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RESEARCH ARTICLE

ASSOCIATED GRAPHS AND CHAIN MAPS

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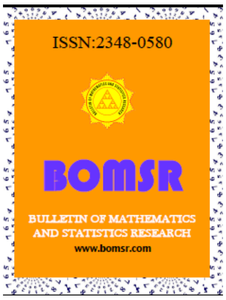

Abstract: In this paper we defined the associated graph constructed to a cellular folding defined on regular CW-complexes. These graphs declare the effect of a cellular folding on the complex.Besides we studied the properties of this graph and we proved that it is connected and vertex transitive if the cellular folding is neat.Finally, by using chain maps and homology groups we obtained the necessary and sufficient conditions for a cellular map to be cellular folding and neat cellular folding respectively.

Key words: Cellular folding, chain map, regular CW-complexes, vertex transitive, neat folding.

1.INTRODUCTION

The study of foldings of a manifold into anther manifold began with S.A. Robertson's work on isometric folding of Riemannian manifolds [11]. After several attempts of generalizing the notion of isometric foldings, regular foldings were first studies by S.A. Robertson, H.R. Forran and E.El-Kholy [2]. The notion of cellular foldings are invented by E.El-Kholy and H.A.AL-Khurassani [1]. Different types of foldings are introduced by E.EL-Kholy and others

[4, 5, 2].

a) A cell decomposition of a topological space X is a decomposition of $\bar{X}$ into disjoint open cells such that for each cell e of the decomposition, the boundary $\partial e = \bar{e}$ - e is a union of lower dimensional cells of the decomposition. The set of cells of a cell decomposition of a topological space is called cell complex, [10].

A pair (x, ζ) consisting of a Hausdorff space X and a cell – decomposition ζ of X is called CW – complex if the following three axioms are satisfied:

1– (Characteistic Maps): For each n – cell e ϵ ζ there is a continuous map $\Phi_{e|int\,(D_n)} \rightarrow \mathrm{int}(\mathrm{D_n})$ → e and taking S^{n-1} into X^{n-1}.

2 – (Closure Finiteness) : For any cell e ϵ ζ the closure $\bar{e}$ intesects only a finite number of othe cells in ζ.

3 – (Weak Topology):A subset A⊆ X is closed iff A ∩ $\bar{e}$ is closed in X for each e ϵ ζ, [9].

A CW – complex is said to be regula if all its attaching maps are homemorphisms. If each closed n – cell is homeomorphic to a closed Euclidean n – cell [9]. A topological space that admits the structure of a regular CW – complex is temed a regular CW – space.

(b) Let K and L be cellular complexes and f : K → L is a cellular map if

(i) for each cell σ ϵ K, f(σ) is a cell in L,

(ii) dim f(σ) ≤ dim (σ), [8].

(c) Let K and L be regular CW – complexes of the same dimension and K be equipped with finite cellular subdivision such that each closed n- cell. A cellular map f : K → L is a cellular folding iff: (i) for each i-cell σ^i ϵ K, f(σ^i) is an i-cell in L, i.e., f maps i-cells to i-cells.

(ii) if $\bar{\sigma}$ contains n vertices, then $\overline{f(\sigma)}$ must contains n distinct vertices.

In the case of directed complexes it is also required that f maps diected i-cells of k to i-cells of L but of the same diection, [6].

A cellular folding f : K → L is neat if L^n - L^{n-1} consists of a single n – cell, interior L. The set of all cellular foldings of K into L is donated by C(K,L) and the set of all neat foldings of K into L by$\mathcal{N}$(K,L).

(d) If f ϵ C(K,L), then x ϵ K is said to be a singularity of f iff f is not a local homeomorphism at x. The set of all singularities of f corresponds to the "folds" of the map. This set associates a cell decomposition C_f of M. if M is a surface, then the edges and vertices of C_f from a graph Γ_f embedded in M, [7].

(e) Let f : K → L be a continuous function. If, for each k – chain C in K , f (C) is a k – chain in L and if the diagram

$$\begin{tikzcd} C_k(K) \arrow[r, "f"] \arrow[d, "\partial"] & C_k(L) \arrow[d, "\partial"] \\ C_{k-1}(K) \arrow[r, "f"] & C_{k-1}(L) \end{tikzcd}$$

Commutes, then f : K → L is a chain function from K to L, [8].

