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RESEARCH ARTICLE

HOMOTOPY ANALYSIS METHOD FOR SOLVING FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND

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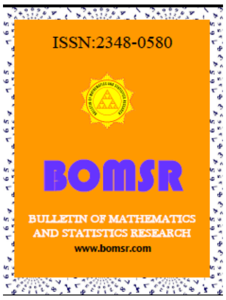

ABSTRACT

This research presents Homotopy Analysis Method to Fredholm integral equations of the second kind. An analytical method solving linear and nonlinear equations. The Homotopy Analysis Method provides an efficient and powerful tool in solving integral equations.The method provides a great freedom in choosing an auxiliary parameter h, an auxiliary linear operator L, an auxiliary nonlinear operator N, to analyze strongly linear and nonlinear problems. Application of the method to Fredholm integral equations of the second kind is analyzed which gives a rapid convergence for the solutions.

Keywords: Homotopy Analysis Method; Fredholm integral equations; Auxiliary parameter h

1. INTRODUCTION

In this research, we apply homotopy analysis method (HAM) which was proposed by Liao in 1992 (Liao,1992, 1997, 2003, 2009). In this method, the solution is considered as the summation of an infinite series, which usually converges rapidly to the exact solution. The HAM is based on homotopy, a fundamental concept in topology and differential geometry. Briefly speaking, by means of the HAM one constructs a continuous mapping of an initial guess approximation to the exact solution of considered equations. An auxiliary linear operator is chosen to construct such kind of continuous mapping and an auxiliary parameter is used to ensure the convergence of solution series. The method enjoys great freedom in choosing initial approximation and auxiliary linear operators. The approximations obtained by the HAM are uniformly valid not only for small parameters, but also for very large parameters. Until recently, the application of the homotopy analysis method in nonlinear problems has been devoted by scientists and engineers. Some different valid methods for solving integral equation have been developed in the last years. In this paper, we present an iterative scheme based on the HAM for linear and non-linear fredholm integral equations of the second kind.

$$y(x) = g(x) + \lambda \int_r^s H(x,t)y(t)dt \tag{1.1}$$

Where H(x, t) is a known continuous function in two variables referred to as the Kernel of the integral equation. g(x) is the free or forcing term, which is also known and continuous; y(x) is the unknown function; r, s - are constants and limits of integration: λ is a numeric parameter.

The integral on the RHS of (1.1) can be considered as an integral of the parameter t. The kernel of the equation $H(x,t)$ is defined on the $x - t$ plane in the square R. where $R: r \leq x \leq s; r \leq t \leq s$. A study of other analytical methods used in solving these problems can be seen in (Wazwaz, 2011)

Two continuous maps $f, g : X \to Y$ between topological spaces X and Y are said to be homotopic, $f \simeq g$, if there exist a continuous map $H : X \times [0,1] \to Y$ with $H(x,0) = f(x) \text{for all } x \in X$ and $H(x,1) = g(x) \text{for all } x \in X$. In this case we write $f \simeq_H g$. For fixed $t \in [0,1]$, we have continuous map $H_t : X \to Y,\ H_t(x) = H(x,t)$ with $H_0 = f$ and $H_1 = f$

If $[0,1]$ is a time interval then

At time $t = 0 : H_t$ has the form f

At time $0 < t < 1 : H_t$ changes continuously its form

At time $t = 1 : H_t$ has the form g

"H is the continuous deformation of the map f into the map g"

Let $f, g : X \to R^2$ be continuous maps, then $f \simeq g$, because the map $H : X \times [0,1] \to R^2$ define by $H(x,t) = (1-t)f(x) + tg(x)$ is a homotopy between f and g. Since H is continuous, $H_0 = f, H_1 = g$. This is called the straight-line homotopy. $f \simeq g$ if R^2 is replaced by any convex set, C. An introduction to the concepts of homotopy and their applications are covered in (Adams and Franzosa, 2008) and (Munkres, 2000)

