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RESEARCH ARTICLE

CLASSES OF VAGUE FILTERS IN RESIDUATED LATTICES

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ABSTRACT

In this paper we further develop the theory of vague filters of a residuated lattice. We introduce and investigate implicative vague filters, positive implicative vague filters, fantastic vague filters, strong vague filter, n-contractive vague filter, divisible vague filter of a residuated lattice and describe their mutual connections.

Key words: non-classical logic, vague filter, Transfer principle, Boolean vague filter.

1.INTRODUCTION

The concept of fuzzy set was introduced by Zadeh (1965) [19]. Since then this idea has been applied to other algebraic structures. Since the fuzzy set is single function, it cannot express the evidence of supporting and opposing. Hence the concept of vague set [6] is introduced in 1993 by W.L.Gau and Buehrer. D.J. In a vague set A, there are two membership functions: a truth membership fucntion t_A and a false membership function f_A, where t_A and f_A are lower bound of the grade of membership respectively and t_A(x) + f_A(x) $\leq$ 1. Thus the grade of membership in a vague set A is a subinterval [t_A(x), 1-f_A(x)] of [0, 1]. Vague sets is an extension of fuzzy sets. The idea of vague sets is that the membership of every elements which can be divided into two aspects including supporting and opposing. With the development of vague set theory, some structure of algebras corresponding to vague set have been studied. R.Biswas [3] initiated the study of vague algebras by studying vague groups. T.Eswarlal [5] study the vague ideals and normal vague ideals in semirings. H.Hkam , etc [13] study the vague relations and its properties. In this paper we introduce implicative vague filters, positive implicative vague filters, Boolean vague filters and fantastic vague filter, strong vague filter, n-contractive vague filter, divisible vague filter of a residuated lattice and analyse their properties.

2.Preliminaries

Definition 2.1: [17]

A residuated lattice is an algebraic structure L = (L, $\vee$, $\wedge$, *, $\rightarrow$, 0, 1) satisfying the following axioms:

1. (L, $\vee$, $\wedge$, 0, 1) is a bounded lattice
2. (L, *, 1) is a commutative monoid .
3. (*, 1) is an adjoint pair, i.e., for any x, y, z, w$\in$L,
   i. if $x \le y$ and $z \le w$, then $x * z \le y * w$.
   ii. if $x \le y$ and $y \to z \le x \to z$ then $z \to x \le z \to y$.
   iii. (adjointness condition) $x * y \le z$ if and only if $x \le y \to z$.

In this paper, denote L as residuation lattice unless otherwise specified.

Theorem 2.2: [17]

In each residuated lattice L, the following properties hold for all x, y, z $\in$ L:

1. $(x * y) \to z = x \to (y \to z)$.
2. $z \le x \to y \Leftrightarrow z * x \le y$.
3. $x \le y \Leftrightarrow z * x \le z * y$.
4. $x \to (y \to z) = y \to (x \to z)$.
5. $x \le y \Rightarrow z \to x \le z \to y$.
6. $x \le y \Rightarrow y \to z \le x \to z$, $y' \le x'$.
7. $y \to z \le (x \to y) \to (x \to z)$.
8. $y \to x \le (x \to z) \to (y \to z)$.
9. $1 \to x = x$, $x \to x = 1$.
10. $x^m \le x^n$, m, n $\in$ N, $m \ge n$.
11. $x \le y \Leftrightarrow x \to y = 1$.
12. $0' = 1$, $1' = 0$, $x' = x^m$, $x \le x^n$.
13. $x \vee y \to z = (x \to z) \wedge (y \to z)$.
14. $x * x' = 0$.
15. $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

Lemma 2.3: [20]