(f) The set S_n of all permutations on n objects forms a group of order n!, called the symmetric goup of degee n, the law of compositionbeing that fo maps of the objects onto themselves. A group of permutations is said to be transitive if, given any pair of letters a, b (which need not be distinct), there exists at least one permutation in the group which transforms a into b, [12]. Otherwise the group is called in transitive. And is said to be 1 – transitive if for any pair of letters a, b, there exists a unique element x of the group such that a ∗ x = b.

2- The associated graph:

Let f : K → L be a cellular folding. By using the cell subdivision C_j of K we can define the associated graph G_j are just the n – cells of C_j and the cellular folding f as follos:

The vertices of G_j are just the n – cells of C_j $and\ if$ σ and σ' are distinct n – cells of C_j such that f (σ) = f (σ'), then there exists an edge E with end points σ and σ'. We then say that E is an edge in G_j $with\ end\ points$ σ and σ'.

The graph G_j can be realized as a $\widetilde{G_j}$ embedded in R^3 as follows. For each n – cells σ , σ' choose any points $\mathfrak{v}$ ϵ σ, $\mathfrak{v}'$ ϵ σ'. If σ and σ' are end points of an edge E, then we can join $\mathfrak{v}$ to $\mathfrak{v}'$ by an arc e in R^3 that runs from v through σ and σ' to $\mathfrak{v}'$ crossing E transversely at a single point [3]. The corespondence σ ↔ v, E ↔ e is trivially a graph isomorphism from G_j to $\widetilde{G_j}$.

It should be noted that the graph G_j has no multiple edges, no loops and generally disconnected.

In this paper by a complex we mean a regula CW – complex.

Examples(2-1):

(a) Let K be a complex with the cellular subdivisions gien in Fig.(1-a).Let f : K → K be cellular foldings defined by: f(v_2 , v_2 , v_5 , v_8 , v_{11}) = (v_4 , v_7 , v_{10} v_{13}),
f(e_1 , e_4 , e_6 . e_9 , e_{11} , e_{14} , e_{16} , e_{19} , e_{21})

$= (e_3, \ e_5 , e_8 . e_{10} , e_{13} , e_{15} , e_{18} , e_{20} , e_{23})$ and f(σ_i) = σ_{i+1}, I = 1,3,5,7,, whee the omitted 0,1,2- cells through this paper will be mapped to themselves.The graph G_j in this case has ten vertices and five edges as shown in Fig. (1 – b).

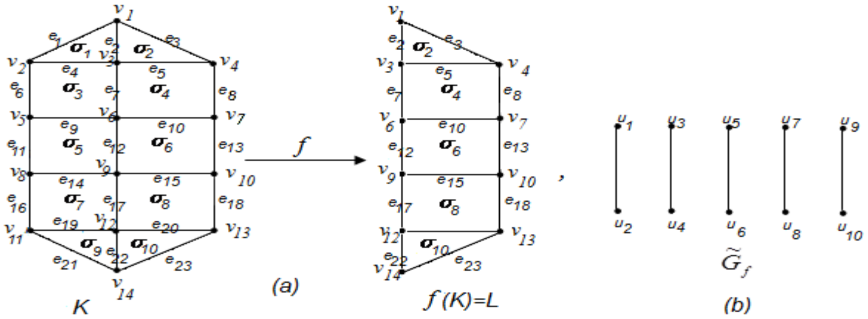

Fig. (1)

(b)Consider that complex K shown in Fig.(2), which consists of one 2 –cell, seen 1-cells and seven 0-cells. Let f : K → K be a cellular folding defined as follow: $f\,(v_5 , v_6 , v_7) = (v_2 , v_3 , v_2)$, f(e_i) = e_2, i=5,6,7 and $f\ (\sigma) = \sigma$. The graph G_f in this case consists of a vertex only with no edges.

