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RESEARCH ARTICLE

FLUID FLOW VISUALIZATION USING COMPLEX VARIABLES IN GROUNDWATER DYNAMICS MODELS

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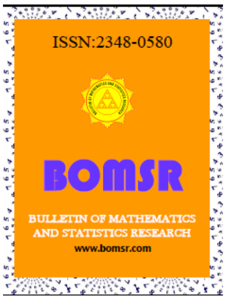

ABSTRACT

Fluid dynamics studies the behaviour of liquids and gases at rest and in motion and their interactions with other objects. One important aspect of fluid dynamics is flowing visualization. Flow visualization can be done using the complex variable method. This method uses the results of complex number theory to obtain solutions for two-dimensional potential incompressible flows. The advantage of this method is that it makes it easier to find solutions to two-dimensional incompressible flows because there is no need to solve differential equations. The aim of this research is to visualize the two-dimensional complex potential of water fluid flow. The parameters to be varied are the potential function and flow function of several simple flow forms. It is hoped that this research can provide information about the application of complex variable methods in studying fluid dynamics, especially fluid flow visualization.

Key words: fluid flow visualization, complex potential, flow function, complex numbers.

1. INTRODUCTION

Fluid dynamics is a branch of science that studies the behavior of liquids and gases at rest and in motion and their interactions with other objects. One important aspect of fluid dynamics is flow visualization. In this paper we will visualize fluid flow in a groundwater flow model. There are several methods that have been used in modeling fluid flow visualization, including: Lattice Boltzman [1], this method is based on the kinetic theory of gases where a fluid is assumed to be an ideal gas that moves randomly in all directions and is related to each other. Smoothed particle hydrodynamics, in this method the flow of a fluid is discretized into small particles of the same diameter and have physical properties such as mass, density, position and speed. Finite element method [2], this method uses the Navier-Stokes equations in solving fluid flow problems, where the N-S equation is discretized in a function of time and position. Line integral convolution is very well used to visualize the vector field of a flow [3].

In this paper, flow visualization will be carried out using the complex variable method. The complex variable method is very well used in studying incompressible Newtonian fluid flow [4] and simulating flow in porous media [5]. This method uses results from complex number theory to obtain solutions to the flow. The advantage of using complex variables in flow visualization is that it makes it easier to find solutions to 2-dimensional incompressible flows because there is no need to solve partial differential equations as in other methods. Therefore, in this paper we will visualize potential fluid flow using complex numbers and determine the shape or pattern of flow lines and equipotential of several forms of flow.

1.1 VISUALIZATION OF FLUID FLOW WITH COMPLEX VARIABLES

Fluid Flow Classification

Based on the flow, fluids can be divided into several groups:

1. 1D, 2D and 3D streams: 1D flow is flow that only occurs in one dimension, 2D flow only occurs in 2-dimensional space, while 3D flow occurs in 3-dimensional space.
2. Compressible flow and incompressible flow: Compressible flow is a flow condition where the fluid mass density changes, while incompressible flow is a flow condition where the fluid mass density does not change or is constant.
3. Steady flow and unsteady flow: Steady flow is a flow condition where the flow components do not change with time, while unsteady flow is a flow condition where the flow components change with time.
4. Uniform flow and non-uniform flow: Uniform flow is a condition where the flow components do not change with distance, while non-uniform flow is a flow condition where the flow components change with distance.
5. Laminar flow and turbulent flow: Laminar flow is flow in which fluid moves in layers with one layer flowing smoothly. Meanwhile, turbulent flow is a flow in which the movement of fluid particles is erratic due to mixing and rotation of particles between layers which results in the exchange of momentum from one part of the fluid to another on a large scale.
6. Viscous flow and inviscid flow: Viscous flow is flow that is influenced by viscosity. The presence of viscosity causes shear stress and energy loss, while inviscid flow is flow that is not influenced by viscosity.
7. Rotational flow and irrotational flow: Rotational flow is a flow where the rotation value or each component of the rotation vector is not equal to zero, while tacrorotational flow is flow where the rotation value or each component of the rotation vector is equal to zero or curl V ▯ 0.