1. $y \le x \to y$
2. $x * y \le x * (x \to y) \le x \wedge y \le x \wedge (x \to y) \le x$
3. $y \le x \to (x * y)$
4. $(x \to y) \to z \le x \to (y \to z)$
5. $(x \to y) * (y \to z) \le x \to z$ (transitivity of $\to$)
6. $y_1 \le y_2$, $x_1 \le x_2$ imply 1. $x \to y_1 \le x \to y_2$(isotonicity of the second variable of $\to$) 2. $x_2 \to y \le x_1 \to y$ (antitonicity of the first variable of $\to$)
7. $x \to (y \to z) = y \to (x \to z) = (x * y) \to z$(Exchange rule)
8. $(y \vee z) * x = (y * x) \vee (z * x)$
9. $(y \vee z) \to x = (y \to x) \wedge (z \to x)$, in particular, $(y \vee z) \to y = z \to y$
10. $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$, in particular, $y \to (y \wedge z) = y \to z$
11. $x \to (y \vee z) \ge (x \to y) \vee (x \to z)$
12. $(y \wedge z) \to x \ge (y \to x) \vee (z \to x)$
13. $(y \to x) * [(x \wedge y) \to z] \le [y \to (x \wedge y)] \wedge (y \to z)$
14. $(x \to y) * (z \to w) \le (x \vee z) \to (y \vee w)$
15. $(x \to y) * (z \to w) \le (x \wedge z) \to (y \wedge w)$
16. $(x \to y) * (z \to w) \le (x * z) \to (y * w)$, in particular, $x \to y \le (x * z) \to (y * z)$
17. $(y \to x) * (z \to w) \le (x \to z) \to (y \to w)$, in particular, $x \le {\sim\sim}x$, $x \le (x \to y) \to y$ and $y \to x \le {\sim}x \to {\sim}y$

18. $\sim\sim(x \to \sim\sim y) = x \to \sim\sim y = \sim\sim x \to \sim\sim y$
19. $\sim\sim x \to \sim\sim y \geq \sim\sim(x \to y)$
20. $[(x \to y) \to y] \to y = x \to y$
21. $x \vee y \leq (x \to y) \to y$.

Definition 2.3: [20]

A non-empty subset F of a residuated lattice L is called a filter of L if it satisfies

1. $x, y \in F \Rightarrow x * y \in F$.
2. $x \in F, x \leq y \Rightarrow y \in F$.

Theorem 2:4: [20]

A non-empty subset F of a residuated lattice L is called a filter of L if it satisfies, for any $x,y \in L$,

1. $1 \in F$.
2. $x \in F, x \to y \in F \Rightarrow y \in F$.

Note 2.5: [20]

A fuzzy set A on a residuated lattice L is a mapping from L' to [0, 1]

Definition 2.6: [20]

A fuzzy set A of a residuated lattice L is called a fuzzy filter, if it satisfies, for any $x, y \in L$

1. $A(1) \geq A(x)$.
2. $A(x * y) \geq \min\{A(x), A(y)\}$.

Theorem 2.7: [20]

A fuzzy set A of a residuated lattice L is a fuzzy filter, if and only if it satisfies, for any $x, y \in L$,

1. $A(1) \geq A(x)$.
2. $A(y) \geq \min\{A(x \to y), A(x)\}$

Definition 2.8:[20]

On any residuated lattice L let us define a unary operation negation ~ by $x^{\sim} = x \to 0$ for any $x \in L$.

Definition 2.9:[20]

A residuated Lattice L is called

1.BL-algebra if and only if M satisfies the identity of pre-linearity $(x \to y) \vee (y \to x) = 1$.

2.MV-algebra if and only if M fulfils the double negation law $x^{\sim\sim} = x$.

3.Heyting algebra if and only if the operations * is idempotent.

Definition 2.10: [6]

A Vague set A in the universe of discourse S is a Pair (t_A, f_A) where $t_A : S \to [0,1]$ and $f_A : S \to [0,1]$ are mappings (called truth membership function and false membership function respectively) where $t_A(x)$ is a lower bound of the grade of membership of x derived from the evidence for x and $f_A(x)$ is a lower bound on the negation of x derived from the evidence against x and $t_A(x) + f_A(x) \leq 1$ $\forall x \in S$.

3.Vague filters of a Residuated Lattice

Definition 3.1:

A vague set V in a Residuated Lattice L is called a vague filter of L if any $x, y \in L$ satisfy

1. $V(x * y) \geq V(x) \wedge V(y)$
2. $x \leq y \Rightarrow V(x) \leq V(y)$
3. $V(1) \geq V(x)$ for every $x \in L$.

Lemma 3.2:

Let V be a vague filter of L. Then for any $x, y \in L$ we have

1. $V(x \vee y) \geq V(x) \wedge V(y)$
2. $V(x \wedge y) = V(x) \wedge V(y)$
3. $V(x * y) = V(x) \wedge V(y)$.

