Vol.4.Issue.1.2016 (January-March)

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

http://www.bomsr.com
Email:editorbomsr@gmail.com

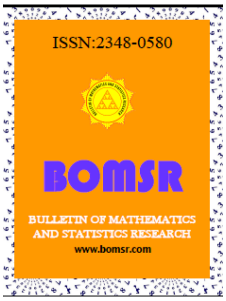
RESEARCH ARTICLE

ISSN:2348-0580

THE SOLUTION OF FRACTIONAL BLACK-SCHOLES EQUATION FOR THE PRICE OF AN OPTION USING LAPLACE TRANSFORM

BRIGHT O.OSU[1], ANGELA I.CHUKWUNEZU[2]

[1]Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Abia State, Nigeria.
email:megaobrait@hotmail.com
[2]Department of Mathematics, Federal Polytechnic, Nekede, Owerri, Nigeria.
email:chukwunezuifeoma@gmail.com



ABSTRACT

In this paper we present a solution for a fractional Black-Scholes equation for the price of an option via Laplace Transform technics. For this purpose, we first derive the Black-Scholes equation for a generic pay-off function whose value is equivalently the worth of the stock at time t. We then show how to reduce the equation to a general parabolic equation by means of change of variables, finally we solve the resulting equation using the Laplace transform method with appropriate boundary conditions.

Key words: option pricing; financial derivatives; Laplace transform; Brownian motion; Fractional Black-Scholes.

MSC: 65C05, 65D30, 98B28.

1. INTRODUCTION

An option provides the holder with the right to buy or sell a specified quantity of an underlying asset at a fixed price (called a strike price or an exercise price) at or before the expiration date of the option. Since it is a right and not an obligation, the holder can choose not to exercise the right and allow the option to expire.

Options are generally defined as contracts between two parties in which one party has the right but not the obligation to do something at a final later time T, usually to buy or sell some underlying asset S_T under protected conditions, see [1]. Having rights without, obligations has financial value, so option holders must purchase these rights, making them assets. These assets derive their value from the primary asset S_T, so they are called derivative assets. More generally, financial derivatives may be viewed as random future payoffs $H_T$which depend somehow on the price of the primary asset, i.e. $H_T = f(S_T)$ Payment for these options takes the form of a flat, up-front sum called premium.

The important feature of an option, either a put or call, is that its purchase / ownership conveys the right to complete the transaction should it be advantageous but not the obligation to do so when not financially advantageous. American-style options give the holder the right to complete the transaction or 'exercise the option' anytime before its expiration, while European-style options can only be executed on the expiration date.The price paid for an option, its premium, differs from its value. Its value is theultimate profit returned to its owner should the transaction be completed. Its price is derived from the difference between the exercise price and the current value of the asset in addition to a premium based on the time remaining until the expiration of the option. The value of an option to its owner (its purchaser or holder) is theoretically unlimited. Conversely, the financial risk to the purchase of an option is limited to the premium, as by definition the owner of an option is not obligated to complete a financially detrimental transaction, see[2].

Laplace transformation is one of the most popular methods of solution of diffusion equations in many areas of science and technology. It has an advantage over other analytical approaches e.g. Fourier or Green's function methods in solving of many practical problems in finance. The most obvious one is a barrier options pricing. Another advantage of Laplace transform method is in straightforward constructing of replications of complex options.See[3].

Black–Scholesmodel is a mathematical model of a financial market containing certain derivative investment instruments. From the model, one can deduce the **Black–Scholes formula**, which gives a theoretical estimate of the price of options. The formula led to a boom in options trading and legitimized scientifically the activities of options markets around the world. It is widely used with some adjustments and corrections, by options market participants. The key financial insight behind the equation is that one can perfectly hedge the option by buying and selling the underlying asset in just the right way and consequently "eliminate risk". This hedge, in turn, implies that there is only one right price for the option which is calculated by the Black–Scholes formula. The Black–Scholes formula calculates the price of **put options** and **call options**.