2.0 Description of the method

Consider the following equation

$$N[y(x)] = 0 \tag{2.1}$$

Where N is an operator, $y(x)$ is unknown function and x the independent variable. Let $y_0(x)$ denote an initial guess of the exact solution y(x),h≠0 an auxiliary parameter (a convergence-control paremeter), H(x)≠0 an auxiliary function,and L an auxiliary linear operator with the property L[r(x)]=0 when r(x)= 0. Then using q∈[0,1] as an embedding parameter, we construct such a homotopy

$$(1-q)L[\phi(x:q) - y_0(x)] - qhH(x)N[\phi(x,q)] = 0 \tag{2.2}$$

It should be emphasized that we have great freedom to choose the initial guess y0(x), the auxiliary linear operator L, the non-zero auxiliary parameter h, and the auxiliary function H(x). We have the so-called zero-order deformation equation

$$(1-q)L[\phi(x:q) - y_0(x)] = qhNH(x)[\phi(x,q)] \tag{2.3}$$

The zero-order deformation equation (2.2) becomes

$\phi(x:0) = y_0(x)$ when $q = 0$

and

$\phi(x:1) = y(x)$, when $q = 1$, since h≠0 and H(x) ≠0.

Thus according to (2.2) and (2.3) as the embedding parameter increases from 0 to 1, $\phi(x:1)$ varies continuously from the initial approximation $y_0(x)$ to the exact solution y(x). such a kind of continuous variation is an analogues of the homotopy deformation concepts in topology.

By Taylor's theorem, $\phi(x:q)$ can be expanded in a power series of q as follows

$$\phi(x:q) = y_0(x) + \sum_{n=1}^{\infty} y_n(x)q^n \tag{2.4}$$

where

$$y_n(x) = \frac{1}{n!}\frac{\delta^n \phi(x:q)}{\delta q^n}\Big|_{q=0} \tag{2.5}$$

If the initial guess $y_0(x)$, the auxiliary linear parameter L, the non-zero auxiliary parameter h, and the auxiliary function H(x) are properly chosen so that the power series (2.4) of $\phi(x:q)$ converges at $q = 1$.Then,we have under these assumptions the solution series

$$y(x) = \phi(x:1) = y_0(x) + \sum_{n=1}^{\infty} y_n(x) \tag{2.6}$$

where the vector is defined

$$\vec{y}_n(x) = \{y_0(x), y_2(x), .., y_n(x)\} \tag{2.7}$$

$$L[y_n(x) - \chi_n y_{n-1}(x) = hH(x)R_n(\vec{y}_{n-1}(x)), \quad y_n(0) = 0 \tag{2.8}$$

Where

$$R_n(\vec{y_{n-1}}(x)) = \frac{1}{(n-1)!}\frac{\delta^{n-1}N[\phi(x:q)]}{\delta q^{n-1}}\Big|_{q=0} \tag{2.9}$$

and

$$\chi_n = \begin{cases} 0, & for\ n \leq 1 \\ 1, & for\ n \geq 1 \end{cases}$$

The high-order deformation equation (2.8) is governing by the linearoperator L, and the term $R_n(\vec{y_{n-1}}(x))$ can be expressed simply by (2.9) for anynonlinear operator N. Therefore, $y_n(x)$ can be easily gained especially by means of computational software such as MATLAB. The solution y(x) given by the aboveapproach is independent of L, h, H(x) and $y_0(x)$ (Liao, *et al.,* 2014). Thus, unlike all previous analytical techniques, the convergence region and rate of solution series given by the above approach might not be uniquely determined. If $\sum_{n=1}^{\infty} y_n(x)$ tends uniformly to a limit as n→∞, this limit is the required solution (Vahdati, *et al*. 2010).

3.0 HAM's Solution to Fredholm Integral Equations

Let consider the equation

$$h(t)u(t) = g(t) + \lambda \int_r^s H(t,x)\, u(x)dx \tag{3.1}$$

where the solution to equation (3.1) of Fredholm integral equations of the second kind.