For 2D non-rotational flow, then:

$$\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y} = 0 \quad (1)$$

Continuity Equation

The continuity equation is obtained from the law of conservation of fluid mass. The continuity equation states the requirements that a fluid must be continuous and that the mass of the fluid is conserved.

$$\frac{\partial \rho}{\partial z} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0 \quad (2)$$

Equation (2) is the fluid continuity equation. For an incompressible flow where the mass density is constant, the continuity equation becomes:

$\frac{\partial(v_x)}{\partial x} + \frac{\partial(v_y)}{\partial y} + \frac{\partial(v_z)}{\partial z} = 0$ (3)

Stream Function

The flow function is a mathematical function that cannot be observed directly in the real world but is very good for calculating and visualizing two-dimensional flows. Together with the potential, the flow function makes it possible to visualize flow shapes that are difficult to produce with other methods [6]. In two-dimensional flow, the equations for streamlines can be described by flow functions. Different values of the flow function ⍰ express different streamlines. Streamline or a streamline is a line that is at any time a tangent to the speed vectors. The characteristics of streamline flow patterns are: no flow intersects a streamline. The distance between streamlines is inversely proportional to speed so that the narrower the distance between streamlines indicates greater speed and the streamlines do not intersect each other. The streamline equation is as follows:

$v_x\, dy - v_v dx = 0$ (4)

The flow function in 2 dimensions has its velocity components defined by the following equation:

$v_x = \frac{\partial \Psi}{\partial \mathrm{y}}, v_y = -\frac{\partial \Psi}{\partial \mathrm{x}}$ (5)

Where v_x and v_y are the velocity components in the y and x directions. If v_x and v_y are substituted into the streamline equation then:

$v_x = \frac{\partial \Psi}{\partial \mathrm{y}}\, dy + \frac{\partial \Psi}{\partial \mathrm{x}} dx = 0$ (6)

⍰ constant along streamlines. When ⍰ (x, y) is known, various constant ⍰ lines can be plotted to obtain various flow streamlines. For irrotational flow the flow function satisfies the Laplace equation:

$\frac{\partial^2 \Psi}{\partial \mathrm{y}^2} + \frac{\partial^2 \Psi}{\partial \mathrm{x}^2} = 0$ (7)

Speed Potential Function

Table 1. The relationship of flow function and potential function

Flow function	Potential Functions
• Continuity equation: $$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial x} = 0$$ • Speed components: $$v_x = \frac{\partial v_x}{\partial y}, v_y = \frac{\partial v_y}{\partial x}$$ $$V = \frac{\partial \Psi}{\partial \mathrm{n}}$$ ∂n is the part of an equipotensil line expressed by $\partial\varphi = 0$ ⍰n perpendicular to the current line • Irotational equation: $\nabla^2\psi = 0$	• Irotational equation: $$\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial x} = 0$$ • Speed components: $$v_x = \frac{\partial v_x}{\partial y}, v_y = \frac{\partial v_y}{\partial x}$$ $$V = \frac{\partial \varphi}{\partial s}$$ ∂s is part of a *Streamline* stated by $\partial\psi = 0$ ∂s perpendicular to the equipotential line • Continuity equation $\nabla^2\varphi = 0$

The potential notation is φ where the flow field is derived from the gradient φ . φ is the potential speed if $v = \pm\nabla\varphi$. Because the velocity vector v is the gradient of the potential velocity, the potential flow is also called irrotational flow. For incompressible fluids, the continuity equation becomes:

$\nabla^2 \varphi = 0$ (8)

so that the potential flow for an incompressible fluid, the potential velocity ⍰ satisfies the Laplace equation so that for the 2-dimensional case it becomes:

$\frac{\partial^2 \varphi}{\partial x_2} + \frac{\partial^2 \varphi}{\partial y_2} = 0$ (9)

After obtaining the solution to the equation, the speeds vx and vy can be determined:

$v_x = \frac{\partial\varphi}{\partial x}, v_y = \frac{\partial\varphi}{\partial y}$ (10)

Complex Numbers

Complex numbers are imaginary numbers that have the following form:

$z = x + iy$, (11)

where x and y are real numbers while i is an imaginary number with the property $i^2 = -1$.

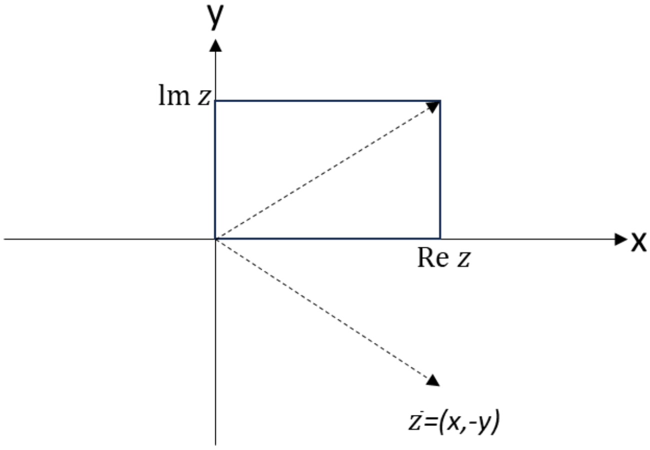

Figure 1. Complex field [6]

Complex number operations are the same as ordinary arithmetic operations, namely addition, subtraction, multiplication and division like real numbers. However, complex numbers also have additional unique properties, for example in every polynomial algebra number, real numbers have complex number solutions, unlike real numbers which only have partial ones.