The Black-Scholes model (BS) for pricing stock options has been applied tomany different commodities and payoff structures. The Black-Scholes model for value of an option is described by the following equation;

$$\frac{\partial v}{\partial t}+\frac{1}{2}\sigma^2 x^2\frac{\partial^2 v}{\partial x^2}+r(t)x\frac{\partial v}{\partial x}-r(t)v=0\,,(x,t)\in R^+\times(0,T)$$

where $v(x;\ t)$ is the European option price at asset price *x* and at time t, T is the maturity, $r(t)$ is the risk free interest rate and $\sigma\_(x;\ t)$ represents the volatility function of underlying asset.

Let us denote by $c(x;\ t)$ and $p(x;\ t)$ the value of the European call and put options, respectively. Then, the payoff functions are

$$c(x;\ t)=\max(x-E,0)\,;$$
$$p(x;\ t)=max(E-x,0);$$

where*E* is the exercise price,see [4].

The Black-Scholes equation relates the recommended price of the option to four other quantities. Three can be measured directly: time, the price of the asset upon which the option is secured and the risk-free interest rate. This is the theoretical interest that could be earned by an investment with zero risk, such as government bonds. The fourth quantity is the volatility of the asset. This is a measure of how erratically its market value changes. The equation assumes that the asset's volatility remains the same for the lifetime of the option, which need not be correct. Volatility can be estimated by statistical analysis of price movements but it can't be measured in a precise, foolproof way, and estimates may not match reality.

In deriving the Black-Scholes equation, the model assumed that the market consists of at least one risky asset, usually called the stock, and one riskless asset, called the money market, cash, or bond, consequent on which we have the following assumptions as well:

- The stock does not pay a dividend.
- There is no arbitrage opportunity (i.e., there is no way to make a riskless profit).
- It is possible to borrow and lend any amount, even fractional, of cash at the riskless rate.
- It is possible to buy and sell any amount, even fractional; of the stock (this includes short selling).
- The above transactions do not incur any fees or costs (i.e., frictionless market).

With these assumptions holding, suppose there is a derivative security also trading in this market. We specify that this security will have a certain payoff at a specified date in the future, depending on the value(s) taken by the stock up to that date. It is a surprising fact that the derivative's price is completely determined at the current time, even though we do not know what path the stock price will take in the future. Black and Scholes showed that "it is possible to create a hedged position, consisting of a long position in the stock and a short position in the option, whose value will not depend on the price of the stock. Their dynamic hedging strategy led to a partial differential equation which governed the price of the option. Its solution is given by the Black–Scholes formula. Several of these assumptions of the original model have been removed in subsequent extensions of the model.

The outline of the paper is the following: In section 2 we derive the Black-Scholespartial differential equation which is parabolic. The parabolic equation is a second order partial differential equation in S-space and first order in time. In section, 3, we introduce the option pricing model for Laplace transform technics. The solution to the Fractional Black-Scholes equation is presented in section 4,and conclusions are given in section 5.

2. Derivation of the Black-Scholes Equation

We base our derivation on replicating portfolio that ensures that no arbitrage opportunities are allowed. As in the discrete case, consider a portfolio$\Lambda = \{\Lambda_t\}_{t>0}$, which is F_t- measurable (we can choose as we go, but at any point in time the choice is deterministic), Λ_t denotes the proportion of shares invested at time t, the rest of the money is invested in the money market account, giving risk-free rate of return, r, say. In what follows, we state:

Lemma 1(Ito's lemma)

Let $f(x,t)$ be a C^2 smooth function of x,t variables. Suppose that the process $\{x(t), t \geq 0\}$satisfies the SDE:

$$dx = \mu(x,t)dt + \sigma(x,t)dB_H(t), \tag{2.1}$$

thenthe first differential of the process $f = f\{x(t), t\}$ is given by

$$df = \frac{\partial f}{\partial x}dx + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}\right)dt \tag{2.2}$$