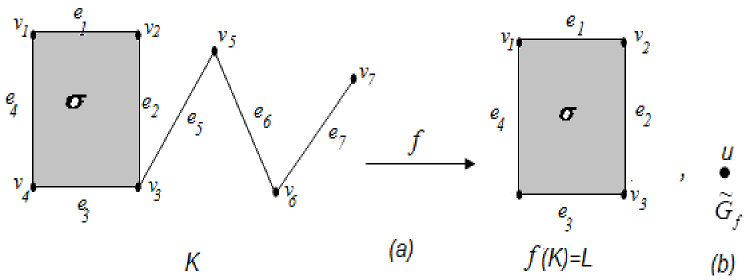

Fig. (2)

(c) Let K be a complex such that |K| is a cylindrical surface with a cellular subdivision consists of eight 0-cells, sixteen 1-cells and eight 2-cells, see Fig.(3). Let f : K → K be a cellular folding defined by : $f(v_5 , v_6 , v_7 , v_8) = (v_1 , v_3 , v_3\ v_3)$,

f($e_1 , \ e_2 , \ e_3 . \ e_4 , \ e_5 , \ e_6 , \ e_8 , \ e_{11} , \ e_{13} , \ e_{14}$)

$= (e_9, \ e_9 , \ e_9 . \ e_9 , \ e_9 , \ e_{15} , \ e_7 , \ e_9 , \ e_{10} , \ e_{16} , \ e_{15})$ and

f($\sigma_1, \ \sigma_2, \ \sigma_3, \ \sigma_4, \ \sigma_5, \ \sigma_8$) = $(\sigma_6, \ \sigma_6, \ \sigma_7 , \ \sigma_7, \ \sigma_6, \ \sigma_7)$.

This can be done by the composition of the following two cellular foldings: $f_1(v_5 , v_8) = (v_1 , v_3), f(e_1 , \ e_2 , \ e_6 . \ e_8 , e_{11} , \ e_{13} , \ e_{14})$

$= (e_3, \ e_4 , \ e_7 . \ e_9 , \ e_{10} , \ e_{15} , \ \ e_{16})$ and

f($\sigma_1, \ \sigma_2, \ \sigma_3, \ \sigma_4$) = $(\sigma_6, \ \sigma_6, \ \sigma_7 , \ \sigma_7, \ \sigma_6, \ \sigma_7)$.

The graph G_j in this case has eight vertices and twelve edges see Fig (3-b).

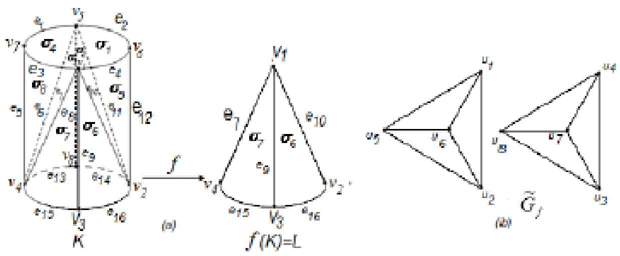

Fig.(3)

(d) Consider a complex K such that |K| is a tous with four 0-cells, eight 1 – cells, eight 1-cells and four 2-cells, see Fig (4-a). Let f : K → K be a cellular folding gien by $f_1(v_i) = (v_i)$, i=1,2,3,4 $f(e_3 , \ e_4) = \ (e_2 . \ e_1)$ and f(σ_2, σ_4) = $(\sigma_1, \ \sigma_3)$. The graph G_f in this case has four vertices and two edges, see Fig. (4-b).

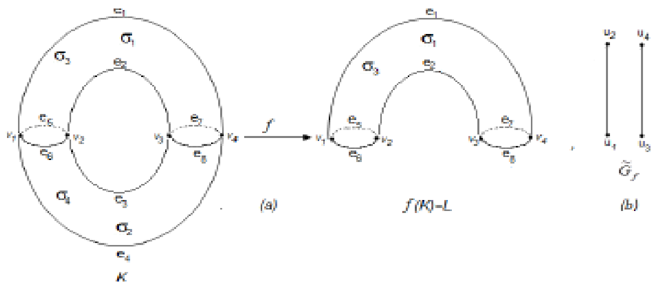

Fig.(4)

3-Properties of the associated graph:

Some of the properties of the associated graph can be characterized by the following theorems:

Theorem(3-1):

Let K and L be complexes of the dimensions n, f ϵ C (K, L). The associated graph G_f is disconnected unless f is a neat cellular folding.

