Vol.4.Issue.1.2016 (January-March)

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

http://www.bomsr.com

Email:editorbomsr@gmail.com

ISSN:2348-0580

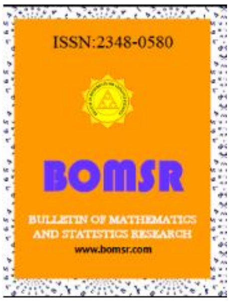
RESEARCH ARTICLE

A COMPARATIVE STUDY OF STATISTICAL HYPOTHESIS TEST FOR 2^2 FACTORIAL EXPERIMENT UNDER FUZZY ENVIRONMENTS

S.PARTHIBAN[1*], P. GAJIVARADHAN[2]

[1]Research Scholar, Department of Mathematics, Pachaiyappa's College,Chennai, Tamil Nadu, India.
Email: selvam.parthiban1979@gmail.com

[2]Department of Mathematics, Pachaiyappa's College, Chennai, Tamil Nadu, India.
Email: drgajivaradhan@gmail.com



ABSTRACT

The 2^2 factorial experiment using Trapezoidal Fuzzy Numbers (tfns.) is proposed here. And the proposed test is analysed under various types of trapezoidal fuzzy models such as Alpha Cut Interval, Membership Function, Ranking Function, Total Integral Value and Graded Mean Integration Representation. Finally a comparative view of the conclusions obtained from various test is given. Moreover, two numerical examples having different conclusions have been illustrated for a concrete comparative study.

Keywords: 2^2 Factorial Design, Trapezoidal Fuzzy Numbers (tfns.), Alpha Cut, Membership Function, Ranking Function, Total Integral Value (TIV), Graded Mean Integration Representation (GMIR).

AMS Mathematics Subject Classification (2010): 62A86, 62F03, 97K80

1. INTRODUCTION

Statistical analysis in traditional form is based on crispness of data, random variables, point estimations, hypotheses and so on. There are many different situations in which such concepts are imprecise. On the other hand, the theory of fuzzy sets [30] is a well-known tool for the formulation and the analysis of imprecise and subjective concepts. Therefore, testing hypotheses with fuzzy data can be important. In traditional statistical testing [11], the observations of sample are crisp and a statistical test leads to a binary decision. However, in the real life, the data sometimes cannot be recorded or collected precisely. The statistical hypotheses testing under fuzzy environments has been studied by many authors using the fuzzy set theory concepts introduced by Zadeh [30]. Viertl [24] investigated some methods to construct confidence intervals and statistical tests for fuzzy data. Wu [28] proposed some approaches to construct fuzzy confidence intervals for the unknown fuzzy parameter. A new approach to the problem of testing statistical hypotheses is introduced by Chachi et al. [8]. Mikihiko Konishi et al. [15] proposed a method of ANOVA for the fuzzy interval data by using the concept of fuzzy sets. Hypothesis testing of one factor ANOVA model for fuzzy data was proposed by Wu [27, 29] using the h-level set and the notions of pessimistic degree and optimistic degree by solving optimization problems. Gajivaradhan and Parthiban analysed one-way ANOVA test using alpha cut interval method for trapezoidal fuzzy numbers [16] and they presented a comparative study of 2-factor ANOVA test [17] under fuzzy environments using various methods also they proposed a comparative study of LSD under fuzzy environments using trapezoidal fuzzy numbers [18].

Liou and Wang ranked fuzzy numbers with total integral value [14]. Wang et al. presented the method for centroid formulae for a generalized fuzzy number [26]. Iuliana Carmen BĂRBĂCIORU dealt with the statistical hypotheses testing using membership function of fuzzy numbers [12]. Salim Rezvani analysed the ranking functions with trapezoidal fuzzy numbers [21]. Wang arrived some different approach for ranking trapezoidal fuzzy numbers [26]. Thorani et al. approached the ranking function of a trapezoidal fuzzy number with some modifications [22]. Salim Rezvani and Mohammad Molani presented the shape function and Graded Mean Integration Representation for trapezoidal fuzzy numbers [20]. Liou and Wang proposed the Total Integral Value of the trapezoidal fuzzy number with the index of optimism and pessimism [14].

In this paper, we propose a new statistical fuzzy hypothesis test for 2^2 factorial experiment in which the designated samples are in terms of fuzzy (trapezoidal fuzzy numbers) data. The main idea in the proposed approach is, when we have some vague data about an experiment, what can be the result when the centroid point/ranking grades of those imprecise data are employed in the hypothesis test? For this reason, we use the centroid point /ranking grades of trapezoidal fuzzy numbers (tfns.) in the hypothesis testing.

Suppose the observed samples are in terms of tfns., we can evenhandedly use the centroid point/ranking grades of tfns. for statistical hypothesis testing. In arriving the centroid point/ranking grades of tfns., various methods are used to test which could be the best fit. Therefore, in the proposed approach, the centroid point point/ranking grades of tfns. are used in 2^2 factorial design. Moreover we provide the decision rules which are used to accept or reject the fuzzy null and alternative hypotheses. In fact, we would like to counter an argument that the alpha cut interval method can be general enough to deal with 2^2 factorial experiment under fuzzy environments. In the decision rules of the proposed testing technique, degrees of optimism, pessimism and h-level sets are not used but they are used in Wu [27]. For better understanding, the proposed fuzzy hypothesis testing technique for 2^2 factorial experiment using tfns., two different kinds of numerical examples are illustrated at each models. And the same concept can also be used when we have samples in terms of triangular fuzzy numbers [5, 27].

2. Preliminaries

Definition 2.1.

Generalized fuzzy number

A generalized fuzzy number $\widetilde{A} = (a, b, c, d; w)$ is described as any fuzzy subset of the real line $\mathbb{R}$, whose membership function $\mu_{\tilde{A}}(x)$ satisfies the following conditions:

i. $\mu_{\tilde{A}}(x)$ is a continuous mapping from $\mathbb{R}$ to the closed interval $[0, \omega]$, $0 \le \omega \le 1$,

ii. $\mu_{\tilde{A}}(x) = 0$, for all $x \in (-\infty, a]$,

iii. $\mu_L(x) = L_{\tilde{A}}(x)$ is strictly increasing on $[a, b]$,

iv. $\mu_{\tilde{A}}(x) = \omega$, for all $[b, c]$, as ω is a constant and $0 < \omega \le 1$,

v. $\mu_R(x) = R_{\tilde{A}}(x)$ is strictly decreasing on $[c, d]$,

vi. $\mu_{\tilde{A}}(x) = 0$, for all $x \in [d, \infty)$ where a, b, c, d are real numbers such that $a < b \le c < d$.

Definition 2.2. A fuzzy set $\tilde{A}$ is called ***normal*** fuzzy set if there exists an element (member) 'x' such that $\mu_{\tilde{A}}(x) = 1$. A fuzzy set $\tilde{A}$ is called ***convex*** fuzzy set if $\mu_{\tilde{A}}(\alpha x_1 + (1 - \alpha) x_2) \ge \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}$ where $x_1, x_2 \in X$ and $\alpha \in [0, 1]$. The set $\tilde{A}_\alpha = \{x \in X / \mu_{\tilde{A}}(x) \ge \alpha\}$ is said to be the ~ ***- cut*** of a fuzzy set $\tilde{A}$.

Definition 2.3. A fuzzy subset $\tilde{A}$ of the real line $\mathbb{R}$ with ***membership function*** $\mu_{\tilde{A}}(x)$ such that $\mu_{\tilde{A}}(x) : \mathbb{R} \to [0, 1]$, is called a fuzzy number if $\tilde{A}$ is normal, $\tilde{A}$ is fuzzy convex, $\mu_{\tilde{A}}(x)$ is upper semi-continuous and $\mathrm{Supp}(\tilde{A})$ is bounded where $\mathrm{Supp}(\tilde{A}) = \mathrm{cl}\{x \in \mathbb{R} : \mu_{\tilde{A}}(x) > 0\}$ and 'cl' is the closure operator.

It is known that for a ***normalized tfn.*** $\tilde{A} = (a, b, c, d; 1)$, there exists four numbers $a, b, c, d \in \mathbb{R}$ and two functions $L_{\tilde{A}}(x), R_{\tilde{A}}(x) : \mathbb{R} \to [0, 1]$, where $L_{\tilde{A}}(x)$ and $R_{\tilde{A}}(x)$ are non-decreasing and non-increasing functions respectively. And its membership function is defined as follows: $\mu_{\tilde{A}}(x) = \{L_{\tilde{A}}(x) = (x-a)/(b-a)$ for $a \le x \le b$; 1 for $b \le x \le c$; $R_{\tilde{A}}(x) = (x-d)/(c-d)$ for $c \le x \le d$ and 0 otherwise}. The functions $L_{\tilde{A}}(x)$ and $R_{\tilde{A}}(x)$ are also called the ***left*** and ***right side*** of the fuzzy number $\tilde{A}$ respectively [9]. In this paper, we assume that $\int_{-\infty}^{\infty} \tilde{A}(x)\,dx < +\infty$ and it is known that the α - cut of a fuzzy number is $\tilde{A}_\alpha = \{x \in \mathbb{R} / \mu_{\tilde{A}}(x) \ge \alpha\}$, for $\alpha \in (0, 1]$ and $\tilde{A}_0 = \mathrm{cl}\left(\bigcup_{\alpha \in (0, 1]} \tilde{A}_\alpha\right)$, according to the definition of a fuzzy number, it is seen at once that every α - cut of a fuzzy number is a closed interval. Hence, for a fuzzy number $\tilde{A}$, we have $\tilde{A}(\alpha) = [\tilde{A}_L(\alpha), \tilde{A}_U(\alpha)]$ where $A_L(\alpha) = \inf\{x \in \mathbb{R} : \mu_{\tilde{A}}(x) \ge \alpha\}$ and $A_U(\alpha) = \sup\{x \in \mathbb{R} : \mu_{\tilde{A}}(x) \ge \alpha\}$. The left and right sides of the fuzzy number $\tilde{A}$ are strictly monotone, obviously, $\tilde{A}_L$ and $\tilde{A}_U$ are ***inverse functions*** of $L_{\tilde{A}}(x)$ and $R_{\tilde{A}}(x)$ respectively.

Another important type of fuzzy number was introduced in [6] as follows:

Let $a, b, c, d \in \mathbb{R}$ such that $a < b \le c < d$. A fuzzy number $\tilde{A}$ defined as $\mu_{\tilde{A}}(x) : \mathbb{R} \to [0, 1]$, $\mu_{\tilde{A}}(x) = \left(\frac{x - a}{b - a}\right)^n$ for $a \le x \le b$; 1 for $b \le x \le c$; $\left(\frac{d - x}{d - c}\right)^n$ for $c \le x \le d$; 0 otherwise where $n > 0$, is denoted by $\tilde{A} = (a, b, c, d)_n$. And $L(x) = \left(\frac{x - a}{b - a}\right)^n$ and $R(x) = \left(\frac{d - x}{d - c}\right)^n$ can also be termed as ***left*** and ***right spread*** of the tfn. [Dubois and Prade in 1981].

If $\tilde{A} = (a, b, c, d)_n$, then[1-4],

$\widetilde{A}_\alpha = \left[\widetilde{A}_L(\alpha),\ \widetilde{A}_U(\alpha)\right] = \left[a + (b - a)\sqrt[n]{\alpha},\ d - (d - c)\sqrt[n]{\alpha}\right];\ \alpha \in [0, 1]$.

When $n = 1$ and $b = c$, we get a triangular fuzzy number. The conditions $r = 1, a = b$ and $c = d$ imply the closed interval and in the case $r = 1, a = b = c = d = t$ (some constant), we can get a crisp number 't'. Since a trapezoidal fuzzy number is completely characterized by $n = 1$ and four real numbers $a \le b \le c \le d$, it is often denoted as $\widetilde{A} = (a, b, c, d)$. And the family of trapezoidal fuzzy numbers will be denoted by $F^T(\mathbb{R})$. Now, for $n = 1$ we have a normal trapezoidal fuzzy number $\widetilde{A} = (a, b, c, d)$ and the corresponding α - cut is defined by

$\widetilde{A}_\alpha = \left[a + \alpha(b - a),\ d - \alpha(d - c)\right];\ \alpha \in [0, 1] - - - (2.4)$. And we need the following results which can be found in [11, 13].

Result 2.1. Let D = {[a, b], $a \le b$ and a, $b \in \mathbb{R}$}, the set of all closed, bounded intervals on the real line $\mathbb{R}$.