Theorem 1:

Let a generic payoff function$G(t) = V(s,t)$, the PDE associated with the price of derivative on the stock price is

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + H\sigma^2 S^2 t^{2H-1}\frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{2.3}$$

Proof:

The stock price S_t follows the fractional Brownian motion process

$$\frac{dS}{S} = \mu dt + \sigma dB_H(t)\,,\; S(0) = s, \tag{2.4}$$

and the wealth of an investor X_t, follows a diffusion driven by (with time suppressed)

$$dX = \Lambda dS + r(X - \Lambda S)dt. \tag{2.5}$$

Putting equation (2.4) into equation (2.5) yields;

$$dX = \{rX + \Lambda S(\mu - r)\}dt + \Lambda S\sigma dB_H(t)\}, \tag{2.6}$$

where$\mu - r$ is the risk premium.

Suppose that the value of this claim at time t is given by

$$G(t) = V(S,t),\; S = S_t. \tag{2.7}$$

Applying the fractional Ito's formula on equation 2.7, we have

$$dG = \left[\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + H\sigma^2 S^2 t^{2H-1}\frac{\partial^2 v}{\partial s^2}\right]dt + \sigma S\frac{\partial V}{\partial S}d\mathrm{B}_H. \tag{2.8}$$

To track $G(t)$ at all times, we have under the assumption of complete market that

$$X(t) \quad = G(t) = V(S,t)\forall t \in [0,T]. \tag{2.9}$$

Thus

$$\frac{\partial V}{\partial t} + \mu s\frac{\partial V}{\partial S} + H\sigma^2 s^2 t^{2H-1}\frac{\partial^2 V}{\partial S^2} = rV + \Lambda_t S(\mu - r) \tag{2.10}$$

and

$$\sigma S\frac{\partial V}{\partial S} = \Lambda_t S\sigma. \tag{2.11}$$

$$\Lambda_t = \frac{\partial v}{\partial s}(s,t). \tag{2.12}$$

While equation (2.10) with $\Lambda_t = \frac{\partial v}{\partial s}$gives

$$\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + H\sigma^2 S^2 t^{2H-1}\frac{\partial^2 V}{\partial S^2} = rV + S\mu\frac{\partial V}{\partial S} - Sr\frac{\partial V}{\partial S} \tag{2.13}$$

$$\frac{\partial V}{\partial t} + H\sigma^2 S^2 t^{2H-1}\frac{\partial^2 V}{\partial S^2} + Sr\frac{\partial V}{\partial S} - rV = 0. \tag{2.14}$$

Thus, the famous Black-Scholes equation for valuing an option with value V is obtained. This parabolic equation is a second order partial differential equation in S-space and first order in time. The properties of this equation and assumptions in the proof are:

- The equation is a linear parabolic partial differential equation with non-constant coefficients.
- The asset price follows the log-normal distribution.
- The risk-free interest rate r and the volatility $\sigma$are both known functions of time over the life of the option.
- Transaction costs associated with hedging are not included.
- There is no dividend payment (the basic model assumes no dividend payment, but a simple modification can be made to include some form of dividend payment).
- It is assumed that there are no arbitrage possibilities.
- Trading of the underlying can take place continuously.
- Short selling is possible, which means that assets may be sold without possessing them.

One of the important drawbacks of this model is that the volatility is assumed to be a constant function. In reality this is not the case, but for many options the Black-Scholes model can still be successfully used. Currently there is a lot of research on more accurate modeling of asset price processes by inclusion of jumps or stochastic volatility in the asset price processes.

