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RESEARCH ARTICLE

DUAL SPACES OF GENERALIZED WEIGHTED CESARO SEQUENCE SPACE AND RELATED MATRIX MAPPING

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ABSTRACT

In this paper we define the generalized weighted Cesaro sequence spaces $\mathrm{ces}(\mathrm{p},\mathrm{q},\mathrm{s})$. We prove the space $\mathrm{ces}(\mathrm{p},\mathrm{q},\mathrm{s})$ is a complete paranorm space. In section-2 we determine its Kothe-Toeplitz dual. In section-3 we establish necessary and sufficient conditions for a matrix A to map $\mathrm{ces}(\mathrm{p},\mathrm{q},\mathrm{s})$ to l_∞ and $\mathrm{ces}(\mathrm{p},\mathrm{q},\mathrm{s})$ to c, where l_∞ is the space of all bounded sequences and c is the space of all convergent sequences. We also get some known and unknown results as remarks.

Keywords: Sequence space, Kothe-Toeplitz dual, Matrix transformation.

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1. INTRODUCTION

Let ω be the space of all (real or complex) sequences and let l_∞, c and c_0 are respectively the Banach spaces of bounded sequences, convergent sequences and null sequences. Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers. Then $l(p)$ was defined by Maddox [7] as

$$l(p) = \left\{x = (x_k) \in \omega : \sum_{k=1}^{\infty} |x_k|^{p_k} < \infty\right\}$$

$$\text{with } 0 < p_k \leq \sup_k p_k = H < \infty.$$

If (q_n) is a bounded sequence of positive real numbers, then for $p = (p_r)$ with $\inf p_r > 0$, we defined the weighted Cesaro sequence space in our recent paper [11] by

$$ces(p,q) = \left\{x = (x_k) \in \omega : \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k|\right)^{p_r} < \infty\right\}$$

$where\ Q_{2^r} = q_{2^r} + q_{2^r+1} + \cdots \ldots\ldots\ldots\ldots\ldots.. + q_{2^{r+1}-1}\ and\ \sum_r$ denotes a sum over the range $2^r \leq k < 2^{r+1}$. In [1] Maji and Srivastava defined this space in a different norm.

The main purpose of this paper to define the generalized weighted Cesaro sequence space $ces(p,q,s)$. We determine the Kothe-Toeplitz dual of $ces(p,q,s)$ and then consider the matrix mapping $ces(p,q,s)\ to\ l_\infty\ and\ ces(p,q,s)\ to\ c$.

In [3] Bulut and Cakar defined and studied the sequence space $l(p,s)$, in [4] Khan and Khan defined and investigated the Cesaro sequence space $ces(p,s)$ and in [12] we defined and studied the Riesz sequence space $r^q(u,p,s)$ of non-absolute type. In the same vein we define generalized weighted Cesaro sequence space $ces(p,q,s)$in the following way.

Definition. For $s \geq 0$ we define

$$ces(p,q,s) = \left\{x = (x_k) \in \omega : \sum_{r=0}^{\infty} (Q_{2^r})^{-s} \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k|\right)^{p_r} < \infty\right\}$$

where (q_k) is a bounded sequence of real numbers, $p = (p_r)$ with $\inf p_r > 0$, $Q_{2^r} = q_{2^r} + q_{2^r+1} + \cdots \ldots\ldots\ldots\ldots\ldots.. + q_{2^{r+1}-1}\ and\ \sum_r$ denotes a sum over the range $2^r \leq k < 2^{r+1}$.

With regard notation, the dual space of $ces(p,q,s)$, that is, the space of all continuous linear functional on $ces(p,q,s)$ will be denoted by $ces^*(p,q,s)$. We write

$$A_r(n) = \max_r \left|\frac{a_{n,k}}{q_k}\right|$$

where for each *n* the maximum with respect to k in $[2^r, 2^{r+1})$.

Throughout the paper the following well-known inequality (see [7] or [8]) will be frequently used. For any positive integer $E > 1$ and any two complex numbers *a* and *b* we have

$$|ab| \leq E(|a|^t E^{-t} + |b|^t) \tag{1}$$

where $p > 1\ and\ \frac{1}{p} + \frac{1}{q} = 1$.