The complex number $z = x + iy$, is a specification of real numbers (x, y) so that complex numbers have a one-to-one relationship with points in one plane. Complex numbers can be visualized as points or position vectors on a two-dimensional coordinate system known as the complex plane.

The Cartesian coordinates of a complex number are the real x axis and the imaginary y axis, while the circular coordinates are $r = |z|$ which is called the modulus and ⍰ ⍰ arg (z) which is called the complex argument of z. After combining it with Euler's formula, we get:

$z = r(cos\theta + i\ sin\theta) = re^{i\theta}$ (12)

The complex argument is unique modulo 2π so that if there are two complex argument values that differ by an integer multiple of 2π then the two arguments are the same or equivalent. Using basic trigonometric identities, we obtain:

$$\begin{cases} sin\theta = \dfrac{e^{i\theta} - e^{i\theta}}{2i} \\ cos\theta = \dfrac{e^{i\theta} + e^{i\theta}}{2} \end{cases}$$

The addition of two complex numbers is like the vector addition of two vectors, and multiplication by a complex number can be visualized as a simultaneous rotation and extension. Multiplication by i is a 90-degree counterclockwise rotation ($\frac{\pi}{2}$ radians). Geometrically i^2 is two 90-degree rotations which is the same as a 180-degree rotation (π radians).

Cauchy Riemann equation

The Cauchy–Riemann equation is a very important equation in complex analysis because this equation is used to test the analyticity of a complex functionw $F(z) = v_x(x.y) + iv_y(x, y)$. To find out the analytical properties of a complex function, limits and continuity of a function in a complex plane can be used. If $F(z)$ has a limit for z ⍰ z_o, then F⍰z⍰ is said to be analytic at z_0 . For v_x and v_y to be real functions of x and y on R, the necessary condition is that v_x and v_y satisfy the Cauchy-Riemann equation, namely:

$$\frac{\partial v_x}{\partial x} = \frac{\partial v_y}{\partial y}, \frac{\partial v_x}{\partial y} = -\frac{\partial v_y}{\partial x} \quad (14)$$

1.2 COMPLEX POTENTIAL FOR 2-DIMENSIONAL POTENTIAL FLOW

Two-dimensional potential flow solutions can be obtained with complex potentials using complex variables with the following conditions:

1. The fluid flow is two-dimensional, steady and irrotational.
2. The fluid is incompressible so it follows the continuity equation for incompressible fluids.
3. It is assumed that the fluid has no viscosity following the properties of an ideal fluid.
4. The potential velocity function □ and flow function □ are connected by the Cauchy-Riemann equation so that they become as follows:

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \Psi}{\partial y}, \frac{\partial \varphi}{\partial y} = -\frac{\partial \Psi}{\partial x} \quad (15)$$

This equation explains the conditions that must be fulfilled by a function $F\ (z)$ if the function is an analytic function with:

$$F(z) = \varphi(x, y) + i\Psi(x, y)$$

$$z = x + iy \quad (16)$$

Visualization of Complex Potentials in Groundwater Models

In general, the working procedure for complex potential visualization in groundwater models is as follows:

1. Conduct a literature review of complex variable applications for potential flows.
2. Determine the complex potential function of water fluid flow F(z) which will be used for the simulation.
3. Create a MATLAB program to simulate the potential distribution of a predetermined complex potential function.
4. Simulate the potential form of the determined potential flow function.
5. Analyse the simulation results.