3. Option Pricing Model For Laplace Transform.

For a call option with maturity date T, strike price K, and payoff function G, the value price $V = V(S,t)$satisfies the following fBm, [5],

$$\frac{\partial V}{\partial t} + Ht^{2H-1}S^2\sigma^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial v}{\partial s} - rV = 0\ , \tag{3.1}$$

$$(S,t) \in (0,\infty)\times(0,T), V(s,0) = h(s),$$

we set$S = e^x \Longrightarrow x = \ln s$,$u(x,t) = V(e^x,t)$ and $h(e^x) = g(x)$, to get

$$\frac{\partial x}{\partial s} = \frac{1}{s} \tag{3.2}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s}$$

$$\frac{\partial u}{\partial s} = \frac{1}{s}\frac{\partial u}{\partial x} \tag{3.3}$$

$$\frac{\partial^2 u}{\partial s^2} = \frac{1}{s}\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial u}{\partial x}\frac{\partial}{\partial s}\left(\frac{1}{s}\right)$$

$$= \frac{1}{s}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial s} + \frac{\partial u}{\partial x}\left(-\frac{1}{s^2}\right)$$

$$= \frac{1}{s^2}\frac{\partial^2 u}{\partial x^2} - \frac{1}{s^2}\frac{\partial u}{\partial x}$$

$$= \frac{1}{s^2}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right). \tag{3.4}$$

Substituting (3.4) in(3.1), we have

$$\frac{\partial u}{\partial t} + Ht^{2H-1}\sigma^2\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right) + r\frac{\partial u}{\partial x} - ru = 0$$

$$\frac{\partial u}{\partial t} + Ht^{2H-1}\sigma^2\frac{\partial^2 u}{\partial x^2} - (Ht^{2H-1}\sigma^2 - r)\frac{\partial u}{\partial x} - ru = 0,$$

this implies that

$$\frac{\partial^2 u}{\partial x^2} - \left(1 - \frac{r}{H\sigma^2}t^{1-2H}\right)\frac{\partial u}{\partial x} - \frac{r}{H\sigma^2}t^{1-2H}u = -\frac{s}{H\sigma^2}t^{1-2H}. \tag{3.5}$$

Let $\lambda = -\left(1 - \frac{r}{H\sigma^2}t^{1-2H}\right)$, $\alpha = -\frac{r}{H\sigma^2}t^{1-2H}$, $x = \frac{-s}{r}$.

Thus, the resulting equation together with some boundary conditions gives

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \lambda\frac{\partial u}{\partial x} + \alpha u = -\alpha x, 0 < x < \infty, t > 0 \\ u(0,t) = 0 \\ u'(0,t) = 0 \end{cases} \tag{3.6}$$

4. Laplace Transform TechnicsFor Fractional Black-Scholes Equation

The Laplace transform of the equation (3.6) is

$$\int_0^\infty \frac{\partial^2 u}{\partial x^2}e^{-st}dt + \lambda\int_0^\infty \frac{\partial u}{\partial x}e^{-st}dt + \alpha\int_0^\infty ue^{-st}dt = -\alpha\int_0^\infty xe^{-st}dt$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}\int_0^\infty u(x,t)e^{-st}dt + \lambda\frac{\partial}{\partial x}\int_0^\infty u(x,t)e^{-st}dt + \alpha\int_0^\infty u(x,t)e^{-st}dt = \frac{-\alpha x}{s}$$

$$\Rightarrow \frac{d^2}{dx^2}\tilde{u}(x,s) + \lambda\frac{d}{dx}\tilde{u}(x,s) + \alpha\tilde{u}(x,s) = \frac{-\alpha x}{s}.$$

Transforming the boundary conditions (3.6), we have:

$$u(0,t) = 0 \Rightarrow \tilde{u}(0,s) = 0$$

$$u'(0,t) = 0 \Rightarrow \tilde{u}'(0,s) = 0,$$

thus, the transformed problem is:

$$\begin{cases} \frac{d^2}{dx^2}\tilde{u}(x,s) + \lambda\frac{d}{dx}\tilde{u}(x,s) + \alpha\tilde{u}(x,s) = \frac{-\alpha x}{s} \\ \tilde{u}(0,s) = 0 \\ \tilde{u}'(0,s) = 0 \end{cases} \tag{4.1}$$

The general solution of (4.1) is of the form:

$$\tilde{u}(x,s) = \tilde{u}_c(x,s) + \tilde{u}_p(x,s)$$

where$\tilde{u}_c(x,s)$ is the corresponding homogeneous solution and $\tilde{u}_p(x,s)$ is the particular solution.