Proof:

For any x, y $\in$ L we have x * y $\leq$ x $\wedge$ y $\leq$ x $\vee$ y. Then from Definition 3.1 we have V(x $\vee$ y) $\geq$ V(x * y) $\geq$ V(x) $\wedge$ V(y). Since x * y $\leq$ x $\wedge$ y $\leq$ x, y, it follows by definition 3.1 that V(x) $\wedge$ V(y) $\leq$ V(x * y) $\leq$ V(x $\wedge$ y) $\leq$ V(x) $\wedge$ V(y).

Proposition 3.3:

A vague set V in a Lattice L is a vague filter of L if and only if it satisfies Definition 3.1(1) and V(x $\vee$ y) $\geq$ V(x) for any x, y $\in$ L.

Proof:

If V is a vague filter of a Lattice L then x $\leq$ x $\vee$ y implies V(x) $\leq$ V(x $\vee$ y). Conversely, if V satisfies Definition 3.1(1) and V(x $\vee$ y) $\geq$ V(x) for any x, y $\in$ L and x $\leq$ y, then V(y) = V(x $\vee$ y) $\geq$ V(x). Hence V is a vague filter of L.

Theorem 3.4:

Let V be a vague set in a Lattice L. Then the following conditions are equivalent.

1. V is a vague filter of L.
2. V satisfies V(1) $\geq$ V(x) and for all x, y $\in$ L, V(y) $\geq$ V(x) $\wedge$ V(x $\rightarrow$ y).

Proof:

(1)$\Rightarrow$ **(2):** Let V be a vague filter of L and let x, y $\in$ L. Then by Lemma 3.2(3),V(y) $\geq$ V(x $\wedge$ y) = V((x $\rightarrow$ y) * x) = V(x $\rightarrow$ y) $\wedge$ V(x). Hence V satisfies the condition (2).

(2) $\Rightarrow$ (1): Let V be a vague set in L satisfying (2). Let x, y $\in$ L, x $\leq$ y. Then x $\rightarrow$ y = 1. Thus V(y) $\geq$ V(x) $\wedge$ V(1) = V(x), hence Definition 3.1(2) holds. Further, since x $\leq$ y $\rightarrow$ (x * y), by (2) and Definition 3.1(2) we get V(x * y) $\geq$ V(y) $\wedge$ V(y $\rightarrow$ (x * y)) $\geq$ V(y) $\wedge$ V(x). Therefore Definition 3.1(1) is also satisfied and hence V is a vague filter of L.

Definition 3.5:

For every subset V of L, we define a formula P_V defined by

P_V : $\forall$x, $\forall$y (V(u_1(x,y)) $\wedge$V(u_2(x,y)) $\wedge$........$\wedge$ V(u_n(x,y))) $\leq$ V(u_n(x,y))).

Definition 3.6:

For every subset V of L, we call V a vague P-set if it satisfies the formula P_V.

Definition 3.7: Transfer Principle: A vague set V defined in a (general) algebra A has a property P (or A is a vague P-set) if and only if all non-empty level subsets U(t, α) and L(1-f, β) have the property P.

4. Positive Implicative and Boolean Vague Filters

Definition 4.1:

A vague set V in a Lattice L is called an implicative vague filter of L if for any x, y, z $\in$ L

1. V(1) $\geq$ V(x)
2. V(x $\rightarrow$ (y $\rightarrow$ z)) $\wedge$ V(x $\rightarrow$ y) $\leq$ V(x $\rightarrow$ z).

Proposition 4.2:

Every implicative vague filter of a Lattice L is a vague filter of L.

Proof:

Let V be an implicative vague filter of L. Let $\alpha \in$ [0, 1] be such that U(t, α) $\neq \Phi$. Then for any x $\in$ U(t, α) we have t(1) $\geq$ t(x), thus 1 $\in$ U(t, α). Also let $\beta \in$ [0, 1] such that L(1-f, β) $\neq \Phi$. Then for any x $\in$ L(1-f, β) we have 1-f(1) $\geq$ 1-f(x), thus 1 $\in$ L(1-f, β) Let x, x $\rightarrow$ y $\in$ U(t, α), i.e. t(x), t(x $\rightarrow$ y) $\geq \alpha$. Then t(1 $\rightarrow$ x), t(1 $\rightarrow$ (x $\rightarrow$ y)) $\geq \alpha$, hence t(1 $\rightarrow$ (x $\rightarrow$ y)) $\wedge$ t(1 $\rightarrow$ x) $\geq \alpha$, thus by Definition 4.1(2), t(1 $\rightarrow$ y) $\geq \alpha$. That means t(y) $\geq \alpha$, and therefore y $\in$ U(t, α). Similarly we can prove 1-f(y) $\geq \beta$ implies y $\in$ L(1-f, β). Hence V is a vague filter of L.