To begin with, we show that the space $ces(p,q,s)$ is a paranorm space paranormed by

$$g(x) = \left(\sum_{r=0}^{\infty} (Q_{2^r})^{-s} \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k|\right)^{p_r}\right)^{1/M} \tag{2}$$

provided $H = \sup_r p_r < \infty\ and\ M = \max\{1, H\}$.

Clearly

$$g(\theta) = 0$$
$$g(-x) = g(x),$$

where $\theta = (0,0,0,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..)$

Since $p_r \leq M,\ M \geq 1\ so\ for\ any\ x,y \in ces(p,q,s)\ we$ have by Minkowski's inequality

$$\left(\sum_{r=0}^{\infty} (Q_{2^r})^{-s} \left(\frac{1}{Q_{2^r}} \sum_r |q_k(x_k + y_k)|\right)^{p_r}\right)^{1/M}$$

$$\leq \left(\sum_{r=0}^{\infty} (Q_{2^r})^{-s} \left(\frac{1}{Q_{2^r}} \sum_r (|q_k x_k| + |q_k y_k|)\right)^{p_r}\right)^{1/M}$$

$$\leq \left(\sum_{r=0}^{\infty} (Q_{2^r})^{-s} \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k|\right)^{p_r}\right)^{1/M} + \left(\sum_{r=0}^{\infty} (Q_{2^r})^{-s} \left(\frac{1}{Q_{2^r}} \sum_r |q_k y_k|\right)^{p_r}\right)^{1/M}$$

which shows that g is subadditive.

Finally we have to check the continuity of scalar multiplication. From the definition of $ces(p,q,s)$, we have $\inf p_r > 0$. So, we may assume that $\inf p_r \equiv \rho > 0$. Now for any complex λ with $||\lambda|| < 1$, we have

$$g(\lambda x) = \left(\sum_{r=0}^{\infty}(Q_{2^r})^{-s}\left(\frac{1}{Q_{2^r}}\sum_{r}|\lambda q_k x_k|\right)^{p_r}\right)^{1/M}$$

$$= |\lambda|^{\frac{p_r}{M}}\left(\sum_{r=0}^{\infty}(Q_{2^r})^{-s}\left(\frac{1}{Q_{2^r}}\sum_{r}|q_k x_k|\right)^{p_r}\right)^{\frac{1}{M}}$$

$$\leq \sup_r \|\lambda\|^{\frac{p_r}{M}} g(x)$$

$$\leq \|\lambda\|^{\frac{\rho}{M}} g(x) \to 0 \ as\ \lambda \to 0$$

It is quite routine to show that $ces(p,q,s)$ is a metric space with the metric $d(x,y) = g(x-y)$ provided that $x, y \in ces(p,q,s)$, where *g* is defined by (2). And using a similar method to that in [5] one can show that $ces(p,q,s)$ is complete under the metric mentioned above.

2. Kothe-Toeplitz duals

If X is a sequence space we define ([2], [6])

$$X^{|+|} = X^{\alpha} = \left\{a = (a_k) \in \omega : \sum_k |a_k x_k| < \infty,\ for\ every\ x \in X\right\}$$

$$X^{+} = X^{\beta} = \left\{a = (a_k) \in \omega : \sum_k a_k x_k\ is\ convergent\ for\ every\ x \in X\right\}$$

Now we are going to give the following theorem by which the generalized Kothe-Toeplitz dual $ces^{+}(p,q,s)$ will be determined.

Theorem 1: If $1 < p_r \leq \sup_r p_r < \infty\ and\ \frac{1}{p_r} + \frac{1}{t_r} = 1,\ for\ r = 0, 1, 2, \ldots \ldots$, then

$ces^{+}(p,q,s) = [ces(p,q,s)]^{\beta}$

$= \left\{a = (a_k) : \sum_{r=0}^{\infty} (Q_{2^r})^{s(t_r-1)}\left(Q_{2^r}\max_r\left|\frac{a_k}{q_k}\right|\right)^{t_r} E^{-t_r} < \infty,\ for\ some\ integer\ E > 1\right\}.$

Proof : Let $1 < p_r \leq \sup_r p_r < \infty\ and\ \frac{1}{p_r} + \frac{1}{t_r} = 1,\ for\ r = 0,1,2, \ldots \ldots$. Define

$$\mu(t,s) = \left\{a = (a_k) : \sum_{r=0}^{\infty} (Q_{2^r})^{s(t_r-1)}\left(Q_{2^r}\max_r\left|\frac{a_k}{q_k}\right|\right)^{t_r} E^{-t_r} < \infty\ for\ some\ integer\ E > 1\right\}. \quad (3)$$

We want to show that $ces^{+}(p,q,s) = \mu(t,s)$.