For the homogeneous equation, the auxiliary equation is:

$$m^2 + \lambda m + \alpha = 0$$

Solving, we have:

$$m = \frac{-\lambda \pm \sqrt{\lambda^2 - 4\alpha}}{2}$$

There are three possibilities,

Case I: If $\lambda^2 - 4\alpha = 0$, then the solution will be:

$$\tilde{u}_c(x,s) = C_1(s)e^{\frac{-\lambda x}{2}} + C_2(s)xe^{\frac{-\lambda x}{2}} \tag{4.2}$$

Case II: If $\lambda^2 - 4\alpha > 0$, then the solution will be:

$$\tilde{u}_c(x,s) = C_1(s)e^{\frac{-\lambda + \sqrt{\lambda^2 - 4\alpha}x}{2}} + C_2(s)xe^{\frac{-\lambda - \sqrt{\lambda^2 - 4\alpha}x}{2}}$$

$$Let \beta = \sqrt{\lambda^2 - 4\alpha}$$

$$\tilde{u}_c(x,s) = C_1(s)e^{\frac{-\lambda + \beta x}{2}} + C_2(s)e^{\frac{-\lambda - \beta x}{2}} \tag{4.3}$$

Case III: If $\lambda^2 - 4\alpha < 0$, then the solution will be:

$$\tilde{u}_c(x,s) = e^{-\frac{\lambda x}{2}}\left[C_1(s)cos\frac{\beta x}{2} + C_2(s)sin\frac{\beta x}{2}\right]. \tag{4.4}$$

For the particular solution: the method of undetermined coefficient is used

Assume

$$\left.\begin{aligned} \tilde{u}_p(x,s) &= Ax + B \\ \tilde{u}_p'(x,s) &= A \\ \tilde{u}_p''(x,s) &= 0 \end{aligned}\right\} (4.5)$$

Substituting (4.5) in equation (4.1), we have:

$$\lambda A + \alpha(\mathrm{A}x + B) = \frac{-\alpha \mathrm{x}}{s}.$$

Solving, we have:

$$\mathrm{A} = \frac{-1}{s} \text{and } \mathrm{B} = \frac{\lambda}{\alpha s}$$

Thus,

$$\tilde{u}_p(x,s) = \frac{-x}{s} + \frac{\lambda}{\alpha s} = \frac{\lambda - \alpha x}{\alpha s}.$$

Hence, the general solution is

$$\tilde{u}(x,s) = c_1(s)e^{\frac{-\lambda x}{2}} + c_2(s)xe^{\frac{-\lambda x}{2}} + \frac{\lambda - \alpha x}{\alpha s} \tag{4.6}$$

OR

$$\tilde{u}(x,s) = c_1(s)e^{\frac{-\lambda + \beta x}{2}} + c_2(s)e^{\frac{-\lambda - \beta x}{2}} + \frac{\lambda - \alpha x}{\alpha s} \tag{4.7}$$

OR

$$\tilde{u}(x,s) = e^{\frac{-\lambda x}{2}}\left[c_1(s)Cos\frac{\beta x}{2} + c_2(s)Sin\frac{\beta x}{2}\right] + \frac{\lambda - \alpha x}{\alpha s} \tag{4.8}$$

$To obtain the values of c_1(s) and c_2(s)$, we apply the boundary conditions.

