Vol.4.Issue.1.2016 (January-March)

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

http://www.bomsr.com

Email:editorbomsr@gmail.com

ISSN:2348-0580

RESEARCH ARTICLE

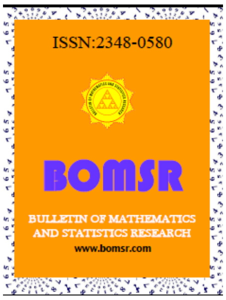
OSCILLATION OF SECOND ORDER NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS

K.V.V SESHAGIRI RAO[1], P.V.H.S SAI KUMAR[2]

[1]Department of Mathematics, Kakatiya Institute of Technology and Science (Autonomous), Warangal, Telangana, India.

[2]Department of Mathematics, Swarnandhra College of Engineering and Technology (Autonomous), Narsapuram, West Godavari, Andhra Pradesh, India.

E-mail address: perasaikumar@gmail.com



ABSTRACT

Sufficient conditions for oscillation of second order nonlinear neutral delay differential equations of the form

$$\frac{d}{dt}\left\{\frac{1}{r(t)}\frac{d}{dt}\{m(t)y(t)+p(t)y(t-\tau)\}\right\}+f(t)G(y(t-\sigma))=0$$

are obtained, where $r(t),\ m(t),\ p(t)$ are positive real valued continuous functions and $f(t)\geq 0.$

AMS Classification: 34C10.

Key Words: Oscillation, Second Order, Neutral Differential Equation.

1. INTRODUCTION

In this paper we consider the nonlinear neutral delay differential equation

$$\frac{d}{dt}\left\{\frac{1}{r(t)}\frac{d}{dt}\{m(t)y(t)+p(t)y(t-\tau)\}\right\}+f(t)G(y(t-\sigma))=0 \qquad (1)$$

where $r(t)\in C([t_0,\infty),(0,\infty))$, $p(t), f(t)\in C([t_0,\infty),[0,\infty))$.

In the equation (1) if $p(t)=0$ we get

$$\frac{d}{dt}\left\{\frac{1}{r(t)}\frac{d}{dt}\{m(t)y(t)\}\right\}+f(t)G(y(t-\sigma))=0 \qquad (2)$$

which is a delay equation and further if we take $p(t)\equiv 0$ and $\sigma=0$ in equation (1) we get

$$\frac{d}{dt}\left\{\frac{1}{r(t)}\frac{d}{dt}\{m(t)y(t)\}\right\}+f(t)G(y(t))=0 \qquad (3)$$

which is an ordinary differential equation.

The study of behavior of solutions of differential equation (2) has been a subject of interest for several researchers. We mention the works of [1, 2, 10 and 12].Oscillatory behavior of delay differential equations is extensively studied by several authors [3, 4, 6, 7, 8, and 9].

Now we see some special case of equation (1) ie when $r(t) \equiv 1$ and $G(y(t)) = y(t)$ equation (1) is reduced to

$$\frac{d^2}{dt^2}\{m(t)y(t) + p(t)y(t-\tau)\} + f(t)y(t-\sigma) = 0 \qquad (4)$$

if further when $p(t) \equiv 0,$ this equation is reduced to

$$\frac{d^2}{dt^2}\{m(t)y(t)\} + f(t)y(t-\sigma) = 0 \qquad (5)$$

and to

$$\frac{d^2}{dt^2}\{m(t)y(t)\} + f(t)y(t) = 0 \text{ if } \quad \sigma = 0 \qquad (6)$$

and we note that, when $m(t) = 1$, this equation reduces to the equation

$$\frac{d^2}{dt^2}\{y(t)\} + f(t)y(t) = 0 \qquad (7)$$

Recently there has been an increasing interest in the study of the oscillation of differential equations e.g. papers [1]-[16]. In particular, differential equations of the form (1) and for special cases when $r(t) \equiv 1$, and $G(y(t)) = y(t)$ is a subject of intensive research.

The oscillation for equation (7) has been discussed by many authors. Here we have some interesting results

(i) Jiqin Deng [6]: If for large $t \in R$,

$$\int_t^\infty f(s)\,ds \geq \frac{\alpha_0}{t} \text{ where } \alpha_0 > \frac{1}{4}$$

then equation (7) is oscillatory.