Let $x \in ces(p,q,s)$ and $a \in \mu(t,s)$. Then using inequality (1) we get

$$\sum_{k=1}^{\infty}|a_k x_k| = \sum_{r=0}^{\infty}\sum_{r}|a_k x_k|$$

$$= \sum_{r=0}^{\infty}\sum_r\left|\frac{a_k}{q_k} q_k x_k\right|$$

$$= \sum_{r=0}^{\infty}\sum_r\left|\frac{a_k}{q_k}\right||q_k x_k|$$

$$\leq \sum_{r=0}^{\infty}\max_r\left|\frac{a_k}{q_k}\right|\sum_r|q_k x_k|$$

$$= \sum_{r=0}^{\infty} Q_{2^r}\max_r\left|\frac{a_k}{q_k}\right| (Q_{2^r})^{\frac{s}{p_r}}\frac{1}{Q_{2^r}} (Q_{2^r})^{-\frac{s}{p_r}}\sum_r|q_k x_k|$$

$$\leq E\sum_{r=0}^{\infty}\left\{\left(Q_{2^r}\max_r\left|\frac{a_k}{q_k}\right|\right)^{t_r}(Q_{2^r})^{\frac{s\,t_r}{p_r}}E^{-t_r} + (Q_{2^r})^{-s}\left(\frac{1}{Q_{2^r}}\sum_r|q_k x_k|\right)^{p_r}\right\}$$

$$= E\left\{\sum_{r=0}^{\infty}\left(Q_{2^r}\max_r\left|\frac{a_k}{q_k}\right|\right)^{t_r}(Q_{2^r})^{s(t_r-1)}E^{-t_r}+\sum_{r=0}^{\infty}(Q_{2^r})^{-s}\left(\frac{1}{Q_{2^r}}\sum_r|q_kx_k|\right)^{p_r}\right\}$$
$$< \infty$$

which implies that the series $\sum_{k=1}^{\infty} a_k x_k$ convergent.

Therefore,

$$a \in dual\ of\ ces(p,q,s) = ces^{+}(p,q,s).$$

This shows, $\mu(t,s) \subset ces^{+}(p,q,s)$

Conversely, suppose that $\sum a_k x_k$ is convergent for all $x \in ces(p,q,s)$ but $a \notin \mu(t,s)$. Then $\sum_{r=0}^{\infty}(Q_{2^r})^{s(t_r-1)}\left(Q_{2^r}\max_r\left|\frac{a_k}{q_k}\right|\right)^{t_r}E^{-t_r} = \infty$, for every integer $E > 1$.

So, we can define a sequence $0 = n(0) < n(1) < n(2) < \cdots \ldots\ldots\ldots.\ldots$

such that $\gamma = 0, 1, 2, \ldots\ldots\ldots\ldots\ldots..$, we have

$$M_\gamma = \sum_{r=n(\gamma)}^{n(\gamma+1)-1}(Q_{2^r})^{s(t_r-1)}\left(Q_{2^r}\max_r\left|\frac{a_k}{q_k}\right|\right)^{t_r}(\gamma+2)^{-t_r/p_r} > 1$$

Now we define a sequence $x = (x_k)$ in the following way:

$$x_k = 0\ if\ k \geq 2^{m_0+1}$$
$$x_{N(r)} = Q_{2^r}^{t_r}\left|a_{N(r)}\right|^{t_r-1}sgn\,a_{N(r)}(Q_{2^r})^{s(t_r-1)}(\gamma+2)^{-t_r}M_\gamma^{-1}$$

for $n(\gamma) \leq r \leq n(\gamma+1)-1$, $\gamma = 0, 1, 2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$

and x_k= 0 for $k \neq N(r)$, where $N(r)$ is such that

$\left|a_{N(r)}\right| = \max_r\left|\frac{a_k}{q_k}\right|$,

the maximum is taken with respect to k in $[2^r, 2^{r+1})$.