3.1 Fredholm integral equations of the second kind

If $h(t) = 1$ is substituted into equation (3.1),we have

$$u(t) = g(t) + \lambda \int_r^s H(t,x)u(x)dx \qquad b \leq t \leq c \tag{3.2}$$

From (2.2) we construct the zeroth-order deformation for this kind of integral equations as

$$(1-q)(u(t,q,h) - g(x)) = hq(g(t)H(t) + \int_r^s H(t,x)u(x,q,h)dx) \tag{3.3}$$

For $q = 0$ and $q = 1$, we have

$$u(t,0,h) = g(t)$$
$$u(t,1,h) = u(t)$$

For Maclaurin series of $u(t,q,h)$ corresponding to q, we have

$$u(t,q,h) = u(t,0,h) + \sum_{n=1}^{+\infty} \frac{u_0^{[n]}(t,h)}{n!} q^n \tag{3.4}$$

Which

$$u_0^{[n]}(t,h) = \frac{\delta^n u(t,q,h)}{\delta q^n}\Big|_{q=0} \tag{3.5}$$

Substituting $q = 1$ into (3.4) give

$$u(t) = g(t) + \sum_{n=1}^{+\infty} \frac{u_0^{[n]}(t,h)}{n!} \tag{3.6}$$

where we obtain the nth-order deformation equation

$$L[u_0^{[n]}(t,h) - \chi_n u_0^{[n]}(t,h)] = hR_n(\vec{u}_{n-1}) \tag{3.7}$$

And the solution of the nth-order deformation equation for $n \geq 1$ yields

$$u_0^{[1]}(t,h) = -h \int_r^s H(t,x) g(x) dx \tag{3.8}$$

and

$$\frac{u_0^{[n]}(t,h)}{n!} = \frac{u_0^{[n-1]}(t,h)}{(n-1)!} + h\frac{(u_0^{[n-1]}(t,h))}{(n-1)!} - h\int_r^s H(x,t)\frac{u_0^{[n-1]}(x,h)}{(n-1)!} dx \ . \tag{3.9}$$

Choosing $h = -1$ the solution of the problem is similar to the Homotopy Perturbation Method, (Vahdati, *et al*,2010) and (Liao,et al 2013)

4.0 Application of HAM

In this section, we apply the HAM for solving Fredholm integral equations of the second kind (1.1) by studying numerical examples with given initial conditions and kernel. MATLAB software will be used to demonstrate the nature of the exact solution that will be provided by the HAM. A MATLAB code for getting these exact solutions are also presented.

Example 1. Let consider the following Fredholm integral equation

$$\phi(x) = 3x + \frac{1}{\alpha}\int_0^1 xt\phi(t)dt. \tag{4.1}$$

To begin with, we choose

$$\phi_0(x) = 3x \tag{4.2}$$

We choose the linear operator

$$L[\phi(x,q)] = \phi(x,q) \tag{4.3}$$

And we now define a nonlinear operator as

$$N[\phi(x,q)] = \phi(x,q) - 3x + \frac{1}{\alpha}\int_0^1 xt\phi(t)dt \tag{4.4}$$

where we construct the nth-order deformation equation

$$L[\phi_n - \chi_n\phi_{n-1}] = hR_n(\vec{\phi}_{n-1}) \tag{4.5}$$

And

$$R_n(\vec{\phi}_{n-1}) = \phi_{n-1}(x) - (1-\chi_n)3x - \frac{1}{\alpha}\int_0^1 xt\phi_{n-1}(t)dt \tag{4.6}$$

where the solution of the nth-order deformation equation(4.5)