Result 2.2. Let A = [a, b] and B = [c, d] in D. Then A = B if a = c and b = d.

3. 2^2 Factorial Design:

A major conceptual advancement in experimental design is exemplified by factorial design. Factorial designs are frequently used in experiments involving several factors where it is necessary to study the joint effect of the factors on a response. In factorial designs, an assessment of each individual factor effect is based on the whole set of measurements so that a more efficient utilization of experimental resources is achieved in these designs. The most importance of these special cases is that of 'k' factors, each at only two levels. These levels may be quantitative such as two values of temperature, time or pressure or they may be qualitative such as two machines, two operators, the 'high' and 'low' level of a factor or perhaps the presence and absence of a factor.

But in the experimental designs either in CRD or RBD or LSD, we are primarily concerned with the comparison and the estimation of the effects of a single set of treatments like varieties of wheat, manure or different methods of cultivation etc. Such experiment which deal with one factor only, called as ***simple experiments***.

3.1. Definition

symmetrical factorial experiment

Suppose that there are factors with $s_1, s_2, \ldots, s_n$ levels respectively which may affect the characteristic in which we are interested. Then we have to estimate (i) the effects of each of the factors (ii) how the effect of one factor varies over the different levels of other factors. To study these effects, we investigate all possible replicate of the experiment. Thus there are $s_1, s_2, \ldots, s_k$ treatment combinations (or composite treatments) to be assigned to the different experimental units. Such an arrangement is called $s_1 \times s_2 \times \cdots \times s_k$ ***factorial experiment***. A factorial experiment in which each of the 'k' factors is at 's' levels is called a ***symmetrical factorial experiment*** and is often known as s^k factorial design. In a symmetrical factorial experiment if each of the k-factors is at two levels is called ***2^k factorial experiment***. And ***2^2 factorial experiment*** means a symmetrical factorial experiment where each of the two factors is at two levels.

3.2. Definition

2^2 Factorial Experiment

Suppose there are 2 factors with 2 levels each which may affect the characteristic in which we are interested. To study their effects, we investigate the 4 possible combinations of the levels of these factors in each complete trial or in the replicate of the experiments. This experiment is called a 2^2 ***factorial experiment*** and can be performed in the form of CRD, RBD and LSD or in other designs as well.

3.3. Definition

2^2 Factorial Design.

A factorial design with two factors, each at two levels is called a ***2^2 – factorial design***.

3.4. Definition

Yate's Notations

The two factors are denoted by the letters A and B. The letters 'a' and 'b' denote one of the two levels of each of the corresponding factors and this will be called the second level. The first level of A and B is generally expressed by the absence of the corresponding letter in the treatment combinations. The four treatment combinations can be enumerated as follows:

a_0b_0 or (1) : Factors A and B, both at first level.

a_1b_0 or a : A at second level and B at first level.

a_0b_1 or b : A at first level and B at second level.

a_1b_1 or ab : A and B both at second levels.

These four treatment combinations can be compared by laying out the experiment in (i) RBD with 'r' replicates (say), each replicate containing 4 units or (ii) 4×4 LSD and ANOVA can be carried out accordingly. In the above cases, there are 3 degrees of freedom associated with treatment effects. In factorial experiment, our main objective is to carry out separate tests for the main effects A, B and the interaction AB, splitting the treatment S.S. with 3 degrees of freedom into three orthogonal components, each with 1 degree of freedom and each associated with the main effects A and B or the interaction AB.

3.5. Yate's method of computing factorial effect totals

For the calculation of various factorial effect totals for 2^2 factorial experiments, the following table provides a special computational rule for the totals of the main effects or the interactions corresponding to the treatment combinations.

Treatment combination (1)	**Total yield from all replicates (2)**	**(3)**	**(4)**	**Effect totals**
1	[1]	[1] + [a]	[1]+[a]+[b]+[ab]	Grand total
a	[a]	[b] + [ab]	[ab]-[b]+[a]-[1]	[A]
b	[b]	[a] - [1]	[ab]+[b]-[a]-[1]	[B]
ab	[ab]	[ab] - [b]	[ab]-[b]-[a]+[1]	[AB]

3.6. **Definition**

Contrast and Orthogonal Contrast

A linear combination $\sum_{i=1}^{k} c_i t_i$ of 'k' treatments means t_i (i=1, 2. ..., k) is called a ***contrast*** (or a comparison) of treatment means t_i (i=1, 2. ..., k) if $\sum_{i=1}^{k} c_i = 0$. In other words, contrast is a linear combination of treatment means, such that the sum of the coefficients is zero. Two contrasts of 'k' treatment means t_i (i=1, 2. ..., k) namely $\sum_{i=1}^{k} c_i t_i$ with $\sum_{i=1}^{k} c_i = 0$ and $\sum_{i=1}^{k} d_i t_i$ with $\sum_{i=1}^{k} d_i = 0$ are said to be ***orthogonal*** if $\sum_{i=1}^{k} c_i d_i = 0$. In other words, the contrasts are orthogonal, if the sum of the product of coefficients of corresponding treatment means is zero.

***3.7.* Definition**

Sign Table

The main effects and interactions in terms of individual and composite treatment means in case of 2^2 factorial experiment can be shown in the following table which contains the divisors also where $M=(1/4)([1]+[a]+[b]+[ab])$ =General Mean.

Factorial Effect	Treatment Combinations				Divisor
	-1	(a)	(b)	(ab)	
M	+	+	+	+	4
A	-	+	-	+	2
B	-	-	+	+	2
AB	+	-	-	+	2

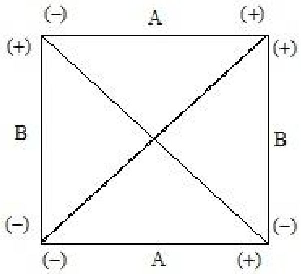


Figure-1

To find any effect, simply multiply the sign in the column of the table by the corresponding treatment combination and add and then divide by the corresponding divisor. For main effects give a plus sign where the corresponding factor is at the second level and a minus sign whenever the factor is at first level. The algebraic signs of the two factor interaction are obtained by multiplying the corresponding signs of the two levels. It is to be noted that two similar signs will give plus sign and the opposite signs will give a minus sign. That is, for A: ab(+), b(-), a(+), 1(-); for B: ab(+), b(+), a(), 1(-); for AB: ab(+), b(-), a(-), 1(+).

4. Statistical Analysis of 2^2 Factorial Design

Factorial experiments are conducted either in CRD or RBD or LSD and they can be analysed in the usual manner except that in this case the treatment S.S. is split into three orthogonal components each with 1 degree of freedom. And the main effects of A, B and the interaction AB are mutually orthogonal contrasts of treatment means [10, 11]. Now, using ***Yate's method***, we would average the observations on the right side of the square in the above figure (1) where A is at the high level and subtract from this the average of the observations on the left side of the square where A is at the low level that is, $A=\overline{Y}_{A+}-\overline{Y}_{A-}=(([a]+[ab])/2r)-(([b]+[1])/2r)$ that is, $A=\frac{1}{2r}([a]+[ab]-[b]-[1])$ similarly, $B=\frac{1}{2r}([b]+[ab]-[a]-[1])$; $AB=\frac{1}{2r}([ab]+[1]-[a]-[b])$.

Here, $\text{Contrast}_A=([a]+[ab]-[b]-[1])$; $\text{Contrast}_B=([b]+[ab]-[a]-[1])$; $\text{Contrast}_{AB}=([ab]+[1]-[a]-[b])$ and $SS_A=[A]^2/4r$; $SS_B=[B]^2/4r$; $SS_{AB}=[AB]^2/4r$ each with 1 degree of freedom where 'r' is the common replication number and SS means sum of squares.

5. 2^2 Factorial design conducted in a CRD

Let x_{ij} = j^{th} observation of i^{th} treatment combination, i = 1, 2, 3, 4; j = 1, 2, …, r (say). That is $x_{1*}=[1]$; $x_{2*}=[a]$; $x_{3*}=[b]$; $x_{4*}=[ab]$ where x_{i*} = total of i^{th} treatment combination. Grand total $G=\sum_i\sum_j x_{ij}$ and total number of observation n = 4r. Then $SS_T=\sum\sum x_{ij}^2-(G^2/4r)$ and $SS_E=SS_T-(SS_A+SS_B+SS_{AB})$.

The ANOVA table for 2^2 Factorial design conducted in CRD

Source of Variation (S.V.)	Sum of Squares (S.S.)	Degrees of freedom (d.f.)	Means Sum Square (M.S.S.)	F-Ratio
A	SS_A	1	$MSS_A = \frac{SS_A}{d.f.}$	$F_A = \frac{MSS_A}{MSS_E}$
B	SS_B	1	$MSS_B = \frac{SS_B}{d.f.}$	$F_B = \frac{MSS_B}{MSS_E}$
AB	SS_{AB}	1	$MSS_{AB} = \frac{SS_{AB}}{d.f.}$	$F_{AB} = \frac{MSS_{AB}}{MSS_E}$
Error	SS_E	4(r-1)	$MSS_E = \frac{SS_E}{d.f.}$	--
Total	SS_T	4r-1	--	--

6. 2^2Factorial experiment conducted in RBD

Let x_{ij} be the observation in j^{th} block for i^{th} treatment i = 1, 2, 3, 4 and j = 1, 2, …, r (say). And x_{i*} = total of i^{th} treatment combination, x_{*j} is the total of j^{th} block. Grand total $G = \sum_i \sum_j x_{ij}$. Total number of observations n = 4r where 'r' is the common replication number. Then $SS_T = \sum\sum x_{ij}^2 - (G^2/4r)$; $SS_{blocks} = (1/4)\sum_j x_{*j}^2 - (G^2/4r)$ and SS_A, SS_B, SS_{AB} can be obtained as in CRD. Now $SS_E = SS_T - (SS_{Blocks} + SS_A + SS_B + SS_{AB})$.

The ANOVA table for 2^2 factorial experiment in RBD with r-replicates

Source of Variation (S.V.)	Sum of Squares (S.S.)	Degrees of freedom (d.f.)	Means Sum Square (M.S.S.)	F-Ratio
Blocks	SS_{Blocks}	r-1	$MSS_{Blocks} = \frac{SS_{Blocks}}{d.f.}$	$F_{Blocks} = \frac{MSS_{Blocks}}{MSS_E}$
Main Effect A	SS_A	1	$MSS_A = \frac{SS_A}{d.f.}$	$F_A = \frac{MSS_A}{MSS_E}$
Main Effect B	SS_B	1	$MSS_B = \frac{SS_B}{d.f.}$	$F_B = \frac{MSS_B}{MSS_E}$
Main Effect AB	SS_{AB}	1	$MSS_{AB} = \frac{SS_{AB}}{d.f.}$	$F_{AB} = \frac{MSS_{AB}}{MSS_E}$
Error	SS_E	3(r-1)	$MSS_E = \frac{SS_E}{d.f.}$	--
Total	SS_T	4r-1	--	--

7. 2^2 Factorial experiment under fuzzy data

The fuzzy test of hypotheses of 2^2 factorial experiment in which the sample data are trapezoidal fuzzy numbers is given here. Using the relation (2.4), we transform the fuzzy 2^2 factorial model to interval 2^2 factorial model. Having the upper limit of the fuzzy interval, we construct upper level crisp 2^2 factorial model and using the lower limit of the fuzzy interval, we construct the lower level crisp 2^2 factorial model. Thus, in this approach, two crisp 2^2 factorial models are designated in terms of upper and lower levels. Finally, we analyse the lower and upper level models using crisp 2^2 factorial experiment technique. For lower level model, from α-cut intervals of tfns. we have, $[a_{ij} + \alpha(b_{ij} - a_{ij})]$ where i = 1, 2, 3, 4; j = 1, 2, …, r and for upper level model, $[d_{ij} - \alpha(d_{ij} - c_{ij})]$ where i = 1, 2, 3, 4; j = 1, 2, …, r. The required formulae of 2^2 factorial experiment for CRD and RBD are given below:

7.1. 2^2 Factorial design in CRD using alpha cut interval method.

Let $\tilde{x}_{ij} = [a_{ij} + \alpha(b_{ij} - a_{ij}), d_{ij} - \alpha(d_{ij} - c_{ij})]; 0 \le \alpha \le 1$, i = 1, 2, 3, 4; j = 1, 2, …, r; we split the above interval into two parts namely lower level data and upper level data viz. $\tilde{x}_{ij}^{L} = [a_{ij} + \alpha(b_{ij} - a_{ij})]$ and $\tilde{x}_{ij}^{U} = [d_{ij} - \alpha(d_{ij} - c_{ij})]$.