(ii) CH.G.Philos [10]: Let *n* be an integer with $n \geq 3$ and ρ be a positive continuously differentiable function on the interval $[t_0, \infty)$ such that

$$Lim_{t\to\infty} \sup \frac{1}{t^{n-1}} \int_{t_0}^{t} \frac{(t-s)^{n-3}}{\rho(s)} \left[(n-1)\rho(s) - (t-s)\rho'(s)\right]^2 ds < \infty$$

Than (7) is oscillatory if

$$Lim_{\,t\to\infty} \sup \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} \rho(s) f(s)\,ds = \infty$$

Motivated by some of these works, we present oscillation criteria under particular type of integral conditions

By a solution of equation (1) we mean a function $y(t) \in C([T_y, \infty))$ where $T_y \geq t_0$ which satisfies(1) on $[T_y, \infty)$. We consider only those solutions of *y(t)* of (1) which satisfy $Sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_y$ and assume that (1) possesses such solutions.

A solution of equation (1) is called oscillatory if it has arbitrary large zeros on $[T_y, \infty)$; otherwise it is called nonoscillatory. Equation (1) is said to be oscillatory it all its solutions oscillate .Unless otherwise stated, when we write a functional inequality, it will be assumed to hold for sufficiently large t in our subsequent discussion.

2. MAIN RESULTS

We need the following in our discussion

$(H_1): r(t) \in C([t_0\ \infty), R)$, $r(t) > 0$

$(H_2): f(t) \in C([t_0, \infty), R)$, $f(t) > 0$

$(H_3): p(t) \in C(t_0, \infty), R)$ and $0 \le p(t) \le 1$

$(H_4): G(u) \in C(R, R)$ and $uG(u) > 0$, $u \ne 0$

(H_5): There exists $q \in C([t_0, \infty), [0, \infty))$ and $f(t)G(x) \ge q(t)x$

$(H_6): m(t-\tau) > p(t)$

We set

$$z(t) = m(t)y(t) + p(t)y(t-\tau) \tag{8}$$

and we have the following lemma

Lemma 2.1 Assume that there exists $T \in [t_0 \infty)$, sufficiently large such that

$$z(t) > 0,\ z'(t) > 0,\ \left(\frac{1}{r(t)} z'(t)\right)' < 0,\ t \in [T, \infty)$$

Then
$$\frac{z(t-\sigma)}{z(t)} \ge \frac{\int_T^{t-\sigma} r(s)ds}{\int_T^t r(s)ds}$$

The details of the proof are omitted.

Now we present the oscillation criteria

Theorem 2.2

Assume that conditions (*H_1*) - (*H_6*) are satisfied and

Let $B_k = \{u \in C'[s_k, t_k] : u(t) \ne 0, u(s_k) = u(t_k) = 0\}$, $k = 1, 2.$ If there exists a function $u \in B_k$, $\beta(t) \in C'([t_0, \infty), (0, \infty))$, and $\alpha(t) \in C'([t_0, \infty), R)$ such that for k=1, 2,

$$J_k(u, \beta, \alpha) = \int_{s_k}^{t_k} \left\{ \beta(t) \left[u^2(t) \left(\frac{1}{m(t-\sigma)} q(t)\{1 - p(t-\sigma)\} \frac{\int_T^{t-\sigma} r(s)ds}{\int_T^t r(s)ds} + \frac{1}{r(t)}\alpha^2(t) - \left\{\frac{1}{r(t)}\alpha(t)\right\}' \right) - \frac{1}{r(t)}\left(u'(t) + \frac{u(t)\beta'(t)}{2\beta(t)} + u(t)\alpha(t) \right)^2 \right] \right\} dt > 0$$

Then every solution of equation (1) is oscillatory.

Proof: Suppose to the contrary. And let y*(t)* be a nonoscillatory solution of equation (1). Without loss of generality we may assume that *y(t)* is eventually positive. From (H_5) and *(1)* we have

$$\frac{d}{dt}\left\{\frac{1}{r(t)}\frac{d}{dt}\{m(t)y(t)+p(t)y(t-\tau)\}\right\}=-f(t)G(y(t-\sigma))$$

$$\frac{d}{dt}\left\{\frac{1}{r(t)}\frac{d}{dt}\{m(t)y(t)+p(t)y(t-\tau)\}\right\}\le -q(t))y(t-\sigma) \tag{9}$$

Further we obtain that

$$m(t)y(t)=z(t)-p(t)y(t-\tau)$$
$$m(t-\sigma)y(t-\sigma)=z(t-\sigma)-p(t-\sigma)y(t-\sigma-\tau)$$
$$m(t-\sigma)y(t-\sigma)\ge z(t-\sigma)-p(t-\sigma)y(t-\sigma)$$
$$m(t-\sigma)y(t-\sigma)\ge [1-p(t-\sigma)]z(t-\sigma)$$
$$y(t-\sigma)\ge \frac{1}{m(t-\sigma)}[1-p(t-\sigma)]z(t-\sigma) \tag{10}$$