Theorem 4.3:

Let F be a vague filter of a Lattice L. Then the following conditions are equivalent.

1. V_F is an implicative vague filter of L.
2. V_F (y → (y → x)) ≤ V_F(y → x) for any x, y ∈ L
3. V_F(z → (y → x)) ≤ V_F((z → y) → (z → x)) for any x, y ∈ L
4. V_F(z → (y → (y → x))) ∧ V_F(z) ≤ V_F(y → x) for any x, y ∈ L
5. V_F(x → (x * x)) = V_F(1).

Proof: It follows from Definitions.

Definition 4.4:

A vague set V in a Lattice L is called a positive implicative vague filter of L if for any x,y,z ∈ L,

1. V(1) ≥ V(x)
2. V(x → ((y → z) → y)) ∧ V(x) ≤ V(y)

Proposition 4.5:

Every positive implicative vague filter of a Lattice L is a vague filter of L.

Proof:

Let V be a positive implicative vague filter of L, α, β ∈ [0, 1] and U(t, α) ≠ Φ , L(1-f, β) ≠ Φ. Then 1 ∈ U(t, α) and 1 ∈ L(1-f, β). Further, let x, x → y ∈ U(t, α). i.e.,t(x), t(x → y) ≥ α and 1-f(x), 1-f(x → y) ≥ β. Then t(x → ((y → 1) → y)) = t(x → (1 → y)) = t(x → y), 1-f(x → ((y → 1) → y)) = 1-f(x → (1 → y)) = 1-f(x → y) hence t(x → ((y → 1) → y) ∧ t(x) ≥ α and 1-f(x → ((y → 1) → y) ∧ 1-f(x) ≥ β and thus by Definition 4.4 we have t(y) ≥ α and 1-f(y) ≥ β. Therefore y ∈ U(t, α) and y ∈ L(1-f, β). Hence V is a vague filter of L.

Theorem 4.6:

A vague filter F of a lattice L is positive implicative if and only if V_F is a positive implicative vague filter of L.

Proof:

Let F be a vague filter of L. Let us suppose that F is positive implicative. Let t_F(x → ((y → z) → y)) ∧ t_F(x) = α and $1 - f_F$(x → ((y → z) → y)) ∧ 1-f_F(x) = β. Then t_F(x → ((y → z) → y)) = α = t_F(x), and 1- f_F(x → ((y → z) → y)) = β = $1 - f_F$(x), thus x → ((y → z) → y), x ∈ F, and hence y ∈ F, that meanst_F(y) = α and 1- f_F(y) = β. Therefore we get that V_F is positive implicative vague filter of L. Conversely, let V_F be a positive implicative vague filter of L. Let x → ((y → z) → y) ∈ F and x ∈ F. Then t_F(x → ((y → z) → y)) = α = t_F(x), 1-f_F(x → ((y → z) → y)) = β = 1-f_F(x), hence t_F(y) = α and 1-f_F(y) = β and so y ∈ F. That means F is a positive implicative filter of L.

Theorem 4.7:

Let F be a vague filter of a lattice L. Then the following conditions are equivalent.

1. V_F is a positive implicative vague filter of L.
2. V_F((x → y) → x) ≤ V_F(x) for any x, y ∈ L.
3. V_F(($x^{\sim}$ → x) → x) = V_F(1).

Proof:

Analogous to that for Theorem 4.3.

Theorem 4.8:

Let V be a vague filter of a lattice L. Then V is a positive implicative vague filter of L if and only if all non-empty level subsets are positive implicative filter of L for every α, β ∈ [0, 1] such that U(t, α) ≠ Φ and L(1-f, β) ≠ Φ.