Setting hypotheses: Let 'k' be the level of significance now, the null hypothesis: $\tilde{H}_0 : \tilde{\mu}_1 = \tilde{\mu}_2 = \cdots = \tilde{\mu}_i$ against the alternative hypothesis: $\tilde{H}_A : \tilde{\mu}_1 \ne \tilde{\mu}_2 \ne \cdots \ne \tilde{\mu}_i$.

$\Rightarrow$ $\left[\tilde{H}_0\right] : \left[\tilde{\mu}_1\right] = \left[\tilde{\mu}_2\right] = \cdots = \left[\tilde{\mu}_i\right]$ against $\left[\tilde{H}_A\right] : \left[\tilde{\mu}_1\right] \ne \left[\tilde{\mu}_2\right] \ne \cdots \ne \left[\tilde{\mu}_i\right]$.

$\Rightarrow$ $\left[H_0^L, H_0^U\right] : \left[\mu_1^L, \mu_1^U\right] = \left[\mu_2^L, \mu_2^U\right] = \cdots = \left[\mu_i^L, \mu_i^U\right]$ against

$\left[H_A^L, H_A^U\right] : \left[\mu_1^L, \mu_1^U\right] \ne \left[\mu_2^L, \mu_2^U\right] \ne \cdots \ne \left[\mu_i^L, \mu_i^U\right]$

$\Rightarrow$ The following two set of hypotheses can be obtained.

(i) The null hypothesis $H_0^L : \mu_1^L = \mu_2^L = \cdots = \mu_i^L$ against the alternative hypothesis $H_A^L : \mu_1^L \ne \mu_2^L \ne \cdots \ne \mu_i^L$.

(ii) The null hypothesis $H_0^U : \mu_1^U = \mu_2^U = \cdots = \mu_i^U$ against the alternative hypothesis $H_A^U : \mu_1^U \ne \mu_2^U \ne \cdots \ne \mu_i^U$.

For lower level model (l.l.m.): Let $\tilde{x}_{ij}^{L} = [a_{ij} + \alpha(b_{ij} - a_{ij})]$ be the j^{th} observation of i^{th} treatment combination i = 1, 2, 3, 4; j = 1, 2, …, r (say). Then $\tilde{x}_{1*}^{L} = [1] = [a_{1j} + \alpha(b_{1j} - a_{1j})]$; $\tilde{x}_{2*}^{L} = [a] = [a_{2j} + \alpha(b_{2j} - a_{2j})]$; $\tilde{x}_{3*}^{L} = [b] = [a_{3j} + \alpha(b_{3j} - a_{3j})]$; $\tilde{x}_{4*}^{L} = [ab] = [a_{4j} + \alpha(b_{4j} - a_{4j})]$ where $\tilde{x}_{i*}^{L}$ = total of i^{th} treatment combination; Grand total $G_L = \sum_i \sum_j \tilde{x}_{ij}^{L}$; total number of observations n = 4r;

$$SS_T = \sum\sum\left(\tilde{x}_{ij}^{L}\right)^2 - (G_L^2 / 4r).$$

For upper level model (u.l.m.): Let $\tilde{x}_{ij}^{U} = [d_{ij} - \alpha(d_{ij} - c_{ij})]$ be the j^{th} observation of i^{th} treatment combination i = 1, 2, 3, 4; j = 1, 2, …, r (say). Then $\tilde{x}_{1*}^{U} = [1] = [d_{1j} - \alpha(d_{1j} - c_{1j})]$; $\tilde{x}_{2*}^{U} = [a] = [d_{2j} - \alpha(d_{2j} - c_{2j})]$; $\tilde{x}_{3*}^{U} = [b] = [d_{3j} - \alpha(d_{3j} - c_{3j})]$; $\tilde{x}_{4*}^{U} = [ab] = [d_{4j} - \alpha(d_{4j} - c_{4j})]$ where $\tilde{x}_{i*}^{U}$ = total of i^{th} treatment combination; Grand total $G_U = \sum_i \sum_j \tilde{x}_{ij}^{U}$; total number of observations n = 4r;

$$SS_T = \sum\sum\left(\tilde{x}_{ij}^{U}\right)^2 - (G_U^2 / 4r).$$

Decision Rules

Lower Level Model:

(i) If $F_A^L < F_t$ at 'k' level of significance with (1, 4(r-1)) degrees of freedom, then the null hypothesis H_0^L is accepted for certain value of $\alpha \in [0,1]$, otherwise the alternative hypothesis H_A^L is accepted.

(ii) If $F_B^L < F_t$ at 'k' level of significance with (1, 4(r-1)) degrees of freedom, then the null hypothesis H_0^L is accepted for certain value of $\alpha \in [0,1]$, otherwise the alternative hypothesis H_A^L is accepted.

(iii) If $F_{AB}^L < F_t$ at 'k' level of significance with (1, 4(r-1)) degrees of freedom, then the null hypothesis H_0^L is accepted for certain value of $\alpha \in [0,1]$, otherwise the alternative hypothesis H_A^L is accepted.

Upper Level Model:

(i) If $F_A^U < F_t$ at 'k' level of significance with (1, 4(r-1)) degrees of freedom, then the null hypothesis H_0^U is accepted for certain value of $\alpha \in [0,1]$, otherwise the alternative hypothesis H_A^U is accepted.

(ii) If $F_B^U < F_t$ at 'k' level of significance with (1, 4(r-1)) degrees of freedom, then the null hypothesis H_0^U is accepted for certain value of $\alpha \in [0,1]$, otherwise the alternative hypothesis H_A^U is accepted.

(iii) If $F_{AB}^U < F_t$ at 'k' level of significance with (1, 4(r-1)) degrees of freedom, then the null hypothesis H_0^U is accepted for certain value of $\alpha \in [0,1]$, otherwise the alternative hypothesis H_A^U is accepted.

8. 2^2 Factorial experiment in RBD using alpha cut interval method.

Let $\tilde{x}_{ij} = [a_{ij} + \alpha(b_{ij} - a_{ij}),\ d_{ij} - \alpha(d_{ij} - c_{ij})];\ 0 \le \alpha \le 1$, i = 1, 2, 3, 4; j = 1, 2, …, r; we split the above interval into two parts namely lower level data and upper level data viz. $\tilde{x}_{ij}^L = [a_{ij} + \alpha(b_{ij} - a_{ij})]$ and $\tilde{x}_{ij}^U = [d_{ij} - \alpha(d_{ij} - c_{ij})]$.

For lower level model (l.l.m.): Here, $\tilde{x}_{i*}^L = [a_{i*} + \alpha(b_{i*} - a_{i*})]$ = total of ith treat combination; $\tilde{x}_{*j}^L = [a_{*j} + \alpha(b_{*j} - a_{*j})]$ = total of jth block; i = 1, 2, 3, 4; j = 1, 2, …, r; Grand total $G_L = \sum_i \sum_j \tilde{x}_{ij}^L$; total no. of observations $n = 4r$; $SS_T = \sum\sum\left(\tilde{x}_{ij}^L\right)^2 - (G_L^2/4r)$; $SS_{Blocks} = (1/4)\sum_j\left(x_{*j}^L\right)^2 - (G_L^2/4r)$; $SS_{Treat.} = (1/4)\sum_j\left(x_{i*}^L\right)^2 - (G_L^2/4r)$; SS_A, SS_B and SS_{AB} can be obtained as in CRD. And $SS_E = SS_T - (SS_{Blocks} + SS_A + SS_B + SS_{AB})$.

For upper level model (u.l.m.): Here, $\tilde{x}^U_{i*} = [d_{i*}\text{-}\alpha(d_{i*}\text{-}c_{i*})]$ = total of i[th] treat combination; $\tilde{x}^U_{*j} = [d_{*j}\text{-}\alpha(d_{*j}\text{-}c_{*j})]$ = total of j[th] block; i = 1, 2, 3, 4; j = 1, 2, …, r; Grand total $G_U = \sum_i \sum_j \tilde{x}^U_{ij}$; total no. of observations $n = 4r$; $SS_T = \sum\sum\left(\tilde{x}^U_{ij}\right)^2 - (G_U^2 / 4r)$; $SS_{Blocks} = (1/4)\sum_j\left(x^U_{*j}\right)^2 - (G_U^2 / 4r)$; $SS_{Treat.} = (1/4)\sum_j\left(x^U_{i*}\right)^2 - (G_U^2 / 4r)$; SS_A, SS_B and SS_{AB} can be obtained as in CRD. And $SS_E = SS_T$ - ($SS_{Blocks} + SS_A + SS_B + SS_{AB}$).

Decision Rules

Lower Level Model:

(i) If $F^L_{Blocks} < F_t$ at 'k' level of significance with ((r-1), 3(r-1)) degrees of freedom, then the null hypothesis H^L_0 is accepted for certain value of $\alpha \in [0,1]$, otherwise the alternative hypothesis H^L_A is accepted.

(ii) If $F^L_{Treat.} < F_t$ at 'k' level of significance with ((r-1), 3(r-1)) degrees of freedom, then the null hypothesis H^L_0 is accepted for certain value of $\alpha \in [0,1]$, otherwise the alternative hypothesis H^L_A is accepted.

(iii) If $F^L_A < F_t$ at 'k' level of significance with (1, 3(r-1)) degrees of freedom, then the null hypothesis H^L_0 is accepted for certain value of $\alpha \in [0,1]$, otherwise the alternative hypothesis H^L_A is accepted.

(iv) If $F^L_B < F_t$ at 'k' level of significance with (1, 3(r-1)) degrees of freedom, then the null hypothesis H^L_0 is accepted for certain value of $\alpha \in [0,1]$, otherwise the alternative hypothesis H^L_A is accepted.

(v) If $F^L_{AB} < F_t$ at 'k' level of significance with (1, 3(r-1)) degrees of freedom, then the null hypothesis H^L_0 is accepted for certain value of $\alpha \in [0,1]$, otherwise the alternative hypothesis H^L_A is accepted.

Upper Level Model:

(i) If $F^U_{Blocks} < F_t$ at 'k' level of significance with ((r-1), 3(r-1)) degrees of freedom, then the null hypothesis H^U_0 is accepted for certain value of $\alpha \in [0,1]$, otherwise the alternative hypothesis H^U_A is accepted.

(ii) If $F^U_{Treat.} < F_t$ at 'k' level of significance with ((r-1), 3(r-1)) degrees of freedom, then the null hypothesis H^U_0 is accepted for certain value of $\alpha \in [0,1]$, otherwise the alternative hypothesis H^U_A is accepted.

(iii) If $F^U_A < F_t$ at 'k' level of significance with (1, 3(r-1)) degrees of freedom, then the null hypothesis H^U_0 is accepted for certain value of $\alpha \in [0,1]$, otherwise the alternative hypothesis H^U_A is accepted.

(iv) If $F_B^U < F_t$ at 'k' level of significance with (1, 3(r-1)) degrees of freedom, then the null hypothesis H_0^U is accepted for certain value of $\alpha \in [0,1]$, otherwise the alternative hypothesis H_A^U is accepted.

(v) If $F_{AB}^U < F_t$ at 'k' level of significance with (1, 3(r-1)) degrees of freedom, then the null hypothesis H_0^U is accepted for certain value of $\alpha \in [0,1]$, otherwise the alternative hypothesis H_A^U is accepted.

Conclusion table:

Acceptance of null hypotheses $\widetilde{H}_0$		
Lower Level Model	**Upper Level Model**	**Conclusion**
If H_0 is accepted for all α $\alpha \in [0,1]$	and H_0 is accepted for all α $\alpha \in [0,1]$	then $\widetilde{H}_0$ is accepted for all α $\alpha \in [0,1]$
If H_0 is accepted for all α $\alpha \in [0,1]$	and H_0 is rejected for all α $\alpha \in [0,1]$	then $\widetilde{H}_0$ is rejected for all α $\alpha \in [0,1]$
If H_0 is rejected for all α $\alpha \in [0,1]$	and H_0 is accepted for all α $\alpha \in [0,1]$	then $\widetilde{H}_0$ is rejected for all α $\alpha \in [0,1]$
If H_0 is rejected for all α $\alpha \in [0,1]$	and H_0 is rejected for all α $\alpha \in [0,1]$	then $\widetilde{H}_0$ is rejected for all α $\alpha \in [0,1]$

Partial acceptance of null hypothesis H_0 at the intersection of certain level of α at both upper level and lower level models can be taken into account for the acceptance of the null hypothesis $\widetilde{H}_0$ **.**

Example-1

The following table gives the plan and yield of a 2^2-factorial experiment conducted in ***CRD***. The observed data are unavoidably trapezoidal fuzzy numbers due to some work fluctuations.