Proof:

Let σ_1 and σ_2 be distinct n – cells of $K^{(n)}$, and let $\sigma_1 \sim \sigma_2$ means $f(\sigma_1) = f(\sigma_2)$. It is clear that the relation $\sim$ is an equivalence relation. Hence the quotient set $K^{(n)}/\sim$ = { [σ] }, σ ϵ $K^{(n)}$ is a partitionon $K^{(n)}$, where [σ] is the equivalence class of any n – cells σ. It follows that G_f has more than one component otherwise all the n – cells of K will be mapped to the same n-cells of L which in fact is the case of cellular neat folding. In the last case there will be a unique equivalence class [σ] and hence the graph G_f is connected.

It follows fom the aboe theorem that the components of the graph G_f is equal to the number of the equivalence classes generated by the relation $\sim$.

Theorem (3.2):

Let K and L be complexes of the same dimension n, f ϵ C(K,L) a cellular folding. Then each component of G_f is vetex transitive on itself.

Proof:

From Theorem (3.1) the equivalence relation defined on the n – cells $K^{(n)}$ of K defines a partition { [σ] } , σ ϵ $K^{(n)}$ on $K^{(n)}$, where each equivalence class represents a component of G_f . Now, consider one of these components G_f^i, with say r vertices, i.e., | | $V(G_f^i)$| = r. Each vertex of G_f^i is adjacent to the other vertices in the component, then any permutation of the set $V(G_f^i)$ is an automorphisms of G_f^i. Thus the set of all permutations (automorphisms) form a group which is the symmetric group S_r acting on the $V(G_f^i)$. The orbit of any σ ϵ $V(G_f^i)$ under S_r is the whole set $V(G_f^i)$. i.e., $V(G_f^i)$ has a single orbit and hence the automorphism group S_r is transitive on $V(G_f^i)$.

Results(3-3):

Let f : K→L be a neat cellular folding:

1) The sysmmetic group S_r , = | $K^{(n)}$| acts 1-transitively on the graph G_f .
2) G_f is vertex transitive.
3) $From$ the above results we conclude that the graph G_f of a neat cellular folding is a complex graph.

Example (3 -4):

Consider the complex K shown in Fig.(5-a), which consists of four 2-cells, eight 1-cells and fie 0-cells. Let f : K → K be a cellular folding defined as follows : $f(v_4, v_5) = (v_3. v_2)$,

$f(e_4, e_5, e_6. e_7, e_8) = (e_3, e_1, e_2. e_2, e_2)$ and f(σ_i) = σ_i , i= 1,2,3,4. The graph G_f in this case is complete, see Fig. (5-b).

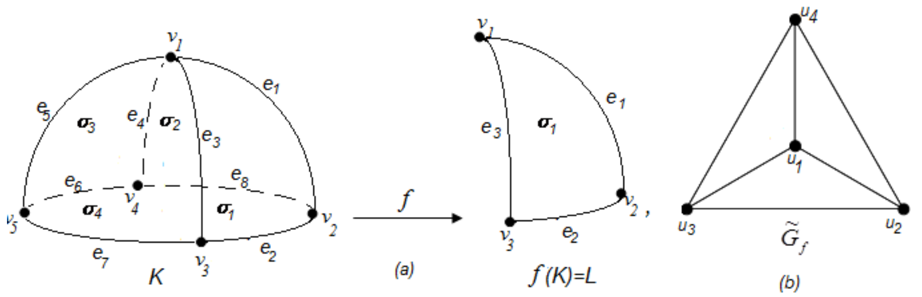

Fig.(5)

(4) Chain maps and cellular folding:

The following theorem gives the necessary and sufficient condition for a cellular map to be a cellular folding.

Theorem(4-1):

Let K and L be complexes of the same dimension n and f : K → L be a cellular map such that f(K)= L + K. Then f is a cellula folding if and only if the map f_p : C_p (K) → C_p (L), between chain complexes (C_p (M), ∂_p), (C_p (M), ∂'_p) is a chain map.

Proof:

Let f : K → L be a cellular n it is a cellular map and for each p-cell σ ϵ K we can define a homomorphism f_p : C_p (K) → C_p (L) by

$$f_p = \begin{cases} f(\sigma), & if\ f(\sigma) is\ a\ p-cell\ in\ L \\ \quad \varphi & if \dim(f(\sigma)) < p \end{cases}$$

And since cellular foldings map p-cells to p-cells [2], f_p (σ_λ) is Thus for a p-chain where and are p-cells in L for all λ. Thus for a p - chain C = $a_1\ \sigma^p{}_1 + a_2\, \sigma^p{}_2 +$...............+ $a_k\ \sigma^p{}_k$ ϵ C_p (K), where a'_λ S ϵ Z and σ'_λ s are p – cells in M, f_p (C) = f_p ($a_1\, \sigma^p{}_1 + a_2\, \sigma^p{}_2 +$.........+ $a_k\ \sigma^p{}_k$) = $a_1\ f_p\, (\sigma^p{}_1) + a_2\, f_p\, (\sigma^p{}_2) +$.........+ $a_k\ f_p\ (\sigma^p{}_k)$ ϵ C_p (L).