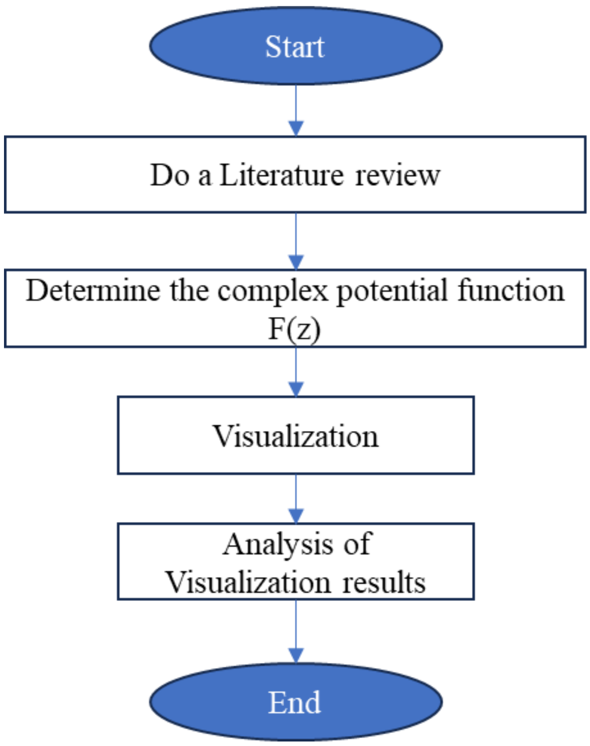

Figure 2. Workflow

2. RESULTS

In this paper, a two-dimensional potential fluid flow simulation is carried out for the case of a well system with several variations in pumping rates. The number of wells used is 3 wells. The simulation is carried out using a complex potential function for base and well flow with the following equation:

Base flow: $F_1(z) = -\overrightarrow{Q_0}z$

Well : $F_2(z) = \frac{Q}{2\pi} \ln{(z - z_1)}$

with $z = x + iy$. F_2 (z) is a function for the type of source flow with the assumption that the source flow is a pumping well. Q is the pumping rate in the well while zl is the position of the well in complex numbers. In visualization, the mirror method is also used, the position of which is determined by the conjugate complex function of complex numbers [7].

The visualization results are shown in figures 3 to 5. The white lines are streamlines which represent the trajectory of fluid particles. Meanwhile, the blue lines are equipotential lines which represent differences in fluid pressure. The black arrow represents the flow field. Stream lines and equipotential lines are always perpendicular to each other. The different colors show the potential distribution with red for the highest potential and blue for the lowest potential [8-9].

Figure 3 shows that the closer the distance between the streamlines, the greater the flow velocity, and vice versa. The flow field moves from areas of high potential to areas of low potential. Visualization was carried out by varying the pumping rate, namely 2 m^3/s and 5 m^3/s, 5 m^3/s and 8 m^3/s, 8 m^3/s and 11 m^3/s. The results show that the greater the pumping rate value, the greater the potential value of each well. This can be seen from the equipotential lines that map each well. Each equipotential line shows a difference in potential value [8].

The following is an image of the visualization results with 3 pumping wells and several variations in pumping rates:

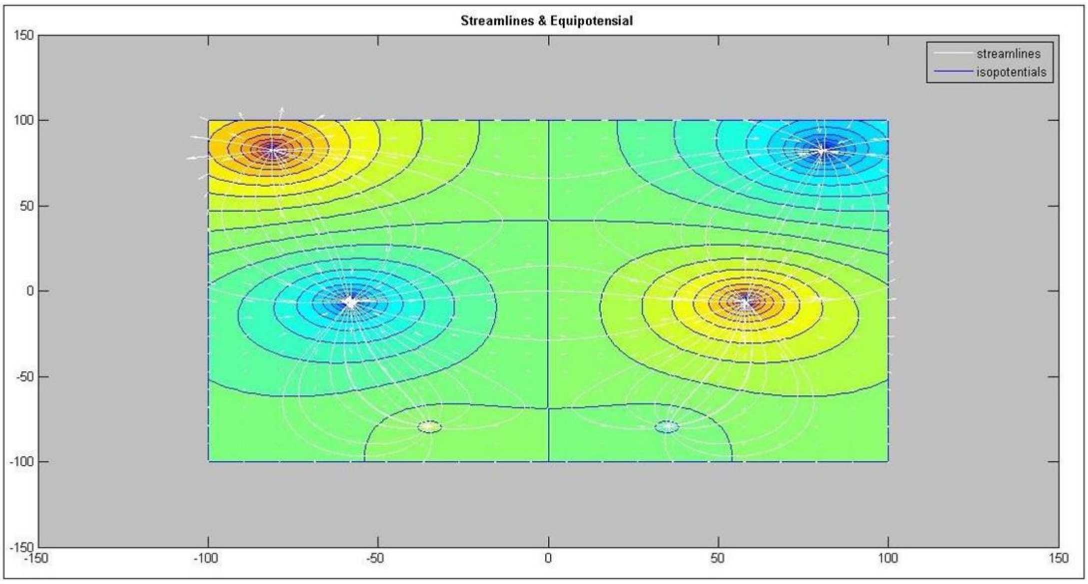

Figure 3. Current line and equipotential line pattern with pumping rates of 2 m/s^3 and 5 m/s^3