$^{(1)}(17, 19, 20, 23)$	$^{\tilde{a}}(24, 25, 26, 28)$	$^{\tilde{a}}(20, 23, 24, 25)$	$^{\tilde{b}}(7, 10, 11, 13)$
$^{\widetilde{ab}}(20, 23, 24, 26)$	$^{\tilde{b}}(7, 8, 11, 13)$	$^{\widetilde{ab}}(20, 22, 25, 27)$	$^{(1)}(14, 16, 17, 20)$
$^{\tilde{a}}(20, 21, 24, 26)$	$^{\tilde{b}}(12, 14, 15, 17)$	$^{\widetilde{ab}}(17, 19, 21, 22)$	$^{(1)}(15, 18, 19, 21)$

We now analyse the 2^2-factorial design for the above tfns.

Null hypothesis $\widetilde{H}_0$: The difference between the main effect A and B is not significant.

Example-2

An experiment was planned to study the effect of sulphates of potash and super phosphate on the yield of potatoes. All the combinations of 2 levels of super phosphate [0 cent (b_0) and 5 cent (b_1) per acre] and two levels of sulphate of potash [0 cent (a_0) and 5 cent (a_1) per acre] were studied in a ***randomized block design*** with 4 replications for each. But unexpectedly the observed data are in terms of trapezoidal fuzzy numbers due to some work congestion and they are tabulated below (lb: per plot = 1/70 acre).

$^{(1)}(20,\ 23, 25, 26)$	$^{\tilde{a}}(24,\ 25, 27, 29)$	$^{\tilde{b}}(20,\ 22, 24, 25)$	$^{\widetilde{ab}}(34,\ 37, 38, 40)$
$^{\tilde{b}}(37,\ 40, 42, 44)$	$^{(1)}(22,\ 23, 25, 27)$	$^{\tilde{a}}(30,\ 33, 34, 36)$	$^{\widetilde{ab}}(36,\ 38, 41, 42)$
$^{(1)}(26,\ 29, 30, 32)$	$^{\tilde{a}}(19,\ 20, 22, 24)$	$^{\widetilde{ab}}(28,\ 30, 33, 34)$	$^{\tilde{b}}(19,\ 20, 21, 24)$
$^{\widetilde{ab}}(33,\ 34, 36, 39)$	$^{\tilde{a}}(29,\ 31, 33, 35)$	$^{\tilde{b}}(21,\ 24, 25, 26)$	$^{(1)}(25,\ 28, 30, 32)$

We now analyse the observed data for conclusion.

Null hypothesis $\widetilde{H}_0$: The data is homogeneous with respect to the blocks and treatments.

2^2-Factorial design using alpha cut interval method

Example 8.1. Let us consider example-1, the interval form of given tfns. using α-cut method is given below:

$^{(1)}(17+2\alpha,\ 23\text{-}3\alpha)$	$^{\tilde{a}}(24+\alpha,\ 28\text{-}2\alpha)$	$^{\tilde{a}}(20+3\alpha,\ 25\text{-}\alpha)$	$^{\tilde{b}}(7+3\alpha,\ 13\text{-}2\alpha)$
$^{\widetilde{ab}}(20+3\alpha,\ 26\text{-}2\alpha)$	$^{\tilde{b}}(7+\alpha,\ 13\text{-}2\alpha)$	$^{\widetilde{ab}}(20+2\alpha,\ 27\text{-}2\alpha)$	$^{(1)}(14+2\alpha,\ 20\text{-}3\alpha)$
$^{\tilde{a}}(20+\alpha,\ 26\text{-}2\alpha)$	$^{\tilde{b}}(12+2\alpha,\ 17\text{-}2\alpha)$	$^{\widetilde{ab}}(17+2\alpha,\ 22\text{-}\alpha)$	$^{(1)}(15+3\alpha,\ 21\text{-}2\alpha)$

H_0^L, H_0^U : The difference between the main effect A and B is not significant.

The upper and lower level data [16, 17] can be tabulatedseparately as per the description in section-7. **Here we have noted only calculated results by omitting repeated tables and surplus explanations.**

For lower level model (l.l.m.) : $SS_A = (\alpha^2\text{-}98\alpha+2401)/12$ **;** $SS_B = (\alpha^2\text{-}54\alpha+729)/12$ **;** $SS_{AB} = (9\alpha^2+78\alpha+169)/12$ **;** $SS_T = (83\alpha^2\text{-}122\alpha+3755)/12$ **;** $SS_E = (6\alpha^2\text{-}4\alpha+38)$ **; Main effect A :** $F_A^L = 2(\alpha^2\text{-}98\alpha+2401)/3(6\alpha^2\text{-}4\alpha+38),\ 0 \le \alpha \le 1$. Here $F_{t(1\%)}(1,8) = 11.26$ and $F_A^L > F_{t(1\%)} \forall\alpha,\ 0 \le \alpha \le 1$. **Main effect B :** $F_B^L = 2(\alpha^2\text{-}98\alpha+2401)/3(6\alpha^2\text{-}4\alpha+38),\ 0 \le \alpha \le 1$. Here $F_{t(1\%)}(1,8) = 11.26$ and $F_B^L > F_{t(1\%)} \forall\alpha,\ 0 \le \alpha \le 1$. **Main effect AB :** $F_{AB}^L = 2(9\alpha^2+78\alpha+169)/3(6\alpha^2\text{-}4\alpha+38),\ 0 \le \alpha \le 1$. Here $F_{t(1\%)}(1,8) = 11.26$ and $F_{AB}^L < F_{t(1\%)} \forall\alpha,\ 0 \le \alpha \le 1$.

For upper level model (u.l.m.) : $SS_A = (16\alpha^2+376\alpha+2209)/12$ **;** $SS_B = (4\alpha^2\text{-}100\alpha+625)/12$ **;** $SS_{AB} = (4\alpha^2\text{-}68\alpha+289)/12$ **;** $SS_T = (16\alpha^2+32\alpha+1177)/4$ **;** $SS_E = (6\alpha^2\text{-}28\alpha+102)/3$ **; Main effect A:** $F_A^U = 2(16\alpha^2+376\alpha+2209)/6\alpha^2\text{-}28\alpha+102,\ 0 \le \alpha \le 1$. Here $F_{t(1\%)}(1,8) = 11.26$ and $F_A^U > F_{t(1\%)} \forall\alpha,\ 0 \le \alpha \le 1$. **Main effect B :** $F_B^U = 2(4\alpha^2\text{-}100\alpha+625)/6\alpha^2\text{-}28\alpha+102,\ 0 \le \alpha \le 1$. Here $F_{t(1\%)}(1,8) = 11.26$ and $F_B^U > F_{t(1\%)} \forall\alpha,\ 0 \le \alpha \le 1$. **Main effect AB :** $F_{AB}^U = 2(4\alpha^2\text{-}68\alpha+289)/6\alpha^2\text{-}28\alpha+102,\ 0 \le \alpha \le 1$. Here $F_{t(1\%)}(1,8) = 11.26$ and $F_{AB}^U < F_{t(1\%)} \forall\alpha,\ 0 \le \alpha \le 1$.

Conclusion 8.1. : From the decision obtained from both l.l.m. and u.l.m., the calculated value of F > tabulated value of F for the main effects A and B. **Hence, we conclude that the main effects A and B both are significantly different at 1% level of significant $\forall\alpha,\ 0 \le \alpha \le 1$.**

Example 8.2. Let us consider example-2, the interval form of given tfns. using α-cut method is given below:

$^{(1)}(20+3\alpha, 26-\alpha)$	$^{\tilde{a}}(24+\alpha, 29-2\alpha)$	$^{\tilde{b}}(20+2\alpha, 25-\alpha)$	$^{\widetilde{ab}}(34+3\alpha, 40-2\alpha)$
$^{\tilde{b}}(37+3\alpha, 44-2\alpha)$	$^{(1)}(22+\alpha, 27-2\alpha)$	$^{\tilde{a}}(30+3\alpha, 36-2\alpha)$	$^{\widetilde{ab}}(36+2\alpha, 42-\alpha)$
$^{(1)}(26+3\alpha, 32-2\alpha)$	$^{\tilde{a}}(19+\alpha, 24-2\alpha)$	$^{\widetilde{ab}}(28+2\alpha, 34-\alpha)$	$^{\tilde{b}}(19+\alpha, 24-3\alpha)$
$^{\widetilde{ab}}(33+\alpha, 39-3\alpha)$	$^{\tilde{a}}(29+2\alpha, 35-2\alpha)$	$^{\tilde{b}}(21+3\alpha, 26-\alpha)$	$^{(1)}(25+3\alpha, 32-2\alpha)$

H_0^L , H_0^U : The data is homogeneous with respect to the blocks and treatments.

For lower level model (l.l.m.) : $SS_T=(188\alpha^2+740\alpha+9215)/16$ **;** $SS_{Blocks}=(12\alpha^2+220\alpha+2499)/16$ **;** $SS_{Treat.}=(20\alpha^2-244\alpha+3563)/16$ **;** $SS_A=(16\alpha^2-344\alpha+1849)/16$ **;** $SS_B=33^2/16$ **;** $SS_{AB}=(4\alpha^2+100\alpha+625)/16$ **;** $SS_E=(156\alpha^2+764\alpha+3153)/16$ **.**

Between Blocks : $F^L_{Blocks}=3(12\alpha^2+220\alpha+2499)/156\alpha^2+764\alpha+3153;\ 0\le\alpha\le 1$**.** Here, $F_{t(5\%)}(3,9)=3.86$ and $F^L_{Blocks}<F_{t(5\%)}\forall\alpha,\ 0\le\alpha\le 1.$

Between Treatments : $F^L_{Treat.}=3(20\alpha^2-244\alpha+3563)/156\alpha^2+764\alpha+3153;\ \ 0\le\alpha\le 1$**.** Here, $F_{t(5\%)}(3,9)=3.86$ and $F^L_{Treat.}<F_{t(5\%)}\forall\alpha,\ 0\le\alpha\le 1.$

Main effect A : $F^L_A=9(16\alpha^2-344\alpha+1849)/156\alpha^2+764\alpha+3153,\ 0\le\alpha\le 1$ Here $F_{t(5\%)}(1,9)=5.12$ and $F^L_A<F_{t(5\%)}\forall\alpha,\ 0\le\alpha\le 1.$

Main effect B : $F^L_B=9801/156\alpha^2+764\alpha+3153,\ 0\le\alpha\le 1$. Here $F_{t(5\%)}(1,9)=5.12$ and $F^L_B<F_{t(5\%)}\forall\alpha,\ 0\le\alpha\le 1.$

Interaction effect AB : $F^L_{AB}=9(4\alpha^2+100\alpha+625)/156\alpha^2+764\alpha+3153,\ 0\le\alpha\le 1.$ Here, $F_{t(5\%)}(1,9)=5.12$ and $F^L_{AB}<F_{t(5\%)}\forall\alpha,\ 0\le\alpha\le 1.$

For upper level model (u.l.m.) : $SS_T=(103\alpha^2-210\alpha+10695)/16$ **;** $SS_{Blocks}=(11\alpha^2+22\alpha+2859)/16$ **;** $SS_{Treat.}=(3\alpha^2+38\alpha+3779)/16$ **;** $SS_A=(\alpha^2-86\alpha+1849)/16$ **;** $SS_B=(\alpha^2+66\alpha+1089)/16$ **;** $SS_{AB}=(\alpha^2+58\alpha+841)/16$ **;** $SS_E=(89\alpha^2-270\alpha+4057)/16$ **.**

Between Blocks : $F^U_{Blocks}=3(11\alpha^2+22\alpha+2859)/89\alpha^2-270\alpha+4057;\ 0\le\alpha\le 1$**.** Here, $F_{t(5\%)}(3,9)=3.86$ and $F^U_{Blocks}<F_{t(5\%)}\forall\alpha,\ 0\le\alpha\le 1.$

Between Treatments : $F^U_{Treat.}=9(3\alpha^2+38\alpha+3779)/89\alpha^2-270\alpha+4057;\ \ 0\le\alpha\le 1$**.** Here, $F_{t(5\%)}(3,9)=3.86$ and $F^U_{Treat.}>F_{t(5\%)}\forall\alpha,\ 0\le\alpha\le 1.$