Proof:

Let us suppose that V is a vague filter of L. Let V be positive implicative, α,β∈ [0,1], U(t, α)≠ Φ and L(1-f, β) ≠ Φ, x, y, z ∈ L , x → ((y → z) → y)∈ U(t, α) and x → ((y → z) → y)∈ L(1-f, β), x ∈ U(t, α) and x ∈ L(1-f, β). Then t(x → ((y → z) → y)) ,t(x) ≥ α and 1-f(x → ((y → z) → y)), 1-f(x) ≥ β. Hence t(x

$\rightarrow ((y \rightarrow z) \rightarrow y)) \wedge t(x) \geq \alpha$. Since $t(y) \geq t(x \rightarrow ((y \rightarrow z) \rightarrow y)) \wedge t(x)$ and $1\text{-}f(y) \geq 1\text{-}f\,(x \rightarrow ((y \rightarrow z) \rightarrow y)) \wedge 1\text{-}f(x)$ we get $y \in U(t, \alpha)$ and $y \in L(1\text{-}f, \beta)$. Therefore $U(t, \alpha)$ and $L(1\text{-}f, \beta)$ are positive implicative filter. Conversely, Let V be such that $U(t, \alpha)$ and $L(1\text{-}f, \beta)$ are positive implicative filter for any $\alpha, \beta \in [0, 1]$ such that $U(t, \alpha) \neq \Phi$ and $L(1\text{-}f, \beta) \neq \Phi$. If x, y, z $\in$ L, then $x \rightarrow ((y \rightarrow z) \rightarrow y)$, $x \in U(t, (x \rightarrow((y \rightarrow z) \rightarrow y)) \wedge x)$ and $x \in L(1\text{-}f, (x \rightarrow((y \rightarrow z) \rightarrow y)) \wedge x)$thus also $y \in L(1\text{-}f, (x \rightarrow((y \rightarrow z) \rightarrow y)) \wedge x)$, hence $V(y) \geq V((x \rightarrow ((y \rightarrow z) \rightarrow y)) \wedge x) = V(x \rightarrow ((y \rightarrow z) \rightarrow y))) \wedge V(x)$. That means V is a positive implicative vague filter.

Theorem 4.9:

Every positive implicative vague filter of a lattice L is implicative.

Proof:

Let V be a positive implicative vague filter of L. Then by Theorem 4.8, if $\alpha, \beta \in [0, 1]$ is such that $U(t, \alpha) \neq \Phi$ and $L(1\text{-}f, \beta) \neq \Phi$ then $U(t, \alpha)$ and $L(1\text{-}f, \beta)$ are positive implicative filters of L. Hence $U(t, \alpha)$ and $L(1\text{-}f, \beta)$ are implicative filters of L. Therefore V is an implicative vague filter of L.

Definition 4.10:

A vague filter V of a lattice L is called a Boolean vague filter of L, if for any x $\in$ L, $V(x \vee x^{\sim}) = V(1)$

Theorem 4.11:

A filter F of a lattice L is Boolean if and only if V_F is Boolean vague filter of L.

Proof:

Let F be a Boolean filter of L and let x $\in$ L. Then $V_F(x \vee x^{\sim}) = V_F(1)$, hence V_F is Boolean vague filter of L. Conversely, let V_F be a Boolean vague filter of L and let x $\in$ F.

Then $V_F(x \vee x^{\sim}) = V_F(1) = \alpha$, then $x \vee x^{\sim} \in F$, that means F is Boolean filter of L.

Definition 4.12:

A vague subset V in a lattice L is called a fantastic vague filter of L if for any x, y, z $\in$ L,

1. $V(1) \geq V(x)$
2. $V(z \rightarrow (y \rightarrow x)) \wedge V(z) \leq V(((x \rightarrow y) \rightarrow y) \rightarrow x)$.

Proposition 4.13:

Every fantastic vague filter of a lattice L is a vague filter of L.

Proof:

Let V be a fantastic vague filter of L. Let $\alpha, \beta \in [0, 1]$, $U(t, \alpha) \neq \Phi$ and $L(1\text{-}f, \beta) \neq \Phi$. Then $1\in U(t, \alpha)$ and $1 \in L(1\text{-}f, \beta)$. Let x, $x \rightarrow y \in U(t, \alpha)$ and $L(1\text{-}f, \beta)$, i.e., $t(x), t(x \rightarrow y) \geq \alpha$, $1\text{-}f(x)$, $1\text{-}f(x \rightarrow y) \geq \beta$. Then $t(x \rightarrow (1 \rightarrow y)) = t(x \rightarrow y) \geq \alpha$, $1\text{-}f(x \rightarrow (1 \rightarrow y)) = 1\text{-}f(x \rightarrow y) \geq \beta$. Hence $t(x \rightarrow (1 \rightarrow y)) \wedge t(x) \geq \beta$, thus by Definition 4.12 we have $t(y) = t(1 \rightarrow y) = t(((y \rightarrow 1) \rightarrow 1) \rightarrow y) \geq t(x \rightarrow (1 \rightarrow y)) \wedge t(x) \geq \alpha$, $1\text{-}f(y) = 1\text{-}f(1 \rightarrow y) = 1\text{-}f(((y \rightarrow 1) \rightarrow 1) \rightarrow y) \geq 1\text{-}f(x \rightarrow (1 \rightarrow y)) \wedge 1\text{-}f(x) \geq \beta$, and so $y \in U(t, \alpha)$ and $y \in L(1\text{-}f, \beta)$. Therefore V is a vague filter of L.

Theorem 4.14:

A vague filter F of a lattice L is fantastic if and only if V_F is a fantastic vague filter of L.

Proof:

Let F be a vague filter of L. Let us suppose that F is fantastic. Let $t_F(z \rightarrow (y \rightarrow x)) \wedge t_F\,(z) = \alpha$ and $1\text{-}f_F(z \rightarrow (y \rightarrow x)) \wedge 1\text{-}f_F\,(z) = \beta$. Then $t_F(z \rightarrow (y \rightarrow x)) = \alpha = t_F(z)$ and $1 - f_F(z \rightarrow (y \rightarrow x)) = \beta = 1\text{-}f_F(z)$, thus $z \rightarrow (y \rightarrow x) \in F$, $z \in F$, and hence $((x \rightarrow y) \rightarrow y) \rightarrow x \in F$, that means $t(((x \rightarrow y) \rightarrow y) \rightarrow x) = \alpha$ and $1\text{-}\ f(((x \rightarrow y) \rightarrow y) \rightarrow x) = \beta$. Therefore we get V_F is a fantastic vague filter of L. Conversely, let V_F be a fantastic vague filter of L. Let $z \rightarrow (y \rightarrow x) \in F$ and $z \in F$. Then $t_F\,(z \rightarrow (y \rightarrow x)) = \alpha = t_F(z)$, $1 - f_F\,(z \rightarrow (y \rightarrow x)) = \beta = 1\text{-}f_F(z)$, hence

$t_F\,(((x \rightarrow y) \rightarrow y) \rightarrow x) = \alpha$, $1\text{-}\ f_F\,(((x \rightarrow y) \rightarrow y) \rightarrow x) = \beta$ and therefore $((x \rightarrow y) \rightarrow y) \rightarrow x\in F$ That means, F is a fantastic filter of L.

Theorem 4.15:
Every positive implicative vague filter of a lattice L is fantastic.
Proof:
If V is a positive implicative vague filter of L, then U(t, α) and L(1-f, β) are positive implicative filter of L for every U(t, α) $\neq \Phi$ and L(1-f, β) $\neq \Phi$. Hence U(t, α) and L(1-f, β) are fantastic filter of L, and hence, by Theorem 4.14, V is fantastic filter of L.
5.New subclasses of vague filters
Definition 5.1:
Let F be a vague filter of L. Then it is called a divisible vague filter if for all x, y $\in$ L ,
V_F((x $\wedge$y) $\rightarrow$ [x * (x $\rightarrow$ y)]) = $V_F(1)$.
Theorem 5.2:
Let F be a vague filter of L. Then the following assertions are equivalent for all x, y,z $\in$ L:
1. V_F is a divisible vague filter.
2. V_F([x $\rightarrow$ (y $\wedge$ z)] $\rightarrow$ {(x $\rightarrow$ y) * [(x $\wedge$ y) $\rightarrow$ z]}) = $V_F(1)$.