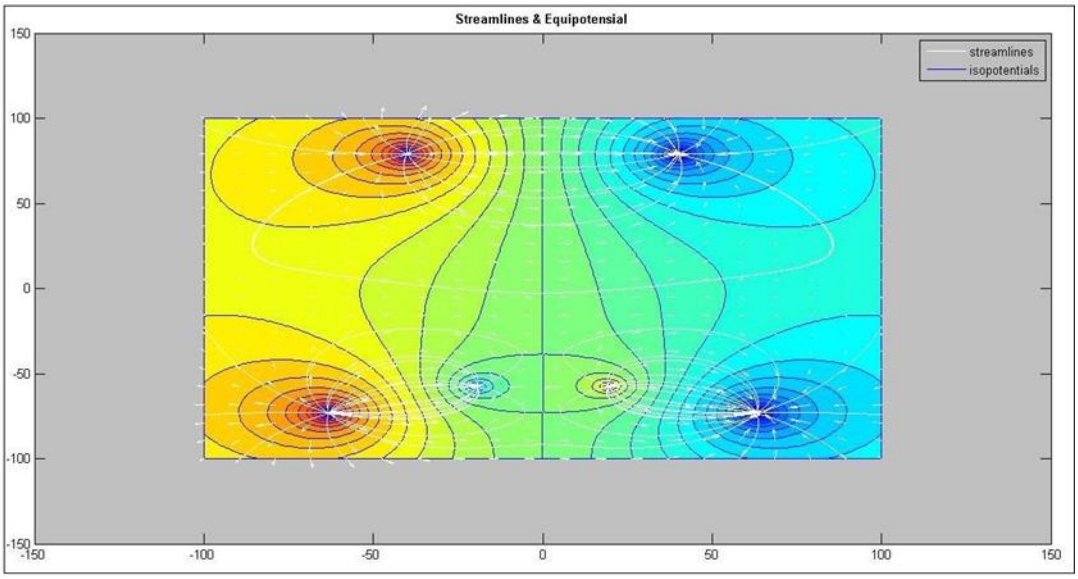

Figure 4. Current line and equipotential line pattern with pumping rates of 5 m/s^3 and 8 m/s^3

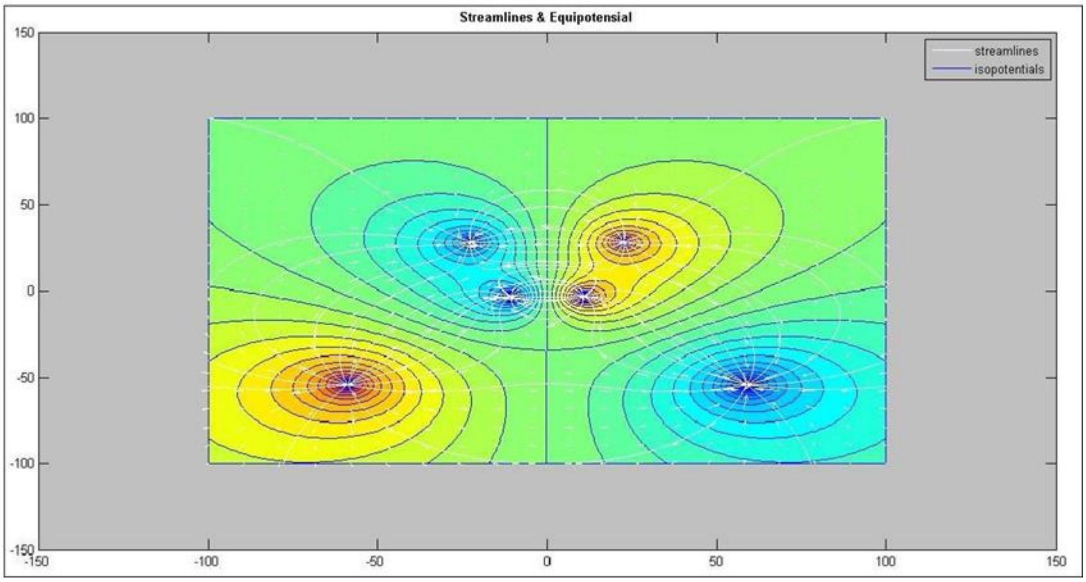

Figure 5. Current line and equipotential line pattern with pumping rates of 8 m/s^3 and 11 m/s^3

2. CONCLUSION

The complex variable method can be an alternative and simple method for visualizing fluid flow in groundwater flow models without having to solve partial differential equations. Streamline patterns and equipotential lines can be visualized well using the complex variable method. Next, other potential shape variations can be carried out to see streamline patterns and equipotential lines.

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