Main effect A : $F^U_A=9(\alpha^2-86\alpha+1849)/89\alpha^2-270\alpha+4057,\ 0\le\alpha\le 1.$ Here $F_{t(5\%)}(1,9)=5.12$ and $F^U_A<F_{t(5\%)}\forall\alpha,\ 0\le\alpha\le 1.$

Main effect B : $F^U_B=9(\alpha^2+66\alpha+1089)/89\alpha^2-270\alpha+4057,\ 0\le\alpha\le 1$. Here $F_{t(5\%)}(1,9)=5.12$ and $F^U_B<F_{t(5\%)}\forall\alpha,\ 0\le\alpha\le 1.$

Interaction effect AB : $F^U_{AB}=9(\alpha^2+58\alpha+841)/89\alpha^2-270\alpha+4057,\ 0\le\alpha\le 1.$ Here, $F_{t(5\%)}(1,9)=5.12$ and $F^U_{AB}<F_{t(5\%)}\forall\alpha,\ 0\le\alpha\le 1.$

Conclusion 8.2. : Here $F_{Blocks} < F_{t(5\%)} \forall\alpha$ at both l.l.m. and u.l.m. $\Rightarrow$ The blocks do not differ significantly. Also $F_A < F_t$ for the partial level of $\alpha \in [0.1,\ 1]$ at both l.l.m. and u.l.m. $\Rightarrow$ The main effect A does not differ significantly at $0.1 \le \alpha \le 1$. And $F_B < F_t,\ F_{AB} < F_t,\ \forall\alpha \Rightarrow$ The effect B and the interaction effect AB do not differ significantly. Moreover, $F_{Treat.} < F_t,\ \forall\alpha$ at l.l.m. on the other hand $F_{Treat.} > F_t,\ \forall\alpha$ at u.l.m. $\Rightarrow$ **The null hypothesis $\widetilde{H}_0$ is rejected in this case. Therefore, the treatment differs significantly for all α.**

Now, we provide here another new technique in 2^2-factorial design for a comparative study when the test is performed under fuzzy environments. More generally, the CRD, RBD and LSD are independent of origin which implies that the arithmetic operations such as addition/subtraction/multiplication or division by non-zero quantity can be performed among the observed data uniformly for all entries in order to simplify the large numerical calculations while the observed data are numerically large. This indicates that ANOVA test stands on the magnitude ratio among each data of the sample observations. Another idea in this paper is, when the test is conducted using natural and vague observations such as fuzzy numbers for instance, we may use ranking grades for all observed fuzzy numbers by using unique method without damaging the magnitude ratios among the fuzzy samples. In fact, the ranking grades of all fuzzy numbers using fuzzy analytic method are crisp in nature and we perform the hypotheses test as usual and better decisions can be obtained.

9. Wang's centroid point and ranking method

Wang et al. [26] found that the centroid formulae proposed by Cheng are incorrect and have led to some misapplications such as by Chu and Tsao. They presented the correct method for centroid formulae for a generalized fuzzy number $\widetilde{A} = (a,\ b,\ c,\ d;\ w)$ as

$$\left(\overline{x}_0,\ \overline{y}_0\right) = \left[\frac{1}{3}\left((a+b+c+d) - \left(\frac{dc - ab}{(d+c)-(a+b)}\right)\right), \left(\frac{w}{3}\right)\left(1 + \left(\frac{c - b}{(d+c)-(a+b)}\right)\right)\right] --- (9.1)$$

And the ranking function associated with $\widetilde{A}$ is $\quad R\left(\widetilde{A}\right) = \sqrt{\overline{x}_0^2 + \overline{y}_0^2} --- (9.2)$

For a normalized tfn, we put w = 1 in equations (9.1) so we have,

$$\left(\overline{x}_0,\ \overline{y}_0\right) = \left[\frac{1}{3}\left((a+b+c+d) - \left(\frac{dc - ab}{(d+c)-(a+b)}\right)\right), \left(\frac{1}{3}\right)\left(1 + \left(\frac{c - b}{(d+c)-(a+b)}\right)\right)\right] --- (9.3)$$

And the ranking function associated with $\widetilde{A}$ is $\quad R\left(\widetilde{A}\right) = \sqrt{\overline{x}_0^2 + \overline{y}_0^2} --- (9.4)$.

Let $\widetilde{A}_i$ and $\widetilde{A}_j$ be two fuzzy numbers, (i) $R\left(\widetilde{A}_i\right) > R\left(\widetilde{A}_j\right)$ then $\widetilde{A}_i > \widetilde{A}_j$ (ii) $R\left(\widetilde{A}_i\right) > R\left(\widetilde{A}_j\right)$ then $\widetilde{A}_i > \widetilde{A}_j$ and (iii) $R\left(\widetilde{A}_i\right) = R\left(\widetilde{A}_j\right)$ then $\widetilde{A}_i = \widetilde{A}_j$.

Example 9.1. Let we consider example 1, using the above relations (9.3) and (9.4), we obtain the ranks of tfns. which are tabulated below:

$^{(1)}19.8132$	$^{\tilde{a}}25.8031$	$^{\tilde{a}}22.8922$	$^{\tilde{b}}10.1976$
$^{\widetilde{ab}}23.1936$	$^{\tilde{b}}9.7879$	$^{\widetilde{ab}}23.5040$	$^{(1)}16.8138$
$^{\tilde{a}}22.7821$	$^{\tilde{b}}14.5052$	$^{\widetilde{ab}}19.7189$	$^{(1)}18.1945$

The ANOVA table values using Wang's rank of tfns. :

$SS_A = 196.682$**;** $SS_B = 53.7282$**;** $SS_{AB} = 19.4308$**;** $SS_T = 302.71$**;** $SS_E = 32.8691$**; Main effect A**: $F_A = 47.8708$ and $F_{t(1\%)}(1,8) = 11.26$, here $F_A > F_{t(1\%)}$. **Main effect B**: $F_B = 13.0770$ and $F_{t(1\%)}(1,8) = 11.26$, here $F_B > F_{t(1\%)}$. **Interaction effect AB**: $F_{AB} = 4.7293$ and $F_{t(1\%)}(1,8) = 11.26$, here $F_{AB} < F_{t(1\%)}$.

Conclusion 9.1. : Here, the calculated value of F > tabulated value of F for the main effects A and B. **Hence, we conclude that the main effects A and B both are significantly different at 1% level of significant.**

Example 9.2. Let we consider example 2, using the above relations (9.3) and (9.4), we obtain the ranks of tfns. which are tabulated below:

$^{(1)}23.4204$	$^{\tilde{a}}26.2892$	$^{\tilde{b}}22.7183$	$^{\widetilde{ab}}37.1924$
$^{\tilde{b}}40.7057$	$^{(1)}24.2895$	$^{\tilde{a}}33.1927$	$^{\widetilde{ab}}39.2247$
$^{(1)}29.1930$	$^{\tilde{a}}21.2900$	$^{\widetilde{ab}}31.2254$	$^{\tilde{b}}21.1147$
$^{\widetilde{ab}}35.5858$	$^{\tilde{a}}32.0027$	$^{\tilde{b}}23.8921$	$^{(1)}28.7066$

The ANOVA table values using Wang's rank of tfns. :

$SS_A = 110.054$**;** $SS_B = 69.2016$**;** $SS_{AB} = 47.7218$**;** $SS_T = 629.29$**;** $SS_{Blocks} = 170.31$**;** $SS_{Treat.} = 226.98$; $SS_E = 232$. **Between Blocks :** $F_{Blocks} = 2.2023$ and $F_{t(5\%)}(3,9) = 3.86$ here, $F_{Blocks} < F_{t(5\%)}$. **Between Treatments :** $F_{Treat.} = 2.9351$ and $F_{t(5\%)}(3,9) = 3.86$ here, $F_{Treaat.} < F_{t(5\%)}$. **Main effect A** : $F_A = 4.2693$; and $F_{t(5\%)}(1,9) = 5.12$ here, $F_A < F_{t(5\%)}$. **Main effect B** : $F_B = 2.6845$ and $F_{t(5\%)}(1,9) = 5.12$ here, $F_B < F_{t(5\%)}$. **Interaction effect AB** : $F_{AB} = 1.8513$ and $F_{t(5\%)}(1,9) = 5.12$ here, $F_{AB} < F_{t(5\%)}$.

Conclusion 9.2. : In each of the cases, the computed value of F is less than the tabulated value of F at 5% level of significance. Therefore, there are no significant main or interaction effects present in the experiment. The blocks as well as treatments do not differ significantly. Since the blocks do not differ significantly, we conclude that there is no special contribution from fluctuations in soil fertility and thus the division of the whole experimental area into blocks does not result in any gain in accuracy.

10. Rezvani's ranking function of tfns.

The centroid of a trapezoid is considered as the balancing point of the trapezoid. Divide the trapezoid into three plane figures. These three plane figures are a triangle (APB), a rectangle (BPQC) and a triangle (CQD) respectively. Let the centroids of the three plane figures be G_1, G_2 and G_3 respectively. The incenter of these centroids G_1, G_2 and G_3 is taken as the point of reference to define the ranking of generalized trapezoidal fuzzy numbers. The reason for selecting this point as a point of reference is that each centroid point are ***balancing points*** of each individual plane figure and the incenter of these centroid points is much more balancing point for a generalized trapezoidal fuzzy number. Therefore, this point would be a better reference point than the centroid point of the trapezoid.

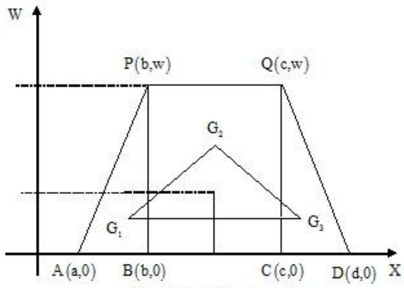


Fig. 2 Centroid of centroids

Consider a generalized trapezoidal fuzzy number $\widetilde{A}=(a, b, c, d; w)$. The centroids of the three plane figures are:

$$G_1=\left(\frac{a+2b}{3},\frac{w}{3}\right),\ G_2=\left(\frac{b+c}{2},\frac{w}{2}\right) \text{ and } G_3=\left(\frac{2c+d}{3},\frac{w}{3}\right)---(10.1)$$

Equation of the line G_1G_3 is $y=\frac{w}{3}$ and G_2 does not lie on the line G_1G_3. Therefore, G_1, G_2 and G_3 are non-collinear and they form a triangle. We define the incenter $I\left(\bar{x}_0, \bar{y}_0\right)$ of the triangle with vertices G_1, G_2 and G_3 of the generalized fuzzy number $\widetilde{A}=(a, b, c, d; w)$ as [21]

$$I_{\widetilde{A}}\left(\bar{x}_0, \bar{y}_0\right)=\left[\frac{\alpha\left(\frac{a+2b}{3}\right)+\beta\left(\frac{b+c}{2}\right)+\gamma\left(\frac{2c+d}{3}\right)}{\alpha+\beta+\gamma},\ \frac{\alpha\left(\frac{w}{3}\right)+\beta\left(\frac{w}{2}\right)+\gamma\left(\frac{w}{3}\right)}{\alpha+\beta+\gamma}\right]---(10.2)$$

$$\text{where } \alpha=\frac{\sqrt{(c-3b+2d)^2+w^2}}{6},\beta=\frac{\sqrt{(2c+d-a-2b)^2}}{3},\gamma=\frac{\sqrt{(3c-2a-b)^2+w^2}}{6}$$

And ranking function of the trapezoidal fuzzy number $\widetilde{A}=(a, b, c, d; w)$ which maps the set of all fuzzy numbers to a set of all real numbers $\left[\text{i.e. } R:\left[\widetilde{A}\right]\to\mathbb{R}\right]$ is defined as $R\left(\widetilde{A}\right)=\sqrt{\bar{x}_0^2+\bar{y}_0^2}---(10.3)$ which is the Euclidean distance from the incenter of the centroids.