From (9) and (10) we have

$$\frac{d}{dt}\left\{\frac{1}{r(t)}\frac{d}{dt}\{m(t)y(t)+p(t)y(t-\tau)\}\right\}\le -\frac{1}{m(t-\sigma)}q(t)[1-p(t-\sigma)]z(t-\sigma) \tag{11}$$

From **Lemma 2.1** we have

$$z(t-\sigma)\ge \frac{\int_T^{t-\sigma} r(s)ds}{\int_T^{t} r(s)ds}z(t)$$

$$-z(t-\sigma)\le -\frac{\int_T^{t-\sigma} r(s)ds}{\int_T^{t} r(s)ds}z(t) \tag{12}$$

Substituting (12) in (11) we get

$$\frac{d}{dt}\left\{\frac{1}{r(t)}\frac{d}{dt}\{m(t)y(t)+p(t)y(t-\tau)\}\right\}\le -\frac{1}{m(t-\sigma)}q(t)[1-p(t-\sigma)]\frac{\int_T^{t-\sigma} r(s)ds}{\int_T^{t} r(s)ds}z(t) \tag{13}$$

Define

$$\omega(t)=-\beta(t)\left[\frac{\frac{1}{r(t)}z'(t)}{z(t)}+\frac{1}{r(t)}\alpha(t)\right] \tag{14}$$

$$\omega'(t)=-\beta'(t)\left[\frac{\frac{1}{r(t)}z'(t)}{z(t)}+\frac{1}{r(t)}\alpha(t)\right]-\beta(t)\left[\frac{\frac{1}{r(t)}z'(t)}{z(t)}+\frac{1}{r(t)}\alpha(t)\right]'$$

$$\omega'(t) = -\beta'(t)\left[\frac{\frac{1}{r(t)}z'(t)}{z(t)} + \frac{1}{r(t)}\alpha(t)\right] - \beta(t)\left[\frac{\frac{1}{r(t)}z'(t)}{z(t)}\right]' - \beta(t)\left[\frac{1}{r(t)}\alpha(t)\right]'$$

$$\omega'(t) = -\beta'(t)\left[\frac{\frac{1}{r(t)}z'(t)}{z(t)} + \frac{1}{r(t)}\alpha(t)\right] - \beta(t)\left[\frac{z(t)\left\{\frac{1}{r(t)}z'(t)\right\}' - \left\{\frac{1}{r(t)}z'(t)\right\}z'(t)}{z^2(t)}\right] - \beta(t)\left[\frac{1}{r(t)}\alpha(t)\right]'$$

$$\omega'(t) = -\beta'(t)\left[\frac{\frac{1}{r(t)}z'(t)}{z(t)} + \frac{1}{r(t)}\alpha(t)\right] - \beta(t)\left[\frac{\left\{\frac{1}{r(t)}z'(t)\right\}'}{z(t)}\right] + \beta(t)\left[\frac{\left\{\frac{1}{r(t)}z'(t)\right\}z'(t)}{z^2(t)}\right] - \beta(t)\left[\frac{1}{r(t)}\alpha(t)\right]' \tag{15}$$

$$\omega'(t) = \beta'(t)\left[\frac{\omega(t)}{\beta(t)}\right] - \beta(t)\left[\frac{\left\{\frac{1}{r(t)}z'(t)\right\}'}{z(t)}\right] + \beta(t)\left[\frac{\frac{1}{r(t)}\{z'(t)\}^2}{z^2(t)}\right] - \beta(t)\left[\frac{1}{r(t)}\alpha(t)\right]' \tag{16}$$

from (14) we have

$$\frac{\omega(t)}{-\beta(t)} = \left[\frac{\frac{1}{r(t)}z'(t)}{z(t)} + \frac{1}{r(t)}\alpha(t)\right]$$

$$\frac{\omega(t)}{-\beta(t)} - \frac{1}{r(t)}\alpha(t) = \frac{\frac{1}{r(t)}z'(t)}{z(t)}$$

$$r(t)\left[\frac{\omega(t)}{-\beta(t)} - \frac{1}{r(t)}\alpha(t)\right] = \frac{z'(t)}{z(t)}$$

i.e.

$$\frac{z'(t)}{z(t)} = r(t)\left[\frac{\omega(t)}{-\beta(t)} - \frac{1}{r(t)}\alpha(t)\right]$$

$$\left\{\frac{z'(t)}{z(t)}\right\}^2 = \left[\frac{\omega(t)r(t)}{-\beta(t)} - \alpha(t)\right]^2$$

$$\left\{\frac{z'(t)}{z(t)}\right\}^2 = \frac{\omega^2(t)r^2(t)}{\beta^2(t)} + \alpha^2(t) + 2\frac{\omega(t)r(t)}{\beta(t)}.\alpha(t) \tag{17}$$