Proof:
Assume that V_F is a divisible vague filter. Then we have V_F([(x $\rightarrow$ y) $\wedge$(x $\rightarrow$ z)] $\rightarrow$ {(x $\rightarrow$ y) * [(x $\rightarrow$ y) $\rightarrow$ (x $\rightarrow$ z)]}) = V_F([x $\rightarrow$ (y $\wedge$z)] $\rightarrow$ {(x $\rightarrow$ y) * {[x* (x $\rightarrow$ y) $\rightarrow$z)}} and hence V_F([x $\rightarrow$ (y $\wedge$z)] $\rightarrow$ {(x $\rightarrow$ y) * {[x* (x $\rightarrow$ y) $\rightarrow$z)}}= $V_F(1)$. Therefore V_F((x $\wedge$y) $\rightarrow$ [x * (x $\rightarrow$ y)]) $\leq$ V_F[{[x * (x $\rightarrow$ y)] $\rightarrow$ z} $\rightarrow$ [(x $\wedge$ y) $\rightarrow$ z)]] $\leq$ V_F[{(x $\rightarrow$ y) * {[x * (x $\rightarrow$ y)] $\rightarrow$ z}} $\rightarrow$ {(x $\rightarrow$ y) * [(x $\wedge$ y) $\rightarrow$ z]} and hence V_F[{(x $\rightarrow$ y) * {[x * (x $\rightarrow$ y)] $\rightarrow$ z}} $\rightarrow$ {(x $\rightarrow$ y) * [(x $\wedge$ y) $\rightarrow$ z]} = $V_F(1)$. By the transitivity of $\rightarrow$, we have V_F([x $\rightarrow$ (y $\wedge$z)] $\rightarrow$ {(x $\rightarrow$ y) * {[x $\wedge$ y) $\rightarrow$z)]}) = $V_F(1)$.
Definition 5.3:
Let F be a vague filter of L. Then it is called a strong filter if for x $\in$ L V_F[~~(~~x $\rightarrow$ x)] = $V_F(1)$.
Proposition 5.4:
Let F be a vague filter of L. Then the following assertions are equivalent for all x, y $\in$ L,
1. V_F is a strong vague filter
2. V_F[(y $\rightarrow$ ~~x) $\rightarrow$ ~~(y $\rightarrow$ x)] = $V_F(1)$.

Proof:
Assume that F is a strong vague filter, that is, V_F[~~(~~x $\rightarrow$ x)] = $V_F(1)$.
Therefore V_F[~~(~~x $\rightarrow$ x)] $\leq$ V_F(~~[(y $\rightarrow$ ~~x) $\rightarrow$(y $\rightarrow$ x)]) $\leq$ V_F(~~[(y $\rightarrow$~~x) $\rightarrow$ ~~(y $\rightarrow$ x)]) = V_F [(y $\rightarrow$ ~~x) $\rightarrow$ ~~(y $\rightarrow$ x)]. Hence V_F[(y $\rightarrow$ ~~x) $\rightarrow$ ~~(y $\rightarrow$ x)] = $V_F(1)$. Conversely, it follows immediately by taking y = ~~x in (2).
Theorem 5.5:
Let F be a vague filter of L. Then the following assertions are equivalent for all x, y $\in$ L.
1. V_F is a strong vague filter
2. V_F[(x $\rightarrow$ y) $\rightarrow$ ~~(~~x $\rightarrow$ y)] = $V_F(1)$.

Proof: It follows from Propostion 5.5
Theorem 5.6:
Let F be a vague filter of L. Then the following assertions are equivalent for all x, y $\in$ L,
1. V_F is a strong vague filter
2. V_F[(~x $\rightarrow$ y) $\rightarrow$ ~~(~y $\rightarrow$ x)] = $V_F(1)$.

Proof:
Assume that F is a strong vague filter, that is, V_F[~~(~~x $\rightarrow$ x)] = $V_F(1)$. V_F[~~(~~x $\rightarrow$ x)] $\leq$ V_F[~~{[(~x $\rightarrow$ y) * ~y] $\rightarrow$ x}] = V_F[~~(~x $\rightarrow$ y) $\rightarrow$ (~y $\rightarrow$ x)] $\leq$ V_F[~~[(~x $\rightarrow$ y) $\rightarrow$~~(~y$\rightarrow$ x)]] = V_F [(~x $\rightarrow$ y) $\rightarrow$ ~~(~y $\rightarrow$ x)] = $V_F(1)$. Conversely, it follows immediately by taking y = ~~x.

Definition 5.7:

Let F be a vague filter of L. Then it is called an n-contractive vague filter if for all x ∈ L, $V_F[x^n \to x^{n+1}]$ = $V_F(1)$.

Theorem 5.8:

Let F be a vague filter of L. Then the following assertions are equivalent for all x, y ∈ L:

1. F is an n-contractive vague filter
2. $V_F[[x^n * (x^n \to y)] \to (x * y)] = V_F(1)$
3. $V_F[[x^n * (x^n \to y)] \to x^{n+1}] = V_F(1)$.