Therefore .

$$\sum_{k=2^{n(\gamma)}}^{2^{n(\gamma+1)-1}} a_k x_k = \sum_{r=n(\gamma)}^{n(\gamma+1)-1}\left(Q_{2^r}\left|a_{N(r)}\right|\right)^{t_r}(Q_{2^r})^{s(t_r-1)}(\gamma+2)^{-t_r}M_\gamma^{-1}$$
$$= M_\gamma^{-1}(\gamma+2)^{-1}\sum_{r=n(\gamma)}^{n(\gamma+1)-1}\left(Q_{2^r}\left|a_{N(r)}\right|\right)^{t_r}(Q_{2^r})^{s(t_r-1)}(\gamma+2)^{-t_r/p_r}$$
$$= M_\gamma^{-1}M_\gamma\ (\gamma+2)^{-1}$$
$$= (\gamma+2)^{-1}$$

It follows that

$$\sum_{k=1}^{\infty} a_k x_k = \sum_{\gamma=0}^{\infty}(\gamma+2)^{-1}$$

diverges.

Moreover

$$\sum_{r=n(\gamma)}^{n(\gamma+1)-1}(Q_{2^r})^{-s}\left(\frac{1}{Q_{2^r}}\sum_r|q_kx_k|\right)^{p_r}$$
$$= \sum_{r=n(\gamma)}^{n(\gamma+1)-1}(Q_{2^r})^{-s}\left(Q_{2^r}^{s(t_r-1)}\,Q_{2^r}^{(t_r-1)}\left|a_{N(r)}\right|^{(t_r-1)}(\gamma+2)^{-t_r}M_\gamma^{-1}\right)^{p_r}$$

$$= \sum_{r=n(\gamma)}^{n(\gamma+1)-1} (Q_{2^r})^{-s} Q_{2^r}^{(s+1)(t_r-1)p_r} \left|a_{N(r)}\right|^{(t_r-1)p_r} (\gamma+2)^{-t_r p_r} M_\gamma^{-p_r}$$

$$= \sum_{r=n(\gamma)}^{n(\gamma+1)-1} (Q_{2^r})^{-s} Q_{2^r}^{(s+1)t_r} \left|a_{N(r)}\right|^{t_r} (\gamma+2)^{-t_r p_r} M_\gamma^{-p_r}$$

$$= (\gamma+2)^{-2} M_\gamma^{-1} \sum_{r=n(\gamma)}^{n(\gamma+1)-1} Q_{2^r}^{s(t_{r-1})} (Q_{2^r}\left|a_{N(r)}\right|)^{t_r} (\gamma+2)^{2-t_r-p_r} M_\gamma^{1-p_r}$$

$$= (\gamma+2)^{-2} M_\gamma^{-1} \sum_{r=n(\gamma)}^{n(\gamma+1)-1} Q_{2^r}^{s(t_{r-1})} (Q_{2^r}\left|a_{N(r)}\right|)^{t_r} (\gamma+2)^{2-t_r/p_r} M_\gamma^{1-p_r} (\gamma+2)^{2-t_r-p_r+t_r/p_r}$$

$$= (\gamma+2)^{-2} M_\gamma^{-1} M_\gamma \ M_\gamma^{1-p_r} (\gamma+2)^{1-p_r}$$

$$= (\gamma+2)^{-2} M_\gamma^{-p_r/t_r} (\gamma+2)^{-p_r/t_r}$$

$$= \frac{(\gamma+2)^{-2}}{M_\gamma^{p_r/t_r} (\gamma+2)^{p_r/t_r}} < (\gamma+2)^{-2} < \infty.$$

Therefore

$$\sum_{r=0}^{\infty} (Q_{2^r})^{-s} \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k|\right)^{p_r} \le (\gamma+2)^{-2} < \infty$$

That is, $x \in ces(p,q,s)$ which is a contradiction to our assumption.