$$\phi_n(x) = \chi_n\phi_{n-1}(x) + hL^{-1}[R_n(\vec{\phi}_{n-1})] \tag{4.7}$$

Finally,we have

$$\phi(x) = \phi_0(x) + \sum_{n=1}^{+\infty}\phi_n(x) \tag{4.8}$$

where

$$\phi_0(x) = 3x$$

$$\phi_1(x) = -\frac{1}{\alpha}\int_0^1 xt\phi_0(t)dt = -h\frac{x}{\alpha}$$

$$\phi_2(x) = -\frac{1}{\alpha}\int_0^1 xt\phi_1(t)dt = h\frac{x}{3\alpha^2}$$

$$\phi_3(x) = -\frac{1}{\alpha}\int_0^1 xt\phi_2(t)dt = -h\frac{x}{9\alpha^3}$$

$$\phi_4(x) = -\frac{1}{\alpha}\int_0^1 xt\phi_3\,(t)dt = h\frac{x}{27\alpha^4}$$

Hence

$\phi(x) = \phi_0(x) + \phi_1(x) + \phi_2(x) + \phi_3(x) + \phi_4(x) + \cdots$

$= 3x - h\frac{x}{\alpha} + h\frac{x}{3\alpha^2} - h\frac{x}{9\alpha^3} + h\frac{x}{27\alpha^4} + \ldots$

If $h = -1$

$= 3x + \frac{x}{\alpha} - \frac{x}{3\alpha^2} + \frac{x}{9\alpha^3} - \frac{x}{27\alpha^4} + \ldots$

$= 3x + \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{3^{n-1}\alpha^n}x$ (4.9)

which is the exact solution of equation(4.1)

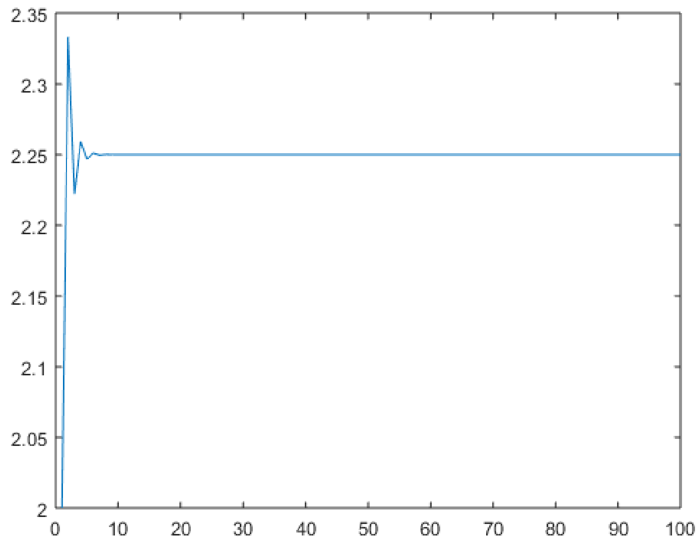

Figure 4.1: Example 1.Exact solution to equation (4.1)(Issaka,2016)

The following algorithm produces **figure 4.1** using the **Matlab** software.

```
function [x, sumc] = solplot1(x, alpha,n)
sumc(1) = 3*x;
for i=1:n
num = (-1)^i; m=i-1;
den = (3^m) * (alpha^i);
rsult = num*x/den;
sumc(i+1) = sumc(i) + rsult;
end
plot(1:n,sumc(2:end))
end
```

Example 2. Consider the following Fredholm integral equation

$\phi(x) = 2x + x\int_0^1 y\phi(y)dy$ (4.10)

To begin with,we choose

$\phi_0(x) = 2x$ (4.11)

We choose the linear operator

$L[\phi(x,q)] = \phi(x,q)$ (4.12)

Thus,we now define the nonlinear operator as

$N[\phi(x,q)] = \phi(x,q) - 2x - x\int_0^1 y\phi(y)dy$ (4.13)

where we construct the nth-order deformation equation

L$[\phi_n$-$\chi_n\phi_{n-1}] = hR_n(\vec{\phi_{n-1}})$ (4.14)

and

$R_n(\vec{\phi_{n-1}})= \phi_{n-1}(x) - (1-\chi_n)2x - x\int_0^1 y\phi_{n-1}(y)dy.$ (4.15)

where the solution of the nth-order deformation equation(4.14)