Now, since the closures of $\sigma^p{}_\lambda$ and $f\ (\sigma^p{}_\lambda)$ have the same number of distinct vertices, the f_{p-1} o $\partial_p = \partial'_p\ o\, f_p$, where ∂_p : C_p (K) →C_{p-1} (K) and ∂'_p : C_p (K) → C_{p-1} (L) are the boundary operators, that is to say that following diagram commutes

$$\begin{tikzcd} C_p(K) \arrow[r, "f_p"] \arrow[d, "\partial_p"'] & C_p(L) \arrow[d, "\partial'_p"] \\ C_{p-1}(K) \arrow[r, "f_{p-1}"] & C_{p-1}(L) \end{tikzcd}$$

and hence f_p is a chain map. Conversely, suppose f is not a cellular folding then there exists a j-cells σ in K such that $f(\sigma)$ is an m – cell in L, where j ≠m. Since f_p is a homomorphism from the p^{th} - chain of K to the p^{th} - chain of L, then

$$f_j\Big(\sum_{j=1}^{n-1}\lambda_i\sigma_i^{(j)}+\lambda_n\sigma\Big)=\sum_{i=1}^{n-1}\lambda_i f_j(\sigma_i^{(j)})+\lambda_n f(\sigma),$$

But $f(\sigma)$ is not a j-cell, then f_j can not be a j-chain map and hence our assumption is false, and we have the result.

Examples (4-2)

Let K be a complex such that |K| is the infinite strip {(x,y): -∞< x< $\infty, 0 \le y \le l$} equipped with an infinite numbe of 2-cells such that the closure of each 2-cell consists of four 0-cells and fou 1cells, P_4 . Let L be a complex with 0-cells, seen 1-cells and two 2-cells, Fig.(6). The cellular map f: K → L defined by : $f\ (v_i) = v'_i$ where I = 1,2,...,6, $f\ (v_i) = v'_j$, whee j = 1,2,...6 and (i-j) is a multiple of 6, $f\ (e_i) = e'_i$, i=1,11,21,….., $f\ (e_i) = e'_i$, I = 2,12,22,….., $f\ (e_i) = e'_3$, i=3,8,13,…………, $f\ (e_i) = e'_6$, i=6,16,26,………….., $f\ (e_i) = e'_7$, I = 7,17,27,…………

and $f_p = \begin{cases} \sigma'_1, & if\ \ i\ \ is\ odd, \\ \sigma'_2 & if\ i\ is\ even \end{cases}$

is a cellular folding.

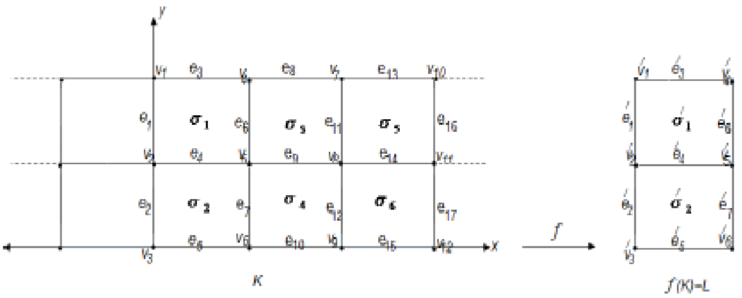

Fig.(6)

(a) Consider a complex K such that |K| = S^2, with cellular subdivision consisting of two 0-cells, four 1-cells and four 2-cells. Let f : K → K be a cellular by : $f(e_2, e_4) = (e_1.\ e_3)$, f(σ_i) = σ_i , i= 1,2,3,4. This map is a cellula folding with image consisting of to 0-cells, two 1-cellsand a single 2-cell, see Fig.(7)

