\rho(t)q(t)\left[1-p(t-\sigma)\right]-\rho(t)\frac{\omega(t)}{\rho(t)}\frac{\frac{\delta(t-\sigma)}{\frac{1}{r(t-\sigma)}}\frac{1}{r_1(t)}\left(\frac{1}{r(t)}z'(t)\right)'}{z(t-\sigma)}$$

$$\omega'(t)\le\frac{\rho'(t)}{\rho(t)}\omega(t)-\rho(t)q(t)\left[1-p(t-\sigma)\right]-\rho(t)\frac{\omega(t)}{\rho(t)}\frac{\omega(t)\delta(t-\sigma)r(t-\sigma)}{\rho(t)}$$

$$\omega'(t)\le-\rho(t)q(t)\left[1-p(t-\sigma)\right]+\frac{\rho'(t)}{\rho(t)}\omega(t)-\frac{\delta(t-\sigma)r(t-\sigma)}{\rho(t)}\omega^2(t)\qquad(2.7)$$

Multiplying the above inequality by *H(t,s),* we get

$$H(t,s)\omega'(t)\le-H(t,s)\rho(t)q(t)\left[1-p(t-\sigma)\right]+H(t,s)\frac{\rho'(t)}{\rho(t)}\omega(t)-H(t,s)\frac{\delta(t-\sigma)r(t-\sigma)}{\rho(t)}\omega^2(t)$$

or

$$H(t,s)\rho(t)q(t)\left[1-p(t-\sigma)\right]\le-H(t,s)\omega'(t)+H(t,s)\frac{\rho'(t)}{\rho(t)}\omega(t)-H(t,s)\frac{\delta(t-\sigma)r(t-\sigma)}{\rho(t)}\omega^2(t)$$

Using the integration by parts formula, we have

$$\int_{T_2}^{t}H(t,s)\rho(t)q(t)\left[1-p(t-\sigma)\right]ds\le-H(t,t)\omega(t)+H(t,T_2)\omega(T_2)+\int_{T_2}^{t}H'(t,s)\omega(s)ds$$
$$+\int_{T_2}^{t}H(t,s)\frac{\rho'(s)}{\rho(s)}\omega(s)ds-\int_{T_2}^{t}H(t,s)\frac{\delta(s-\sigma)r(s-\sigma)}{\rho(s)}\omega^2(s)ds\ (2.8)$$

Since *H(t,t)=0,* we obtain

$$\int_{T_2}^{t}H(t,s)\rho(t)q(t)\left[1-p(t-\sigma)\right]ds\le H(t,T_2)\omega(T_2)+\int_{T_2}^{t}\left[H'(t,s)\rho(s)+H(t,s)\rho'(s)\right]\frac{\omega(s)}{\rho(s)}ds$$
$$-\int_{T_2}^{t}H(t,s)\frac{\delta(s-\sigma)r(s-\sigma)}{\rho(s)}\omega^2(s)ds\ (2.9)$$

$$\int_{T_2}^{t} H(t,s)\rho(t)q(t)[1-p(t-\sigma)]ds \le H(t,T_2)\omega(T_2) + \int_{T_2}^{t} [\phi(t,s)]\frac{\omega(s)}{\rho(s)}ds - \int_{T_2}^{t} H(t,s)\frac{\delta(s-\sigma)r(s-\sigma)}{\rho(s)}\omega^2(s)ds$$

$$\le H(t,T_2)\omega(T_2) + \int_{T_2}^{t} \frac{1}{4}\frac{1}{H(t,s)\delta(s-\sigma)r(s-\sigma)\rho(s)}\phi^2(t,s)ds$$

$$- \int_{T_2}^{t}\left[\sqrt{H(t,s)\delta(s-\sigma)r(s-\sigma)\rho(s)}\,\frac{\omega(s)}{\rho(s)} - \frac{1}{2}\sqrt{\frac{1}{H(t,s)\delta(s-\sigma)r(s-\sigma)\rho(s)}}\phi(t,s)\right]^2 ds$$

Hence we obtain

$$\int_{T_2}^{t} H(t,s)\rho(t)q(t)[1-p(t-\sigma)]ds \le H(t,T_2)\omega(T_2) + \int_{T_2}^{T} \frac{k(s)}{H(t,s)}\phi^2(t,s)ds \qquad (2.10)$$

where $k(s) = \frac{1}{4}\frac{1}{\delta(s-\sigma)r(s-\sigma)\rho(s)}$. Then for all $t \ge T_2$ we have

$$\int_{T_2}^{t}\left[H(t,s)\rho(s)q(s)[1-p(s-\sigma)] - \frac{k(s)}{H(t,s)}\phi^2(t,s)\right]ds \le H(t,T_2)\omega(T_2) \qquad (2.11)$$

and this implies that

$$\operatorname*{Lim\,sup}_{t\to\infty}\frac{1}{H(t,T_2)}\int_{T_2}^{t}\left[H(t,s)\rho(s)q(s)[1-p(s-\sigma)] - \frac{k(s)}{H(t,s)}\phi^2(t,s)\right]ds \le \omega(T_2) \qquad (2.12)$$

which contradicts (2.2). This contradiction completes the proof.

Example 2.3: Consider the following neutral delay differential equation

$$\frac{d}{dt}\left\{\frac{1}{t}\frac{d}{dt}\left\{\frac{1}{t}\left\{\frac{d}{dt}\left[y(t)+\frac{1}{t}y(t-2\pi\right]\right\}\right\}\right\} + \frac{2}{t}G(y(t-\pi)=0 \qquad (2.13)$$

where

$r_1(t)=t$, $r(t)=t$, $p(t)=\frac{1}{t}$, $f(t)=\frac{2}{t}$

$\tau = 2\pi$, $\sigma=\pi$, $q(t)=\frac{1}{t^2}$

We set $\rho(t)=\frac{1}{t^3}$, $H(t,s)=(t-s)^2$

We observe that the conditions of the theorem are satisfied

Hence there exists an oscillatory solution in equation (2.13)

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