Hence $a \in \mu(t,s)$. That is, $\mu(t,s) \supset ces^+(p,q,s)$.

Then combining the two results, we get

$ces^+(p,q,s) = \mu(t,s)$.

The continuous dual of $ces(p,q,s)$ is determined by the following theorem.

Theorem 2: Let $1 < p_r \le \sup_r p_r < \infty$. Then continuous dual $ces^*(p,q,s)$ is isomorphic to $\mu(t,s)$, which is defined by (3)

Proof: It is easy to check that each $x \in ces(p,q,s)$ can be written in the form

$$x = \sum_{k=1}^{\infty} x_k e_k \text{ , where } e_k = (0,0,0,\dots\dots\dots 0,1,0,\dots\dots\dots\dots\dots\dots..)$$

and the 1 appears at the k-th place. Then for any $f \in ces^*(p,q,s)$ we have

$$f(x) = \textstyle\sum_{k=1}^{\infty} x_k f(e_k) = \sum_{k=1}^{\infty} x_k \, a_k. \qquad (4)$$

where $f(e_k) = a_k$. By theorem 1, the convergence of $\sum a_k x_k$ for every x in $ces(p,q,s)$ implies that $a \in \mu(t,s)$.

If $x \in ces(p,q,s)$ and if we take $a \in \mu(t,s)$, then by theorem 1, $\sum a_k x_k$ converges and clearly defines a linear functional on $ces(p,q,s)$. Using the same kind of argument as in theorem 1, it is easy to check that

$$\sum_{k=1}^{\infty} |a_k x_k| \le E\left(\sum_{r=0}^{\infty} Q_{2^r}^{s(t_{r-1})} \left(Q_{2^r} \max_r \left|\frac{a_k}{q_k}\right|\right)^{t_r} E^{-t_r} + 1\right) g(x)$$

whenever $g(x) \le 1$, where $g(x)$ is defined by (2).

Hence $\sum a_k x_k$ defines an element of $ces^*(p,q,s)$.

Furthermore, it is easy to see that representation (4) is unique. Hence we can define a mapping

$$T: \ ces^*(p,q,s) \to \mu(t,s).$$

By $T(f) = (a_1, a_2, \dots\dots\dots\dots\dots\dots)$ where the a_k appears in representation (4). It is evident that T is linear and bijective. Hence $ces^*(p,q,s)$ is isomorphic to $\mu(t,s)$.

3. Matrix Transformations

In the following theorems we shall characterize the matrix classes $(ces(p,q,s),l_\infty)$ and $(ces(p,q,s),c)$. Let $A=(a_{n,k})\ n,k=1,2,\dots\dots$ be an infinite matrix of complex numbers and *X, Y* two subsets of the space of complex sequences. We say that the matrix A defines a matrix transformation from X into Y and denote it by $A\in(X,Y)$ if for every sequence $x=(x_k)\in X$ the sequence $A(x)=A_n(x)$ is in Y, where

$$A_n(x)=\sum_{k=1}^{\infty}a_{n,k}x_k$$

provided the series on the right is convergent.

Theorem 3: Let $1<p_r\le \sup_r p_r<\infty$. Then $A\in(ces(p,q,s),l_\infty)$ if and only if there exists an integer $E>1$, such that $U(E,s)<\infty$, where

$U(E,s)=\sup_n\sum_{r=0}^{\infty}\left(Q_{2^r}A_r(n)\right)^{t_r}(Q_{2^r})^{s(t_r-1)}E^{-t_r}$ and $\frac{1}{p_r}+\frac{1}{t_r}=1,\ r=0,1,2,\dots\dots\dots\dots\dots$