Proof:

(1) ⇔ (2)

Assume that F is an n-contractive vague filter. $V_F[x^n \to x^{n+1}] \le V_F[[x^n * (x^n \to y)] \to [x^{n+1} * (x^n \to y)]] \le V_F[[x^n * (x^n \to y)] \to (x * y)]$ and hence $V_F[[x^n * (x^n \to y)] \to (x * y)] = V_F(1)$. Conversely, it follows immediately by taking y = x^n that $V_F[x^n \to x^{n+1}] = V_F(1)$. Thus F is an n-contractive vague filter .

(1) ⇔ (3) Consider the inequality $V_F\{[x^n * (x^n \to y)] \to [x^{n+1} * (x^n \to y)]\} \le V_F\{[x^n * (x^n \to y)] \to x^{n+1}\}$, then it can be similarly proved as that of (1) ⇔ (2).

Theorem 5.9:

Let F be a vague filter of L. Then F is an n-contractive vague filter if and only if $V_F[x^n \to x^{2n}]$ = $V_F(1)$.

Proof:

Assume that F is an n-contractive vague filter. It follows immediately that $V_F[x^n \to x^{n+1}] \le V_F[x^{n+1} \to x^{n+2}]$, and hence $V_F[x^{n+1} \to x^{n+2}] = V_F(1)$. Similarly, we have $V_F[x^{n+2} \to x^{n+3}]$, ...$V_F[x^{2n-1} \to x^{2n}] = V_F(1)$. The transitivity of → leads that $V_F[x^n \to x^{n+1}]$ *.....$V_F[x^{2n-1} \to x^{2n}] \le V_F[x^n \to x^{2n}]$, and hence $V_F[x^n \to x^{2n}] = V_F(1)$. Conversely, since $V_F[x^{2n}] \le V_F[x^{n+1}]$, it follows immediately from the isotonicity of the second variable of → that $V_F[x^n \to x^{2n}] \le V_F[x^n \to x^{n+1}]$, and hence $V_F[x^n \to x^{n+1}] = V_F(1)$.

Remark 5.10:

Let F be a vague filter of L. Then the following assertions are equivalent for all x, y ∈ L,

1. F is an n-contractive vague filter
2. $V_F[[x^n * (x^n \to y)] \to (x^n * y)] = V_F(1)$
3. $V_F[[x^n * (x^n \to y)] \to x^{2n}] = V_F(1)$.

Theorem 5.11:

Let F be a divisible vague filter of L. Then $V_F[[(x * y) \wedge (x * z)] \to [x * (y \wedge z)]] = V_F(1)$, for all x, y, z ∈ L.

Proof:

Assume that F is a divisible vague filter. Then we get $V_F[[(x * y) \wedge (x * z)] \to \{(x * y) * [(x * y) \to (x * z)]\}] = V_F(1)$. (x * y) * [(x * y) → (x * z)]} = x * y * {y → [x → (x * z)]} ≤ x * {y ∧ [x → (x * z)]}. Therefore we have $V_F[[(x * y) \wedge (x * z)] \to \{(x * y) * [(x * y) \to (x * z)]\}] = V_F[[(x * y) \wedge (x * z)] \to \{x * \{y \wedge [x \to (x * z)]\ ]$ = $V_F(1)$. By applying the definition of divisible vague filter we get $V_F[\{y \wedge [x \to (x * z)]\} \to \{[x \to (x * z)] * \{[x \to (x * z)] \to y\}\}] \le V_F[x * \{y \wedge [x \to (x * z)]\}] \to x * \{[x \to (x * z)] * \{[x \to (x * z)] \to y\}\}]$ and hence $V_F[x * \{y \wedge [x \to (x * z)]\}] \to x * \{[x \to (x * z)] * \{[x \to (x * z)] \to y\}\}] = V_F(1)$. Therefore by the transitivity of → , we get $V_F[[(x * y) \wedge (x * z)] \to \{x * [x \to (x * z)] * \{[x \to (x * z) \to y\}\}] = V_F(1)$. Since x * [x → (x * z)] * {[x → (x * z) → y} ≤ x * z * (z → y) ≤ x * (y ∧ z). Hence $V_F[[(x * y) \wedge (x * z)] \to x * (y \wedge z)] = V_F(1)$.

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