Applying the transformed boundary conditions (4.1) to (4.6), we have

$$\tilde{u}(0,s) = c_1(s) + \frac{\lambda}{\alpha s} = 0$$

$$\tilde{u}'(0,s) = \frac{-\lambda}{2}c_1(s) + c_2(s) - \frac{1}{s} = 0.$$

Solving, we have:

$$c_1(s) = \frac{-\lambda}{\alpha s}$$

$$c_2(s) = \frac{2\alpha-\lambda^2}{2\alpha s}.$$

Thus, substituting in (4.6) we have:

$$\tilde{u}(x,s) = \frac{-\lambda}{\alpha s} e^{\frac{-\lambda x}{2}} + \left(\frac{2\alpha-\lambda^2}{2\alpha s}\right) x e^{\frac{-\lambda x}{2}} + \frac{\lambda-\alpha x}{\alpha s}$$

$$=\frac{-2\lambda e^{-\frac{-\lambda x}{2}}+(2\alpha-\lambda^2)xe^{\frac{-\lambda x}{2}}+\lambda-\alpha x}{2\alpha s}. \qquad (4.9)$$

To find our solution, we apply the inverse Laplace transform:

$$u(x,t) = L^{-1}[\tilde{u}(x,s)]$$

$$u(x,t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \tilde{u}(x,s)e^{st}ds$$

$$u(x,t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+\infty} [\frac{-2\lambda e^{-\frac{-\lambda x}{2}}+(2\alpha-\lambda^2)xe^{\frac{-\lambda x}{2}}+\lambda-\alpha x}{2\alpha s}]e^{st}ds.$$

To evaluate this, we find the sum of residues at the poles of$\tilde{u}(x,s)e^{st}$.

Here, the pole is at s=0.

Recall: Res$[f(s),s_0]$ $=\lim_{s\to s_0} \frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}(s-s_0)^n f(s)$

$$\text{Res }(at s=0) = \lim_{s\to 0}\left\{\frac{s\left[-2\lambda e^{\frac{-\lambda x}{2}}+(2\alpha-\lambda^2)xe^{\frac{-\lambda x}{2}}+\lambda-\alpha x\right]e^{st}}{2\alpha s}\right\}$$

$$= \lim_{s\to 0}[\frac{-2\lambda e^{-\frac{\lambda x}{2}}+(2\alpha-\lambda^2)xe^{-\frac{-\lambda x}{2}}+\lambda-\alpha x}{2\alpha}]e^{st}$$

$$= [\frac{-2\lambda e^{-\frac{\lambda x}{2}}+(2\alpha-\lambda^2)xe^{-\frac{-\lambda x}{2}}+\lambda-\alpha x}{2\alpha}],$$

this implies that;

$$u(x,t) = [\frac{-2\lambda e^{-\frac{-\lambda x}{2}}+(2\alpha-\lambda^2)xe^{-\frac{-\lambda x}{2}}+\lambda-\alpha x}{2\alpha}]. \qquad (4.10)$$

Similarly, applying the transformed boundary condition (4.1) to (4.7),we have:

$$\tilde{u}(0,s) = c_1(s)e^{\frac{-\lambda}{2}} + c_2(s)e^{\frac{-\lambda}{2}} + \frac{\lambda}{\alpha s} = 0$$

$$\tilde{u}'(0,s) = \frac{\beta}{2}c_1(s)e^{\frac{-\lambda}{2}} - \frac{\beta}{2}c_2(s)e^{\frac{-\lambda}{2}} - \frac{1}{s} = 0.$$