Proof: Sufficiency: Suppose there exists an integer $E>1$, such that $U(E,s)<\infty$. Then by inequality (1), we have

$$\sum_{k=1}^{\infty}\left|a_{n,k}x_k\right|=\sum_{r=0}^{\infty}\sum_r\left|\frac{a_{n,k}}{q_k}q_kx_k\right|=\sum_{r=0}^{\infty}\sum_r\left|\frac{a_{n,k}}{q_k}\right||q_kx_k|$$

$$\le\sum_{r=0}^{\infty}\max_r\left|\frac{a_{n,k}}{q_k}\right|\sum_r|q_kx_k|$$

$$=\sum_{r=0}^{\infty}(Q_{2^r})^{\frac{s}{p_r}}\ Q_{2^r}\max_r\left|\frac{a_{n,k}}{q_k}\right|\ (Q_{2^r})^{-\frac{s}{p_r}}\frac{1}{Q_{2^r}}\sum_r|q_kx_k|$$

$$\le E\sum_{r=0}^{\infty}\left\{(Q_{2^r})^{\frac{s\,t_r}{p_r}}(Q_{2^r}A_r(n))^{t_r}E^{-t_r}+\left((Q_{2^r})^{-\frac{s\,t_r}{p_r}}\frac{1}{Q_{2^r}}\sum_r|q_kx_k|\right)^{p_r}\right\}$$

$$\le E\left\{\sum_{r=0}^{\infty}(Q_{2^r})^{s\,(t_r-1)}(Q_{2^r}A_r(n))^{t_r}E^{-t_r}+\sum_{r=0}^{\infty}(Q_{2^r})^{-s}\left(\frac{1}{Q_{2^r}}\sum_r|q_kx_k|\right)^{p_r}\right\}$$

$$<\infty.$$

Therefore, $A\in(ces(p,q,s),l_\infty)$.

Necessity: Suppose that $A\in(ces(p,q,s),l_\infty)$, but

$$\sup_n\sum_{r=0}^{\infty}\left(Q_{2^r}A_r(n)\right)^{t_r}(Q_{2^r})^{s\,(t_r-1)}E^{-t_r}=\infty\ for\ every\ integer\ E>1.$$

Then $\sum_{k=1}^{\infty}a_{n,k}x_k$ converges for every n and $x\in ces(p,q,s)$,

whence $(a_{n,k})_{k=1,2,\dots\dots}\in ces^{+}(p,q,s)$ for every *n*. By theorem 1, it follows that each A_n defined by

$$A_n(x)=\sum_{k=1}^{\infty}a_{n,k}x_k$$

is an element of $ces^{*}(p,q,s)$. Since $ces(p,q,s)$ is complete and since $\sup_n|A_n(x)|<\infty$ on $ces(p,q,s)$, by the uniform boundedness principle there exists a number *L* independent of *n* and a number $\delta<1$, such that

$$|A_n(x)|\le L \qquad (5)$$

for every *n* and $x\in S[\theta,\delta]$, where $S[\theta,\delta]$ is the closed sphere in $ces(p,q,s)$ with centre at the origin θ and radius δ.

Now choose an integer $G > 1$, such that

$$G\delta^{M} > L.$$

Since

$$\sup_{n} \sum_{r=0}^{\infty} (Q_{2^r} A_r(n))^{t_r} \, (Q_{2^r})^{s\,(t_r-1)} \, G^{-t_r} = \infty$$

there exists an integer $m_0 > 1$, such that

$$R = \sum_{r=0}^{m_0} (Q_{2^r} A_r(n))^{t_r} \, (Q_{2^r})^{s\,(t_r-1)} \, G^{-t_r} > 1 \qquad (6)$$

Define a sequence $x = (x_k)$ as follows:

$$x_k = 0 \;\; if \;\; k \geq 2^{m_0+1}$$

$$x_{N(r)} = Q_{2^r}^{t_r} \delta^{M/p_r} \left(sgn\, a_{n,N(r)}\right) \left|a_{n,N(r)}\right|^{t_r-1} R^{-1} G^{-t_r/p_r} (Q_{2^r})^{s\,(t_r-1)}$$

and $x_k = 0 \;\; if \; k \neq N(r)$ for $0 \leq r \leq m_0$, where $N(r)$ is the smallest integer such that

$$\left|a_{n,N(r)}\right| = \max_{r} \left|\frac{a_{n,k}}{q_k}\right|$$

Then one can easily show that $g(x) \leq \delta$ but $|A_n(x)| > L$, which contradicts (5).This complete the proof of the theorem.