For a normalized tfn, we put w = 1 in equations (1), (2) and (3) so we have,

$$G_1=\left(\frac{a+2b}{3},\frac{1}{3}\right),\ G_2=\left(\frac{b+c}{2},\frac{1}{2}\right) \text{ and } G_3=\left(\frac{2c+d}{3},\frac{1}{3}\right)---(10.4)$$

$$I_{\widetilde{A}}\left(\bar{x}_0, \bar{y}_0\right)=\left[\frac{\alpha\left(\frac{a+2b}{3}\right)+\beta\left(\frac{b+c}{2}\right)+\gamma\left(\frac{2c+d}{3}\right)}{\alpha+\beta+\gamma},\ \frac{\alpha\left(\frac{1}{3}\right)+\beta\left(\frac{1}{2}\right)+\gamma\left(\frac{1}{3}\right)}{\alpha+\beta+\gamma}\right]---(10.5)$$

$$\text{where } \alpha=\frac{\sqrt{(c-3b+2d)^2+1}}{6},\beta=\frac{\sqrt{(2c+d-a-2b)^2}}{3} \text{ and } \gamma=\frac{\sqrt{(3c-2a-b)^2+1}}{6}$$

And ranking function of the trapezoidal fuzzy number $\widetilde{A}=(a, b, c, d; 1)$ is defined as $R\left(\widetilde{A}\right)=\sqrt{\bar{x}_0^2+\bar{y}_0^2}---(10.6)$.

2^2-Factorial experiment using Rezvani's ranking function

We now analyse the 2^2-factorial experiment by assigning rank for each normalized trapezoidal fuzzy numbers and based on the ranking grades, the decisions are observed.

Example 10.1. Let we consider example 1, using the above relations (10.4), (10.5) and (10.6), we obtain the ranks of tfns. which are tabulated below:

$^{(1)}19.5058$	$^{\tilde{a}}25.5057$	$^{\tilde{a}}23.5$	$^{\tilde{b}}10.5069$
$^{\widetilde{ab}}23.5024$	$^{\tilde{b}}9.5097$	$^{\widetilde{ab}}23.5037$	$^{(1)}16.5066$
$^{\tilde{a}}22.5044$	$^{\tilde{b}}14.506$	$^{\widetilde{ab}}20.0033$	$^{(1)}18.5034$

The ANOVA table values using Rezvani's rank of tfns. :

$SS_A = 204.032$; $SS_B = 49.9959$; $SS_{AB} = 20.0015$; $SS_T = 305.514$; $SS_E = 31.4849$; **Main effect A** : $F_A = 51.8427$ and $F_{t(1\%)}(1,8) = 11.26$, here $F_A > F_{t(1\%)}$. **Main effect B** : $F_B = 12.7035$ and $F_{t(1\%)}(1,8) = 11.26$, here $F_B > F_{t(1\%)}$. **Interaction effect AB** : $F_{AB} = 5.0822$ and $F_{t(1\%)}(1,8) = 11.26$, here $F_{AB} < F_{t(1\%)}$.

Conclusion 10.1. : Here, the calculated value of F > tabulated value of F for the main effects A and B. **Hence, we conclude that the main effects A and B both are significantly different at 1% level of significant.**

Example 10.2. Let we consider example 2, using the above relations (10.4), (10.5) and (10.6), we obtain the ranks of tfns. which are tabulated below:

$^{(1)}24.0019$	$^{\tilde{a}}26.0044$	$^{\tilde{b}}23.0027$	$^{\widetilde{ab}}37.501$
$^{\tilde{b}}41.0014$	$^{(1)}24.0046$	$^{\tilde{a}}33.5013$	$^{\widetilde{ab}}39.5016$
$^{(1)}29.5016$	$^{\tilde{a}}21.0052$	$^{\widetilde{ab}}31.5022$	$^{\tilde{b}}20.5079$
$^{\widetilde{ab}}35.0042$	$^{\tilde{a}}32.0027$	$^{\tilde{b}}24.4999$	$^{(1)}29.0023$

The ANOVA table values using Rezvani's rank of tfns. :

$SS_A = 102.517$; $SS_B = 70.1276$; $SS_{AB} = 50.7439$; $SS_T = 646.03$; $SS_{Blocks} = 175.61$; $SS_{Treat.} = 223.39$; $SS_E = 247.03$. **Between Blocks :** $F_{Blocks} = 2.1327$ and $F_{t(5\%)}(3,9) = 3.86$ here, $F_{Blocks} < F_{t(5\%)}$. **Between Treatments :** $F_{Treat.} = 2.7129$ and $F_{t(5\%)}(3,9) = 3.86$ here, $F_{Treaat.} < F_{t(5\%)}$. **Main effect A** : $F_A = 3.7350$; and $F_{t(5\%)}(1,9) = 5.12$ here, $F_A < F_{t(5\%)}$. **Main effect B** : $F_B = 2.5549$ and $F_{t(5\%)}(1,9) = 5.12$ here, $F_B < F_{t(5\%)}$.**Interaction effect AB** : $F_{AB} = 1.8487$ and $F_{t(5\%)}(1,9) = 5.12$ here, $F_{AB} < F_{t(5\%)}$.

Conclusion 10.2. : In each of the cases, the computed value of F is less than the tabulated value of F at 5% level of significance. Therefore, there are no significant main or interaction effects present in the experiment. The blocks as well as treatments do not differ significantly. Since the blocks do not differ significantly, we conclude that there is no special contribution from fluctuations in soil fertility and thus the division of the whole experimental area into blocks does not result in any gain in accuracy.

11. Graded mean integration representation (GMIR)

Let $\widetilde{A}=(a, b, c, d; w)$ be a generalized trapezoidal fuzzy number, then the **GMIR** [20] of $\widetilde{A}$ is defined by $P(\widetilde{A}) = \int_0^w h\left[\frac{L^{-1}(h)+R^{-1}(h)}{2}\right]dh \Big/ \int_0^w hdh$.

Theorem 11.1. Let $\widetilde{A}=(a, b, c, d; 1)$ be a tfn. with normal shape function, where a, b, c, d are real numbers such that $a < b \le c < d$. Then the graded mean integration representation (GMIR) of $\widetilde{A}$ is $P(\widetilde{A}) = \frac{(a+d)}{2} + \frac{n}{2n+1}(b - a - d + c)$.

Proof : For a trapezoidal fuzzy number $\widetilde{A}=(a, b, c, d; 1)_n$, we have $L(x) = \left(\frac{x-a}{b-a}\right)^n$ and $R(x) = \left(\frac{d-x}{d-c}\right)^n$ Then,

$$h = \left(\frac{x-a}{b-a}\right)^n \Rightarrow L^{-1}(h) = a + (b-a)h^{1/n};\ h = \left(\frac{d-x}{d-c}\right)^n \Rightarrow R^{-1}(h) = d - (d-c)h^{1/n}$$

$$\therefore P(\widetilde{A}) = \left(\frac{1}{2}\int_0^1 h\left[\left(a + (b-a)h^{1/n}\right) + \left(d - (d-c)h^{1/n}\right)\right]dh\right) \Big/ \int_0^1 hdh$$

$$= \left(\frac{1}{2}\left[\frac{(a+d)}{2} + \frac{n}{2n+1}(b - a - d + c)\right]\right) \Big/ \left(1/2\right)$$

Thus, $P(\widetilde{A}) = \frac{(a+d)}{2} + \frac{n}{2n+1}(b - a - d + c)$ Hence the proof.

Result 11.1. If n =1 in the above theorem, we have $P(\widetilde{A}) = \frac{a + 2b + 2c + d}{6}$

2^2-Factorial experiment using GMIR of tfns.

Example 11.1. Let we consider example 1, using the result-11.1 from above theorem-11.1, we get the GMIR of each tfns. which are tabulated below:

$^{(1)}$19.6667	$^{\tilde{a}}$25.6667	$^{\tilde{a}}$23.1667	$^{\tilde{b}}$10.3333
$^{\widetilde{ab}}$23.3333	$^{\tilde{b}}$9.6667	$^{\widetilde{ab}}$23.5	$^{(1)}$16.6667
$^{\tilde{a}}$22.6667	$^{\tilde{b}}$14.5	$^{\widetilde{ab}}$19.8333	$^{(1)}$18.3333

The ANOVA table values using GMIR of tfns. :

SS_A =200.083; SS_B =52.0842; SS_{AB} =19.5923; SS_T =303.741; SS_E =31.9818; **Main effect A** : $F_A = 50.0495$ and $F_{t(1\%)}(1,8) = 11.26$, here $F_A > F_{t(1\%)}$. **Main effect B** : $F_B = 13.0285$ and $F_{t(1\%)}(1,8) = 11.26$, here $F_B > F_{t(1\%)}$. **Interaction effect AB** : $F_{AB} = 4.9009$ and $F_{t(1\%)}(1,8) = 11.26$, here $F_{AB} < F_{t(1\%)}$.

Conclusion 11.1. : Here, the calculated value of F > tabulated value of F for the main effects A and B. **Hence, we conclude that the main effects A and B both are significantly different at 1% level of significant.**

Example 11.2. Let we consider example 2, using the result-11.1 from above theorem-11.1, we get the GMIR of each tfns. which are tabulated below:

$^{(1)}23.6667$	$^{\tilde{a}}26.1667$	$^{\tilde{b}}22.8333$	$^{\widetilde{ab}}37.3333$
$^{\tilde{b}}40.8333$	$^{(1)}24.1667$	$^{\tilde{a}}33.3333$	$^{\widetilde{ab}}39.3333$
$^{(1)}29.3333$	$^{\tilde{a}}21.1667$	$^{\widetilde{ab}}31.3333$	$^{\tilde{b}}20.8333$
$^{\widetilde{ab}}35.3333$	$^{\tilde{a}}32$	$^{\tilde{b}}24.1667$	$^{(1)}28.8333$

The ANOVA table values using GMIR of tfns. :

$SS_A = 106.778$; $SS_B = 69.4435$; $SS_{AB} = 48.9997$; $SS_T = 636.22$; $SS_{Blocks} = 172.72$; $SS_{Treat.} = 225.22$; $SS_E = 238.28$. **Between Blocks :** $F_{Blocks} = 2.1746$ and $F_{t(5\%)}(3,9) = 3.86$ here, $F_{Blocks} < F_{t(5\%)}$. **Between Treatments :** $F_{Treat.} = 2.8356$ and $F_{t(5\%)}(3,9) = 3.86$ here, $F_{Treaat.} < F_{t(5\%)}$. **Main effect A** : $F_A = 4.0331$; and $F_{t(5\%)}(1,9) = 5.12$ here, $F_A < F_{t(5\%)}$. **Main effect B** : $F_B = 2.6229$ and $F_{t(5\%)}(1,9) = 5.12$ here, $F_B < F_{t(5\%)}$. **Interaction effect AB** : $F_{AB} = 1.8507$ and $F_{t(5\%)}(1,9) = 5.12$ here, $F_{AB} < F_{t(5\%)}$.

Conclusion 11.2. : In each of the cases, the computed value of F is less than the tabulated value of F at 5% level of significance. Therefore, there are no significant main or interaction effects present in the experiment. The blocks as well as treatments do not differ significantly. Since the blocks do not differ significantly, we conclude that there is no special contribution from fluctuations in soil fertility and thus the division of the whole experimental area into blocks does not result in any gain in accuracy.

12. 2^2-Factorial experiment using total integral value (TIV) of tfns.

The TIV for a normalized tfn. $\widetilde{A} = (a, b, c, d; 1)$ is calculated by the relation [12]

$$\int_{Supp(\widetilde{A})} \mu_{\widetilde{A}} x dx = \int_a^b \left(\frac{x - a}{b - a}\right) dx + \int_b^c dx + \int_c^d \left(\frac{x - d}{c - d}\right) dx --- (12.1)$$

Example 12.1. Let us consider example 1, the TIV for the first member is calculated as follows

$$\int_{Supp(\widetilde{A}_i)} \mu_{\widetilde{A}_i}(x) dx = \int_{17}^{19} \left(\frac{x - 17}{2}\right) dx + \int_{19}^{20} dx + \int_{20}^{23} \left(\frac{x - 23}{-3}\right) dx = 3.5 = I$$

Similarly we can calculate the TIV of all other entries using $\int_{Supp(\widetilde{A}_i)} \mu_{\widetilde{A}_i}(x) dx = I$ for the given tfns. which has been tabulated below.