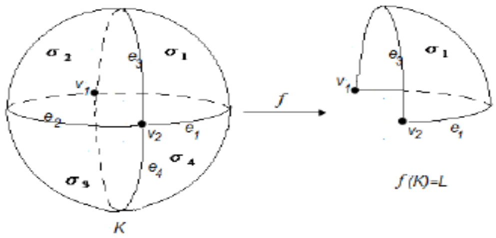

Fig.(7)

(b) Consider a complex K such that |K| is a tours with cellula subdivision consisting of three 0-cells, six 1-cells, six 1-cells and three 2-cells. Any cellular map f :K→ K which has two vertices in the image is not a cellula folding since f_1 in this case is not a chain map, see Fig.(8).

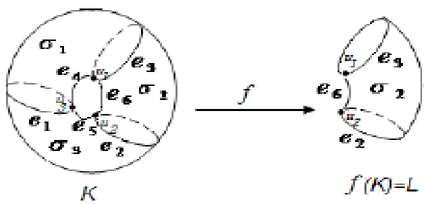

Fig.(8)

(c) Consider a complex K such that |K| = S^2, with cellular subdivision consisting of four 0-cells, six 1-cells, four 2-cells, see Fig.(9).

Let be f :K→ K be a cellular map defined by : $f(v_i) = v_i,\ \ i = 1, \dots\dots, 4$,

$f(e_2, e_3) = (e_1.\ e_4)$, f(σ_1) = σ_2 , i= 1,2,3,4. This map is not a cellular folding since $\overline{\sigma_1}$ and $\overline{f(\sigma_1)}$ do not containthe same number of vertices.

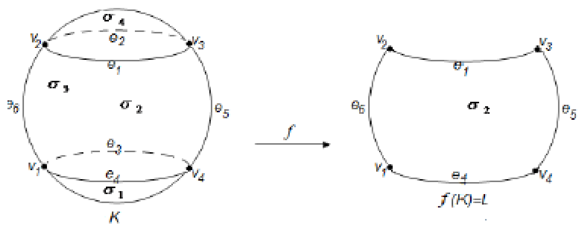

Fig.(9)

Result (4-3):

Let f : K →L, be a cellular folding. Then the induced homomorphism f_p^* : H_p(K) → H_p(L) with maps the generators of H_p(K) to either the generators of L or to zeroes. This follows directly from the fact the chain map f_j : C_p (K) → C_p (L) defines a homomorphism which has this property [2].

(5)Homology groups and neat cellular foldings:

The following theorem gives the necessary and sufficient condition for a cellular map to be a neat cellular folding.

Theoem (5.1):

Let K and L be complexes of the same dimension n. If f ϵ C (K , L), then f is neat if and only if the map f_p : C_p (K) → C_p (L) between chain complexes (C_p (M), ∂_p), (C_p (M), $\partial_p^{'}$) is a chain map and H_p (L) ≅ Ker f_* : H_p (K) → H_p (L), p ≥ 1 is the induced homomorphisms.

Proof:

Assuming that f is a neat folding, then it is a cellula folding and hence the map f_p : H_p (K) → H_p (L) between the chain complexes (C_p (M), ∂_p), (C_p (M), $\partial_p^{'}$) is a chain map. Now consider the induced homomorphism f_p : H_p (K) → H_p (L), there is a shot exact sequence

$$0 \rightarrow \ker\ f_* \xrightarrow{i^*} H_p\ (\ K\ ) \xrightarrow{f_*} \mathrm{Im}\ \ f_*$$

Where i^* is the induced homomorphism by the inclusion. Since f surjective, we have Im f_* ≅ H_p (L) but H_p (L) = 0, for neat cellular foldings, hence the aboe sequence will take the form

$$0 \rightarrow \ker\ f_* \xrightarrow{i^*} H_p\ (\ K\ ) \rightarrow\ 0$$

The exactness of this sequence implies that H_p (K) ≅ ker f_*.

Conversely, suppose f is a chain map between chain complexes, and H_p (K) ≅ ker f_*, but f is not neat, L^n - L^{n-1} consists of more than one n-cell. Thus H_o(K) ≅ Z^j , H_p (L) = 0

For p = 1,2,…..,n and H_p (K) ≅ H_p (K) ⊕ ker f_* ≅ ker f_* for p = 0, and hence the assumption is false and f is neat.

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