Solving, we have:

$$c_1(s) = \frac{2\alpha-\beta\lambda}{2\alpha\beta s\, e^{\frac{-\lambda}{2}}}$$

$$c_2(s) = \frac{-(2\alpha+\beta\lambda)}{2\alpha\beta s\, e^{\frac{-\lambda}{2}}},$$

substituting in (4.7), we have:

$$\tilde{u}(x,s) = \frac{2\alpha-\beta\lambda}{2\alpha\beta s\, e^{\frac{-\lambda}{2}}}e^{\frac{-\lambda+\beta x}{2}} - \frac{(2\alpha+\beta\lambda)}{2\alpha\beta s\, e^{\frac{-\lambda}{2}}}e^{\frac{-\lambda-\beta x}{2}} + \frac{\lambda-\alpha x}{\alpha s}$$

$$u(x,t) = L^{-1}\{\tilde{u}(x,s)\}$$

$$=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{(2\alpha-\beta\lambda)e^{\frac{-\lambda+\beta x}{2}}-(2\alpha+\beta\lambda)e^{\frac{-\lambda-\beta x}{2}}+2\beta e^{\frac{-\lambda}{2}}(\lambda-\alpha x)e^{st}}{2\alpha\beta s\, e^{\frac{-\lambda}{2}}}ds$$

Residue at s= 0 is given by

$$\lim_{s\to 0} \quad \frac{s[(2\alpha-\beta\lambda)e^{\frac{-\lambda+\beta x}{2}} - [2\alpha+\beta\lambda]e^{\frac{-\lambda-\beta x}{2}} + 2\beta e^{\frac{-\lambda}{2}}(\lambda-\alpha x)]\, e^{st}}{2\alpha\beta s e^{\frac{-\lambda}{2}}}$$

$$= \lim_{s\to 0} \quad \frac{(2\alpha-\beta\lambda)e^{\frac{-\lambda+\beta x}{2}} - \ [2\alpha+\beta\lambda]e^{\frac{-\lambda-\beta x}{2}} + 2\beta e^{\frac{-\lambda}{2}}(\lambda-\alpha x)\, e^{st}}{2\alpha\beta e^{\frac{-\lambda}{2}}}$$

$$= \frac{(2\alpha-\beta\lambda)e^{\frac{-\lambda+\beta x}{2}} - [2\alpha+\beta\lambda]e^{\frac{-\lambda-\beta x}{2}} + 2\beta e^{\frac{-\lambda}{2}}(\lambda-\alpha x)}{2\alpha\beta e^{\frac{-\lambda}{2}}}.$$

This implies that;

$$u(x,t) = \frac{(2\alpha-\beta\lambda)e^{\frac{-\lambda+\beta x}{2}} - [2\alpha+\beta\lambda]e^{\frac{-\lambda-\beta x}{2}} + 2\beta e^{\frac{-\lambda}{2}}(\lambda-\alpha x)}{2\alpha\beta e^{\frac{-\lambda}{2}}}. \qquad (4.11)$$

Also applying transformed boundary conditions (4.1) to(4.7), we have:

$$\tilde{u}(o,s) = c_1(s) + \frac{\lambda}{\alpha s} = 0$$

$$\tilde{u}'(o,s) = \frac{\beta}{2}c_2(s) - \frac{\lambda}{2}c_1(s) - \frac{1}{s} = 0,$$

solving , we have :

$$c_1(s) = -\frac{\lambda}{\alpha s}\, and\, c_2(s) = \frac{2\alpha-\lambda^2}{\alpha\beta s},$$

substituting in (4.8) we have:

$$\tilde{u}(x,s) = e^{\frac{-\lambda x}{2}}\left[\frac{-\lambda}{\alpha s}cos\frac{\beta x}{2} + \frac{(2\alpha-\lambda^2)}{\alpha\beta s}sin\frac{\beta x}{2}\right] + \frac{\lambda-\alpha x}{\alpha s}$$

$$\tilde{u}(x,s) = \frac{e^{\frac{-\lambda x}{2}}\left[-\lambda\beta cos\frac{\beta x}{2} + (2\alpha-\lambda^2)sin\frac{\beta x}{2}\right] + \beta(\lambda-\alpha x)}{\alpha\beta s}.$$