Substituting (13) and (17) in (16) we get

$$\omega'(t) \geq \frac{\beta'(t)}{\beta(t)}\omega(t) + \beta(t)\left[\frac{1}{m(t-\sigma)}q(t)[1-p(t-\sigma)]\frac{\int_T^{t-\sigma} r(s)ds}{\int_T^{t} r(s)ds}\right] \tag{18}$$

$$+\beta(t)\frac{1}{r(t)}\left[\frac{\omega^2(t)r^2(t)}{\beta^2(t)} + \alpha^2(t) + 2\frac{\omega(t)r(t)}{\beta(t)}.\alpha(t)\right] - \beta(t)\left[\frac{1}{r(t)}\alpha(t)\right]'$$

$$\omega'(t) \geq \beta(t)\left\{\left[\frac{1}{m(t-\sigma)}q(t)[1-p(t-\sigma)]\frac{\int_T^{t-\sigma} r(s)ds}{\int_T^{t} r(s)ds}\right] + \left\{\frac{1}{r(t)}\alpha^2(t)\right\} - \left[\frac{1}{r(t)}\alpha(t)\right]'\right\} \tag{19}$$

$$+\left\{\frac{\beta'(t)}{\beta(t)} + 2.\alpha(t)\right\}\omega(t) + \left\{\frac{r(t)}{\beta(t)}\right\}\omega^2(t)$$

Let $u \in B_1$ be given as in the hypothesis. Multiplying (19) by u^2 and integrating the resulting inequality from s_1 to t_1 we have

$$\int_{s_1}^{t_1} u^2(t)\omega'(t)dt \geq$$

$$\int_{s_1}^{t_1} u^2\beta(t)\left\{\left[\frac{1}{m(t-\sigma)}q(t)[1-p(t-\sigma)]\frac{\int_T^{t-\sigma} r(s)ds}{\int_T^{t} r(s)ds}\right] + \left\{\frac{1}{r(t)}\alpha^2(t)\right\} - \left[\frac{1}{r(t)}\alpha(t)\right]'\right\}dt \tag{20}$$

$$+\int_{s_1}^{t_1} u^2\left\{\frac{\beta'(t)}{\beta(t)} + 2.\alpha(t)\right\}\omega(t)\,dt + \int_{s_1}^{t_1} u^2\left\{\frac{r(t)}{\beta(t)}\right\}\omega^2(t)dt$$

Integrating (20) by parts and using the fact that $u(s_1) = u(t_1) = 0$ we deduce that

$$-\int_{s_1}^{t_1} 2u(t)u'(t)\omega(t)dt \geq$$

$$\int_{s_1}^{t_1} u^2\beta(t)\left\{\left[\frac{1}{m(t-\sigma)}q(t)[1-p(t-\sigma)]\frac{\int_T^{t-\sigma} r(s)ds}{\int_T^{t} r(s)ds}\right] + \left\{\frac{1}{r(t)}\alpha^2(t)\right\} - \left[\frac{1}{r(t)}\alpha(t)\right]'\right\}dt$$

$$+\int_{s_1}^{t_1} u^2\left\{\frac{\beta'(t)}{\beta(t)} + 2.\alpha(t)\right\}\omega(t)\,dt + \int_{s_1}^{t_1} u^2\left\{\frac{r(t)}{\beta(t)}\right\}\omega^2(t)dt$$

$$\int_{s_1}^{t_1} u^2\left\{\frac{r(t)}{\beta(t)}\right\}\omega^2(t)dt + 2u\omega(t)\left[u'+u\left(\frac{\beta'(t)}{2\beta(t)}+\alpha(t)\right)\right]dt$$

$$+\int_{s_1}^{t_1} u^2\beta(t)\left\{\left[\frac{1}{m(t-\sigma)}q(t)[1-p(t-\sigma)]\frac{\int_T^{t-\sigma} r(s)ds}{\int_T^{t} r(s)ds}\right]+\left\{\frac{1}{r(t)}\alpha^2(t)\right\}-\left[\frac{1}{r(t)}\alpha(t)\right]'\right\}dt \le 0$$