Theorem 4. Let $1 < p_r \leq \sup_{r} p_r < \infty$. Then $A \in (ces(p,q,s), c)$ if and only if

(i) $a_{n,k} \to \alpha_k (n \to \infty, k \; is \; fixed)$ and

(ii) there exists an integer $E > 1$, such that $U(E,s) < \infty$, where

$U(E,s) = \sup_{n} \sum_{r=0}^{\infty} \left(Q_{2^r} A_r(n)\right)^{t_r} \; (Q_{2^r})^{s\,(t_r-1)} E^{-t_r}$ and $\frac{1}{p_r} + \frac{1}{t_r} = 1,\; r = 0,1,2,\ldots\ldots\ldots\ldots\ldots$

Proof: Necessity. Suppose $A \in (ces(p,q,s), c)$. Then $A_n(x)$exists for each $n \geq 1$ *and* $\lim_{n \to \infty} A_n(x)$ exists for every $x \in ces(p,q,s)$. Therefore by an argument similar to that in theorem 3 we have condition (ii). Condition (i) is obtained by taking $x = e_k \in ces(p,q,s)$, where e_k is a sequence with 1 at the k-th place and zeros elsewhere.

Sufficiency. The conditions of the theorem imply that

$$\sum_{r=0}^{\infty} \left(Q_{2^r} \max_{r} \left|\frac{\alpha_k}{q_k}\right|\right)^{t_r} (Q_{2^r})^{s\,(t_r-1)} E^{-t_r} \leq U(E,s) < \infty \qquad (7)$$

By (7) it is easy to check that $\sum_k \alpha_k \, x_k$ is absolutely convergent for each $x \in ces(p,q,s)$. For each $x \in ces(p,q,s)$ and $\varepsilon > 0$, we can choose an integer $m_0 > 1$, such that

$$g_{m_0}(x) = \sum_{r=m_0}^{\infty} (Q_{2^r})^{-s} \left(\frac{1}{Q_{2^r}} \sum_{r} |q_k x_k|\right)^{p_r} < \varepsilon^{M}$$

Then by inequality (1), we have

$$\sum_{k=2^{m_0}}^{\infty} \left|a_{n,k} - \alpha_k\right| |x_k| \leq E\left(\sum_{r=m_0}^{\infty} (Q_{2^r})^{s\,(t_r-1)} \, (Q_{2^r} B_r(n))^{t_r} \, E^{-t_r} + 1\right) \left(g_{m_0}(x)\right)^{1/M}$$

$$< E(2U(E,s) + 1)\varepsilon,$$

where $B_r(n) = \max_{r} \left|\frac{a_{n,k} - \alpha_k}{q_k}\right|$ and

$$\sum_{r=m_0}^{\infty} (Q_{2^r})^{s\,(t_r-1)} \left(Q_{2^r} B_r(n)\right)^{t_r} E^{-t_r} \leq 2U(E,s) < \infty$$

It follows immediately that $\lim_{n \to \infty} \sum_{k=1}^{\infty} a_{n,k}\, x_k = \sum_{k=1}^{\infty} \alpha_k x_k$

This shows that $A \in (ces(p,q,s), c)$ which proved the theorem.

Corollary 1. Let $1 < p_r \leq \sup_r p_r < \infty$. Then $A \in (ces(p,q,s), c_0)$ if and only if

(i) $a_{n,k} \to 0$ $(n \to \infty,\ k\ is\ fixed)$

(ii) there exists an integer *E >1* such that $U(E,s) < \infty$, where

$$U(E,s) = \sup_n \sum_{r=0}^{\infty} (Q_{2^r} A_r(n))^{t_r}\, (Q_{2^r})^{s\,(t_r - 1)} E^{-t_r}\ and\ \frac{1}{p_r} + \frac{1}{t_r} = 1, \qquad r = 0, 1, 2, \dots \dots \dots$$

Remarks:

(1) If $s = 0$ then we get the results of Rahman and Karim [11]

(2) If $s = 0,\ q_n = 1$ for every n then we get the results of Lim [10]

(3) When $s = 0,\ q_n = 1\ and\ p_n = p$ for all n then the results of Lim [9] follows.

(4) If $s \geq 1$ then specializing the sequences (p_n) and (q_n) we get many unknown results.

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