The inversion is

$$u(x,t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{\frac{-\lambda x}{2}}\left[-\lambda\beta cos\frac{\beta x}{2} + (2\alpha-\lambda^2)sin\frac{\beta x}{2}\right] + \beta(\lambda-\alpha x)}{\alpha\beta s} e^{st}\, ds$$

To evaluate this, we find the residue at s =0

$$= \lim_{s\to 0} \quad \frac{s\{e^{\frac{-\lambda x}{2}}\left[-\lambda\beta cos\frac{\beta x}{2} + (2\alpha-\lambda^2)sin\frac{\beta x}{2}\right] + \beta(\lambda-\alpha x)\}e^{st}}{\alpha\beta s}$$

$$= \lim_{s\to 0} \quad \frac{e^{\frac{-\lambda x}{2}}\left[-\lambda\beta cos\frac{\beta x}{2} + (2\alpha-\lambda^2)sin\frac{\beta x}{2}\right] + \beta(\lambda-\alpha x)e^{st}}{\alpha\beta}$$

$$= \frac{e^{\frac{-\lambda x}{2}}\left[-\lambda\beta cos\frac{\beta x}{2} + (2\alpha-\lambda^2)sin\frac{\beta x}{2}\right] + \beta(\lambda-\alpha x)}{\alpha\beta}.$$

This implies that;

$$u(x,t) = \frac{e^{\frac{-\lambda x}{2}}\left[-\lambda\beta cos\frac{\beta x}{2} + (2\alpha-\lambda^2)sin\frac{\beta x}{2}\right] + \beta(\lambda-\alpha x)}{\alpha\beta}. \qquad (4.12)$$

Combining (4.10), (4.11), (4.12) we have:

$$u(x,t) = \begin{cases} \left[\frac{-2\lambda e^{-\frac{-\lambda x}{2}} + (2\alpha-\lambda^2)xe^{-\frac{-\lambda x}{2}} + \lambda - \alpha x}{2\alpha}\right], \lambda^2 - 4\alpha = 0 \\ \frac{(2\alpha-\beta\lambda)e^{\frac{-\lambda+\beta x}{2}} - [2\alpha+\beta\lambda]e^{\frac{-\lambda-\beta x}{2}} + 2\beta e^{\frac{-\lambda}{2}}(\lambda-\alpha x)}{2\alpha\beta e^{\frac{-\lambda}{2}}}, \lambda^2 - 4\alpha > 0 \\ \frac{e^{\frac{-\lambda x}{2}}\left[-\lambda\beta cos\frac{\beta x}{2} + (2\alpha-\lambda^2)sin\frac{\beta x}{2}\right] + \beta(\lambda-\alpha x)}{\alpha\beta}, \lambda^2 - 4\alpha < 0 \end{cases} \qquad (4.13)$$

5. Conclusion

In this paper we considered Laplace Transform method in obtaining the solution of a fractional Black-Scholes formula for the price of an option. Furthermore, we demonstrate that the price of an option not only depends on time $T-t$, but also on the stock price $S(t)$.The reason is based on the fact that the fractional Brownian motion has a long memory. Hence, the growth rate $V(S,t)$ depends largely on how$S \to \infty$or how$S \to 0$.

REFERENCES

[1]. Leentvaar, C.W. Numerical solution of the Black-Scholes equation with a small number of grid points.Mastersthesis, Delfty University of Technology, December 2003.

[2]. Richard, C.T. Risk measurement, allocation and pricing in network schedule system. A Dissertation submitted to the Department of Civil Engineering, School of Engineering of the Catholic University of America,2012.

[3]. IgorSkachkov. Black-Scholes equation in Laplace transform domain. Published on Wilmott.com in March 2002.

[4]. Mohammad Ali et al. European option pricing of Fractional Black-Scholes model with new Lagrange multipliers. Computational methods for Differential Equations; vol 2, No 1, 2014.

[5]. Francois Coppex. Solving the Black- Scholes equation: a demystification. November 2009.