Hence

$$\int_{s_1}^{t_1} u^2\beta(t)\left\{\left[\frac{1}{m(t-\sigma)}q(t)[1-p(t-\sigma)]\frac{\int_T^{t-\sigma} r(s)ds}{\int_T^{t} r(s)ds}\right]+\left\{\frac{1}{r(t)}\alpha^2(t)\right\}-\left[\frac{1}{r(t)}\beta(t)\right]'\right\}dt$$

$$+\left\{\int_{s_1}^{t_1} u(t)\omega(t)\frac{\sqrt{r(t)}}{\sqrt{\beta(t)}}+\frac{\sqrt{\beta(t)}}{\sqrt{r(t)}}\left[u'+u\frac{\beta'(t)}{2\beta(t)}+u\alpha(t)\right]\right\}^2 dt - \frac{\beta(t)}{r(t)}\left[u'+u\frac{\beta'(t)}{\beta(t)}+u\alpha(t)\right]^2 dt \le 0$$

which is equivalent to

$$\left\{\int_{s_1}^{t_1} u(t)\omega(t)\frac{\sqrt{r(t)}}{\sqrt{\beta(t)}}+\frac{\sqrt{\beta(t)}}{\sqrt{r(t)}}\left[u'+u\frac{\beta'(t)}{2\beta(t)}+u\alpha(t)\right]\right\}^2 dt + J_1(u,\beta,\alpha) \le 0 \qquad (21)$$

where $J_1(u,\beta,\alpha)$ is as in **Theorem 2.2.**Since $J_1(u,\beta,\alpha) > 0$, inequality (21) yields

$$\left\{\int_{s_1}^{t_1} u(t)\omega(t)\frac{\sqrt{r(t)}}{\sqrt{\beta(t)}}+\frac{\sqrt{\beta(t)}}{\sqrt{r(t)}}\left[u'+u\frac{\beta'(t)}{2\beta(t)}+u\alpha(t)\right]\right\}^2 dt \le -J_1(u,\beta,\alpha) < 0$$

which is a contradiction. Hence y(t) is oscillatory.

REFERENCES

[1]. R.P. Agarwal, S. R. Grace and D. O Regan, Oscillation Theory for Difference and functional Differential Equations, Kluwer Dordrecht, 2000.

[2]. Baculikova, B, Džurina, J: Oscillation theorems for higher order neutral differential equations. Appl. Math. Comput.219, 3769-3778 (2012).

[3]. L. Erbe, A. Peterson and S. H. Saker, Kamenev-type oscillation criteria for Second order linear delay dynamic equations, *Dynam. Syst. Appl.* 15 (2006) 65–78.

[4]. A. F. G¨uvenilir and A. Zafer, Second order oscillation of forced functional Differential equations with oscillatory potentials, *Comp. Math. Appl.* 51 (2006) 1395–1404.

[5]. Hale, JK: Theory of Functional Differential Equations. Springer, New York (1977)

[6]. Jiqin Deng, Oscillation criteria for second order linear differential equations, J.Math.Appl.271 (2002) 283-287.

[7]. I. V. Kamenev, An integral criterion for oscillation of linear differential equations of second order, Math. Zametki **23** (1978) 249-251.

[8]. A. H. Nasr, Sufficient conditions for the oscillation of forced super-linear secondorder differential equations with oscillatory potential, *Proc. Amer. Math. Soc.* 126(1998) 123–125.

[9]. A¨Ozbekler, J. S. W. Wong and A. Zafer, Forced oscillation of second-order Nonlinear drifferential equations with positive and negative coefficients, Appl.Math. Letters **24** (2011)1225-1230.

[10]. CH..G Philos Oscillation of second order linear ordinary differential equations with alternating coefficients, Bull.Austral.Math.Soc.Vol.27(1983),307-313.

[11]. Yu. V. Rogovchenko, On oscillation of a second order nonlinear delay differential equation, Funkcial. Ekvac.**43**(2000), 1-29.

[12]. Y. G. Sun and J. S. Wong, Oscillation criteria for second order forced ordinarydifferential equations with mixed nonlinearities, *J. Math. Anal. Appl.* 334 (2007); 549–560.

[13]. J. S.W.Wong, Oscillation criteria for forced second order linear differential equations, *J. Math. Anal. Appl.* 231 (1999) 235–240.

[14]. P. G. Wang, Oscillation criteria for second order neutral equations with distributed deviating arguments, Comput. Math. Appl. **47** (2004) 1935-1946.

[15]. Zhong, J, Ouyang, Z, Zou, S: An oscillation theorem for a class of second-order forced neutral delay differentialequations with mixed nonlinearities. Appl. Math. Lett. 24, 1449- 1454 (2011).

[16]. Q. Zhang and L. Wang, Oscillatory behavior of solutions for a class of second order nonlinear differential equation with perturbation, Acta Appl. Math. **110** (2010) 885-893