$^{(1)}3.5$	$^{\tilde{a}}2.5$	$^{\tilde{a}}3$	$^{\tilde{b}}3.5$
$^{\widetilde{ab}}3.5$	$^{\tilde{b}}4.5$	$^{\widetilde{ab}}5$	$^{(1)}3.5$
$^{\tilde{a}}4.5$	$^{\tilde{b}}3$	$^{\widetilde{ab}}3.5$	$^{(1)}3.5$

The ANOVA table values using TIV of tfns. :

$SS_A = 0.02083$; $SS_B = 0.52083$; $SS_{AB} = 0.1875$; $SS_T = 5.5625$; $SS_E = 4.83334$; **Main effect A** : $F_A = 0.0345$ and $F_{t(1\%)}(1,8) = 11.26$, here $F_A < F_{t(1\%)}$. **Main effect B** : $F_B = 0.8621$ and

$F_{t(1\%)}(1,8) = 11.26$, here $F_B < F_{t(1\%)}$. **Interaction effect AB** : $F_{AB} = 0.3103$ and $F_{t(1\%)}(1,8) = 11.26$, here $F_{AB} < F_{t(1\%)}$.

Conclusion 12.1. : Here, the calculated value of F < tabulated value of F for the main effects A and B. **Hence, we conclude that the main effects A and B both are homogeneous at 1% level of significant.**

Example 12.2. Let we consider example 2, using the relation (12.1) the TIV of tfns. are tabulated below:

$^{(1)}4$	$^{\tilde{a}}3.5$	$^{\tilde{b}}3.5$	$^{\widetilde{ab}}3.5$
$^{\tilde{b}}4.5$	$^{(1)}3.5$	$^{\tilde{a}}3.5$	$^{\widetilde{ab}}4.5$
$^{(1)}3.5$	$^{\tilde{a}}3.5$	$^{\widetilde{ab}}4.5$	$^{\tilde{b}}3$
$^{\widetilde{ab}}4$	$^{\tilde{a}}4$	$^{\tilde{b}}3$	$^{(1)}4.5$

The ANOVA table values using TIV of tfns. :

$SS_A = 0.14063$; $SS_B = 0.01563$; $SS_{AB} = 0.76563$; $SS_T = 3.9844$; $SS_{Blocks} = 0.4219$; $SS_{Treat.} = 0.9219$; $SS_E = 2.6406$. **Between Blocks :** $F_{Blocks} = 0.4792$ and $F_{t(5\%)}(3,9) = 3.86$ here, $F_{Blocks} < F_{t(5\%)}$. **Between Treatments :** $F_{Treat.} = 1.0474$ and $F_{t(5\%)}(3,9) = 3.86$ here, $F_{Treaat.} < F_{t(5\%)}$. **Main effect A** : $F_A = 0.4793$; and $F_{t(5\%)}(1,9) = 5.12$ here, $F_A < F_{t(5\%)}$. **Main effect B** : $F_B = 0.0533$ and $F_{t(5\%)}(1,9) = 5.12$ here, $F_B < F_{t(5\%)}$.**Interaction effect AB** : $F_{AB} = 2.6095$ and $F_{t(5\%)}(1,9) = 5.12$ here, $F_{AB} < F_{t(5\%)}$.

Conclusion 12.2. : In each of the cases, the computed value of F is less than the tabulated value of F at 5% level of significance. Therefore, there are no significant main or interaction effects present in the experiment. The blocks as well as treatments do not differ significantly. Since the blocks do not differ significantly, we conclude that there is no special contribution from fluctuations in soil fertility and thus the division of the whole experimental area into blocks does not result in any gain in accuracy.

13. Liou and Wang's centroid point method

Liou and Wang [14] ranked fuzzy numbers with total integral value. For a fuzzy number defined by definition (2.3), the total integral value is defined as

$$I_T^{\alpha}\left(\widetilde{A}\right) = \alpha I_R\left(\widetilde{A}\right) + (1 - \alpha) I_L\left(\widetilde{A}\right) --- (13.1)$$

$$I_R\left(\widetilde{A}\right) = \int_{Supp\left(\widetilde{A}\right)} R_{\widetilde{A}}(x)dx = \int_c^d \left(\frac{x-d}{c-d}\right)dx --- (13.2) \ \& \ I_L\left(\widetilde{A}\right) = \int_{Supp\left(\widetilde{A}\right)} L_{\widetilde{A}}(x)dx = \int_a^b \left(\frac{x-a}{b-a}\right)dx --- (13.3)$$

are the ***right*** and ***left integral*** values of $\widetilde{A}$ respectively and $0 \le \alpha \le 1$.

(i) $\alpha \in [0,1]$ is the ***index of optimism*** which represents the ***degree of optimism*** of a decision maker. (ii) If $\alpha = 0$, then the total value of integral represents a **pessimistic decision maker's view point** which is equal to left integral value. (iii) If $\alpha = 1$, then the total integral value represents an **optimistic decision maker's view point** and is equal to the right integral value. (iv)If $\alpha = 0.5$ then the total integral value represents a **moderate decision maker's view point** and is equal to the mean of right and left integral values. For a decision maker, the larger the value of α is, the higher is the degree of optimism.

2^2-Factorial experiment using Liou and Wang's centroid point method:

Example 13.1. Let us consider example 1, using the above equations (13.1), (13.2) and (13.3), we get the centroid point of first member as follows:

$$I_L(\widetilde{A}) = \int_{17}^{19}\left(\frac{x-17}{2}\right)dx = 1; \quad I_R(\widetilde{A}) = \int_{20}^{23}\left(\frac{x-23}{-3}\right)dx = 1.5 \text{ Therefore } I_T^{\alpha}(\widetilde{A}) = (1+0.5\alpha).$$

Similarly we can find the centroid point for all other members and the calculated values are tabulated below:

$^{(1)}(1+0.5\alpha)$	$^{\tilde{a}}(0.5+0.5\alpha)$	$^{\tilde{a}}(1.5-\alpha)$	$^{\tilde{b}}(1.5-0.5\alpha)$
$^{\widetilde{ab}}(1.5-0.5\alpha)$	$^{\tilde{b}}(0.5+0.5\alpha)$	$^{\widetilde{ab}}1$	$^{(1)}(1+0.5\alpha)$
$^{\tilde{a}}(0.5+0.5\alpha)$	$^{\tilde{b}}1$	$^{\widetilde{ab}}(1-0.5\alpha)$	$^{(1)}(1.5-0.5\alpha)$

The ANOVA table values using Liou and Wang's centroid point of tfns. :

$SS_A=(2.25\alpha^2+1.5\alpha+0.25)/12$; $SS_B=(2.25\alpha^2-1.5\alpha+0.25)/12$; $SS_{AB}=(0.25\alpha^2-1.5\alpha+2.25)/12$; $SS_T=(155\alpha^2-190\alpha+83)/48$; $SS_E=(17\alpha^2-23\alpha+9)/6$; **Main effect A** : $F_A=(9\alpha^2+6\alpha+1)/(17\alpha^2-23\alpha+9)$ and $F_{t(1\%)}(1,8)=11.26$, here $F_A < F_{t(1\%)}$. **Main effect B** : $F_B=(9\alpha^2-6\alpha+1)/(17\alpha^2-23\alpha+9)$ and $F_{t(1\%)}(1,8)=11.26$, here $F_B < F_{t(1\%)}$. **Interaction effect AB** : $F_{AB}=(\alpha^2-6\alpha+9)/(17\alpha^2-23\alpha+9)$ and $F_{t(1\%)}(1,8)=11.26$, here $F_{AB} < F_{t(1\%)}$ $\forall\alpha,\ 0\le\alpha\le 1$.

Conclusion 13.1. : Here, the calculated value of F < tabulated value of F for the main effects A and B. **Hence, we conclude that the main effects A and B both are homogeneous at 1% level of significant.**

Example 13.2. Let us consider example 2, using the above equations (13.1), (13.2) and (13.3), we get the centroid points of tfns. as follows:

$^{(1)}(1.5\text{-}\alpha)$	$^{\tilde{a}}(0.5\text{+}0.5\alpha)$	$^{\tilde{b}}(1\text{-}0.5\alpha)$	$^{\widetilde{ab}}(1.5\text{-}0.5\alpha)$
$^{\tilde{b}}(1.5\text{-}0.5\alpha)$	$^{(1)}(0.5\text{+}0.5\alpha)$	$^{\tilde{a}}(1.5\text{-}0.5\alpha)$	$^{\widetilde{ab}}(1\text{-}0.5\alpha)$
$^{(1)}(1.5\text{-}0.5\alpha)$	$^{\tilde{a}}(0.5\text{+}0.5\alpha)$	$^{\widetilde{ab}}(1\text{-}0.5\alpha)$	$^{\tilde{b}}(0.5\text{+}\alpha)$
$^{\widetilde{ab}}(0.5\text{+}\alpha)$	$^{\tilde{a}}1$	$^{\tilde{b}}(1.5\text{-}\alpha)$	$^{(1)}(1.5\text{-}0.5\alpha)$

The ANOVA table values using Liou and Wang's centroid point of tfns. :

$SS_A=(25\alpha^2-30\alpha+9)/64$; $SS_B=\alpha^2/64$; $SS_{AB}=(9\alpha^2-12\alpha+4)/64$; $SS_T=(407\alpha^2-492\alpha+188)/64$; $SS_{Blocks}=(35\alpha^2-36\alpha+12)/64$; $SS_{Treat.}=(35\alpha^2-52\alpha+20)/64$; $SS_E=(346\alpha^2-426\alpha+167)/64$. **Between Blocks :** $F_{Blocks}=3(35\alpha^2-36\alpha+12)/346\alpha^2-426\alpha+167$ and $F_{t(5\%)}(3,9)=3.86$ here, $F_{Blocks} < F_{t(5\%)}$. **Between Treatments :** $F_{Treat.}=3(35\alpha^2-52\alpha+20)/346\alpha^2-426\alpha+167$ and $F_{t(5\%)}(3,9)=3.86$ here, $F_{Treaat.} < F_{t(5\%)}$. **Main effect A** : $F_A=9(25\alpha^2-30\alpha+9)/346\alpha^2-426\alpha+167$; and $F_{t(5\%)}(1,9)=5.12$ here, $F_A < F_{t(5\%)}$. **Main effect B** : $F_B=9\alpha^2/346\alpha^2-426\alpha+167$ and $F_{t(5\%)}(1,9)=5.12$ here, $F_B < F_{t(5\%)}$. **Interaction effect AB** : $F_{AB}=9(9\alpha^2-12\alpha+4)/346\alpha^2-426\alpha+167$ and $F_{t(5\%)}(1,9)=5.12$ here, $F_{AB} < F_{t(5\%)}$ $\forall\alpha,\ 0\le\alpha\le 1$.

Conclusion 13.2. : In each of the cases, the computed value of F is less than the tabulated value of F at 5% level of significance. Therefore, there are no significant main or interaction effects present in the experiment. The blocks as well as treatments do not differ significantly. Since the blocks do not differ significantly, we conclude that there is no special contribution from fluctuations in soil fertility and thus the division of the whole experimental area into blocks does not result in any gain in accuracy.

14. 2^2-Factorial experiment using Thorani's ranking method of tfns.