$\phi_n(x)= \chi_n\phi_{n-1}(x)+hL^{-1}[R_n(\vec{\phi_{n-1}})]$ (4.16)

Finally,we have

$\phi(x)=\phi_0(x)+\sum_{n=1}^{+\infty}\phi_n(x)$ (4.17)

where

$\phi_0(x) = 2x$

$\phi_1(x)= -x\int_0^1 y\phi_0(y)dy = -x[\frac{2}{3}y^3]_0^1= -h\frac{2}{3}x$

$\phi_2(x)= -x\int_0^1 y\phi_1(y)dy = -x[-\frac{2}{9}y^3]_0^1= h\frac{2}{9}x$

$\phi_3(x)= -x\int_0^1 y\phi_2(y)dy = -x[\frac{2}{27}y^3]_0^1= -h\frac{2}{27}x$

$\phi_4(x)= -x\int_0^1 y\phi_3(y)dy = -x[-\frac{2}{81}y^3]_0^1= h\frac{2}{81}x$

Hence

$\phi(x)= \phi_0(x) + \phi_1(x)+\phi_2(x)+\phi_3(x)+\phi_4(x) + \cdots$

$=2x - h\frac{2}{3}x + h\frac{2}{9}x - h\frac{2}{27}x + h\frac{2}{81}x + \ldots$

If $h = -1$

$=2x + \frac{2}{3}x - \frac{2}{9}x + \frac{2}{27}x - \frac{2}{81}x + \ldots$

$= 2x + 2\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{3^n}(x)$

Which is the exact solution to equation (4.10)

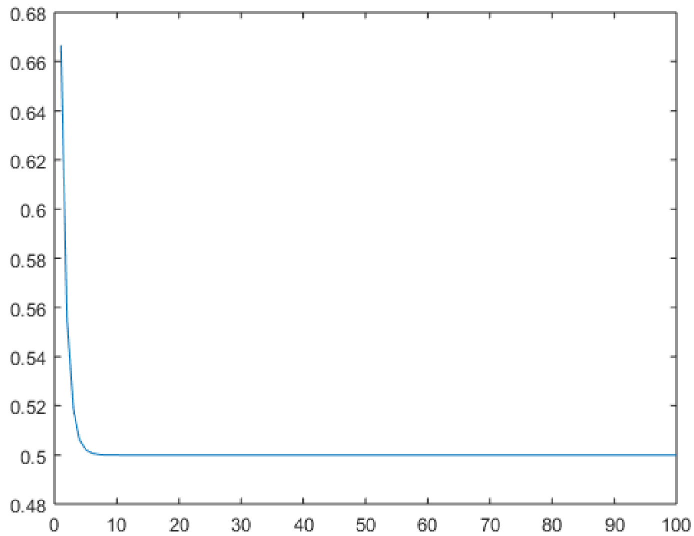

Figure 4.2: Example 2.Exact solution to equation (4.11)(Issaka,2016).

The following algorithm produces **figure 4.2** using the **Matlab** software.

```
function [x,sumc] = solplot2(x,n)
sumc(1) = 2*x;
for i=1:n
num = (-2);
den = 3^i;
rsult = num*x/den;
```

```
sumc(i+1) = sumc(i) + rsult;
end
%plot(1:n+1,sumc)
plot(1:n,sumc(2:end))
end
% % Script to run
solplot1(1,1,100)
solplot2(1,100)
\end{verbatim}
```

Conclusion

In this paper, Fredholm integral equation of the second kind has been solved successfully by Homotopy analysis method (HAM). Choosing the convergence control parameter h properly, greatly influences the convergence of the solution series and the convergence rate. We analytically showed that the HAM is an efficient method for solving Fredholm integral equations of the second kind.

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