As per the description in Salim Rezvani's ranking method, we presented a different kind of centroid point and ranking function of tfns. The incenter $I_{\tilde{A}}\left(\overline{x}_0, \overline{y}_0\right)$ of the triangle [Fig. 2] with vertices G_1, G_2 and G_3 of the generalized tfn. $\tilde{A}=(a, b, c, d; w)$ is given by,

$$I_{\tilde{A}}\left(\overline{x}_0, \overline{y}_0\right)=\left[\frac{\alpha\left(\frac{a+2b}{3}\right)+\beta\left(\frac{b+c}{2}\right)+\gamma\left(\frac{2c+d}{3}\right)}{\alpha+\beta+\gamma}, \frac{\alpha\left(\frac{w}{3}\right)+\beta\left(\frac{w}{2}\right)+\gamma\left(\frac{w}{3}\right)}{\alpha+\beta+\gamma}\right] --- (14.1)$$

$$\text{where } \alpha=\frac{\sqrt{(c-3b+2d)^2+w^2}}{6}, \beta=\frac{\sqrt{(2c+d-a-2b)^2}}{3}, \gamma=\frac{\sqrt{(3c-2a-b)^2+w^2}}{6}$$

And the ranking function of the generalized tfn. $\tilde{A}=(a, b, c, d; w)$ which maps the set of all fuzzy numbers to a set of real numbers is defined as $R\left(\tilde{A}\right)=x_0\times y_0 --- (14.2)$. For a normalized tfn., we put w = 1 in equations (1) and (2) so we have,

$$I_{\tilde{A}}\left(\overline{x}_0, \overline{y}_0\right)=\left[\frac{\alpha\left(\frac{a+2b}{3}\right)+\beta\left(\frac{b+c}{2}\right)+\gamma\left(\frac{2c+d}{3}\right)}{\alpha+\beta+\gamma}, \frac{\alpha\left(\frac{1}{3}\right)+\beta\left(\frac{1}{2}\right)+\gamma\left(\frac{1}{3}\right)}{\alpha+\beta+\gamma}\right] --- (14.3)$$

$$\text{where } \alpha=\frac{\sqrt{(c-3b+2d)^2+1}}{6}, \beta=\frac{\sqrt{(2c+d-a-2b)^2}}{3} \text{ and } \gamma=\frac{\sqrt{(3c-2a-b)^2+1}}{6}$$

And for $\tilde{A}=(a, b, c, d; 1)$, $R\left(\tilde{A}\right)=x_0\times y_0 --- (14.4)$

The ANOVA table values using Thorani's ranking method of tfns. :

Example 14.1. Let us consider example 1, using the above relations (14.3) and (14.4), we get the ranks of each tfns. which are tabulated below:

$^{(1)}8.1192$	$^{\tilde{a}}10.611$	$^{\tilde{a}}9.7794$	$^{\tilde{b}}4.3710$
$^{\widetilde{ab}}9.7834$	$^{\tilde{b}}3.9572$	$^{\widetilde{ab}}9.7888$	$^{(1)}6.8701$
$^{\tilde{a}}9.3720$	$^{\tilde{b}}6.0356$	$^{\widetilde{ab}}8.3277$	$^{(1)}7.7017$

The ANOVA table values using Thorani's ranking method of tfns. :

$SS_A=35.3891$; $SS_B=8.6525$; $SS_{AB}=3.4827$; $SS_T=52.9691$; $SS_E=5.4448$; **Main effect A** $F_A=51.9969$ and $F_{t(1\%)}(1,8)=11.26$, here $F_A>F_{t(1\%)}$. **Main effect B** : $F_B=12.7130$ and $F_{t(1\%)}(1,8)=11.26$, here $F_B>F_{t(1\%)}$. **Interaction effect AB** : $F_{AB}=5.1171$ and $F_{t(1\%)}(1,8)=11.26$, here $F_{AB}<F_{t(1\%)}$.

Conclusion 14.1. : Here, the calculated value of F > tabulated value of F for the main effects A and B. **Hence, we conclude that the main effects A and B both are significantly different at 1% level of significant.**

Example 14.2. Let us consider example 2, using the above relations (14.3) and (14.4), we get the ranks of each tfns. which are tabulated below:

$^{(1)}9.9941$	$^{\tilde{a}}10.827$	$^{\tilde{b}}9.5770$	$^{\widetilde{ab}}15.6122$
$^{\tilde{b}}17.076$	$^{(1)}9.9942$	$^{\tilde{a}}13.9468$	$^{\widetilde{ab}}16.4524$
$^{(1)}12.2814$	$^{\tilde{a}}8.745$	$^{\widetilde{ab}}13.1202$	$^{\tilde{b}}8.5338$
$^{\widetilde{ab}}14.5765$	$^{\tilde{a}}13.3267$	$^{\tilde{b}}10.1956$	$^{(1)}12.078$

The ANOVA table values using Thorani's ranking method of tfns. :

$SS_A = 17.8014$; $SS_B = 12.1635$; $SS_{AB} = 8.8225$; $SS_T = 112.15$; $SS_{Blocks} = 30.491$; $SS_{Treat.} = 38.787$; $SS_E = 42.872$. **Between Blocks :** $F_{Blocks} = 2.1336$ and $F_{t(5\%)}(3,9) = 3.86$ here, $F_{Blocks} < F_{t(5\%)}$. **Between Treatments :** $F_{Treat.} = 2.7141$ and $F_{t(5\%)}(3,9) = 3.86$ here, $F_{Treaat.} < F_{t(5\%)}$. **Main effect A** : $F_A = 3.7370$; and $F_{t(5\%)}(1,9) = 5.12$ here, $F_A < F_{t(5\%)}$. **Main effect B** : $F_B = 2.5534$ and $F_{t(5\%)}(1,9) = 5.12$ here, $F_B < F_{t(5\%)}$.**Interaction effect AB** : $F_{AB} = 1.8521$ and $F_{t(5\%)}(1,9) = 5.12$ here, $F_{AB} < F_{t(5\%)}$.

Conclusion 14.2. : In each of the cases, the computed value of F is less than the tabulated value of F at 5% level of significance. Therefore, there are no significant main or interaction effects present in the experiment. The blocks as well as treatments do not differ significantly. Since the blocks do not differ significantly, we conclude that there is no special contribution from fluctuations in soil fertility and thus the division of the whole experimental area into blocks does not result in any gain in accuracy.

15. General conclusion

The decisions obtained from various methods are tabulated below for the acceptance of null hypothesis.

S.V.	Acceptance of null hypotheses $\widetilde{H}_0$															
	~ cut method				Wang		Rezvani		GMIR		TIV		L & W		Thorani	
	Eg.1		Eg.2		Eg.1	Eg.2	Eg.1	Eg.2	Eg.1	Eg.2	Eg.1	Eg.2	Eg.1	Eg.2	Eg.1	Eg.2
	L	U	L	U												
Between Blocks	-	-	✓	✓	--	✓	--	✓	--	✓	--	✓	--	✓	--	✓
Between Treat.	-	-	✓	×	--	✓	--	✓	--	✓	--	✓	--	✓	--	✓
Main effect A	×	×	✓	✓	×	✓	×	✓	×	✓	✓	✓	✓	✓	×	✓
Main effect B	×	×	✓	✓	×	✓	×	✓	×	✓	✓	✓	✓	✓	×	✓
Int. act. effect AB	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

For example-1, from the decision obtained from both l.l.m. and u.l.m., the calculated value of F > tabulated value of F for the main effects A and B. **Hence, we conclude that the main effects A and B both are significantly different at 1% level of significant** $\forall\alpha,\ 0 \le \alpha \le 1$.

For example-2, we see that $F_{Blocks} < F_{t(5\%)} \forall\alpha$ at both l.l.m. and u.l.m. $\Rightarrow$ **The blocks do not differ significantly**. Also $F_A < F_t$ for the partial level of $\alpha \in [0.1,\ 1]$ at both l.l.m. and u.l.m. $\Rightarrow$ **The main effect A does not differ significantly at** $0.1 \le \alpha \le 1$. And $F_B < F_t,\ F_{AB} < F_t,\ \forall\alpha \Rightarrow$ **The effect B and the interaction effect AB do not differ significantly**. Moreover, $F_{Treat.} < F_t,\ \forall\alpha$ at l.l.m. on the other hand $F_{Treat.} > F_t,\ \forall\alpha$ at u.l.m. $\Rightarrow$ **The null hypothesis** $\widetilde{H}_0$ **is rejected in this case. Therefore, the treatment differs significantly for all** α.

Moreover, the decisions obtained from the ranking grades of Wang's method, Rezvani's method, GMIR and Thorani's method provide parallel and reliable discussions. And the decisions obtained from TIV and Liou & Wang's method (L & W) do not provide a reliable decisions as they accept the null hypothesis in all the cases.

REFERENCES

[1]. S. Abbasbandy, B. Asady, The nearest trapezoidal fuzzy number to a fuzzy quantity, *Appl. Math. Comput.,* 156 (2004) 381-386.

[2]. S. Abbasbandy, M. Amirfakhrian, The nearest approximation of a fuzzy quantity in parametric form, *Appl. Math. Comput.,* 172 (2006) 624-632.

[3]. S. Abbasbandy, M. Amirfakhrian, The nearest trapezoidal form of generalized left right fuzzy number, *Internat. J. Approx. Reason.,* 43 (2006) 166-178.

[4]. S. Abbasbandy, T. Hajjari, Weighted trapezoidal approximation-preserving cores of a fuzzy number, *Computers and Mathematics with Applications,* 59 (2010) 3066-3077.

[5]. E. Baloui Jamkhaneh and A. Nadi Ghara, Testing statistical hypotheses for compare means with vague data*, International Mathematical Forum, 5 (2010) 615-620*.

[6]. S. Bodjanova, Median value and median interval of a fuzzy number, *Inform. Sci. 172 (2005) 73-89.*

[7]. J. J. Buckley, Fuzzy statistics, *Springer-Verlag, New York,* 2005.

[8]. J. Chachi, S. M. Taheri and R. Viertl, Testing statistical hypotheses based on fuzzy confidence intervals, *Forschungsbericht SM-2012-2, Technische Universitat Wien, Austria,* 2012.

[9]. D. Dubois and H. Prade, Operations on fuzzy numbers, *Int. J. Syst. Sci.,* 9 (1978) 613-626.

[10]. S. C. Gupta, V. K. Kapoor, Fundamentals of mathematical statistics, *Sultan Chand & Sons, New Delhi, India.*

[11]. R. R. Hocking, Methods and applications of linear models: regression and the analysis of variance, New York: *John Wiley & Sons,* 1996.

[12]. Iuliana Carmen BĂRBĂCIORU, Statistical Hypothesis Testing Using Fuzzy Linguistic Variables, *Fiabilitatesi Durabilitate-Fiability & Durability, Supplement,* 1 (2012*) Editura "Academica Brâncusi", Târgu Jiu, ISSN 1844-640X.*

[13]. George J. Klir and Bo Yuan, Fuzzy sets and fuzzy logic, *Theory and Applications, Prentice-Hall, New Jersey,* 2008.

[14]. T. S. Liou and M. J. Wang, Ranking Fuzzy Numbers with Integral Value, *Fuzzy Sets and Systems,* 50 (1992) 247-225.

[15]. Mikihiko Konishi, Tetsuji Okuda and Kiyoji Asai, Analysis of variance based on Fuzzy interval data using moment correction method, *International Journal of Innovative Computing, Information and Control,* 2 (2006) 83-99.

[16]. S. Parthiban, and P. Gajivaradhan, One-factor ANOVA model using trapezoidal fuzzy numbers through alpha cut interval method, *Annals of Pure and Applied Mathematics,* 11(1) (2016) 45-61.

[17]. S. Parthiban, and P. Gajivaradhan, A comparative study of two factor ANOVA model under fuzzy environments using trapezoidal fuzzy numbers, *Int. J. of Fuzzy Mathematical Archive,* 10(1) (2016) 1-25.

[18]. S. Parthiban and P. Gajivaradhan, A comparative study of LSD under fuzzy environments using trapezoidal fuzzy numbers, *IOSR J. of Maths., Vol. 12 Issue 1 Ver. II (Jan-Feb.2016), pp. 57-75.*

[19]. S. Salahsour, S. Abbasbandy and T. Allahviranloo, Ranking Fuzzy Numbers using Fuzzy Maximizing-Minimizing points, *EUSFLAT-LFA: July 2011, Aix-les-Bains, France.*

[20]. Salim Rezvani and Mohammad Molani, Representation of trapezoidal fuzzy numbers with shape function, to appear in *Annals of Fuzzy mathematics and Informatics.*

[21]. Salim Rezvani, Ranking Generalized Trapezoidal Fuzzy Numbers with Euclidean Distance by the Incentre of Centroids, *Mathematica Aeterna,* 3 (2) (2013) 103-114.

[22]. Y. L. P. Thorani, et al., Ordering Generalized Trapezoidal Fuzzy Numbers, *Int. J. Contemp. Math. Sciences,* 7(12) (2012) 555-573.

[23]. T. Veerarajan, Probability, statistics and random process, *Tata McGraw Hill Education Pvt. Ltd., New Delhi, India.*

[24]. R. Viertl, Statistical methods for fuzzy data, *John Wiley and Sons, Chichester, 2011.*

[25]. R. Viertl, Univariate statistical analysis with fuzzy data, *Computational Statistics and Data Analysis,* 51 (2006) 33-147.

[26]. Y. M. Wang et al., On the centroids of fuzzy numbers, *Fuzzy Sets and Systems,* 157 (2006) 919-926.

[27]. H. C. Wu, Analysis of variance for fuzzy data, *International Journal of Systems Science,* 38 (2007) 235-246.

[28]. H. C. Wu, Statistical confidence intervals for fuzzy data, *Expert Systems with Applications,* 36 (2009) 2670-2676.

[29]. H. C. Wu, Statistical hypotheses testing for fuzzy data, *Information Sciences,* 175 (2005) 30-56.

[30]. L. A. Zadeh, *Fuzzy sets, Information and Control,* 8 (1965) 338-353.