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RESEARCH ARTICLE

SOME PARTIAL CONVEXITY RESULTS OF HEAT EQUATION IN R^n

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ABSTRACT
In this paper, we are concerned with partial convexity of smooth solutions to heat equation. We prove that partial convexity of these solutions to the heat equation are preserved along the heat equation. Consequently we give a proof that some convex cones of these solutions Γ_k (see the definition in Section 2) are invariant cones.

Keywords: Heat equation; Convexity; Partial convexity.
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1. INTRODUCTİON

The convexity has been studied for a long time in partial differential equations and it is intimately related to geometric properties of solutions to partial differential equations. There are macroscopic and microscopic convexity principle in general to yield convex solutions. The macroscopic convexity principle developed from 1980s, which was obtained by Korevaar [1], Kennington [2] and for the general nonlinear partial differential equations by Alvarez-Lasry-Lions[3]. However this method has difficulties in some geometric partial differential equations on compact manifold. The microscopic convexity principle concentrates on establishing the constant rank theorem for convex solutions to partial differential equations. It is a powerful tool in producing convex solutions to partial differential equations via the continuity methods. Caffarelli-Friedman [4] proved a constant rank theorem for convex solutions of quasilinear elliptic equations in R^2, a similar result was also discovered by Yau [5] at the same time. Korevaar-Lewis [6] generalized these results to R^n. Convexity plays an important role in geometric flow. For example, Huisken [7] proved that the mean curvature flow deforms initial surface with positive curvature into a point, while the curvature remains positive along the mean curvature flow. For general geometric flow under some structural condition, Bian-Guan [8] proved that the evolving hypersurfaces are strictly convex if the initial hypersurface is convex.

In addition to convexity, partial convexity is also an interesting and important subject in analysis and geometry. So far as partial convexity of solution is concerned, there are usually two definitions of partial convexity for a function u : one is that the sum of the smallest k eigenvalues of the Hessian matrix $\{D^2u\}$ of u is positive; the other is that there exists a positive integer k, such that $\sigma_l(D^2u) > 0(\text{or} \geq 0)$ for $1 \leq l \leq k$, where $\sigma_l(D^2u)$ is the l-th elementary symmetric function of the eigenvalues of D^2u. In this paper, we will mainly prove that these partial convexity properties are preserved for smooth solutions to the heat equation and therefore we will give a direct proof that these convex cones $\Gamma_k = \{\lambda = \lambda(D^2u) \in R^n : \sigma_l(\lambda) > 0, 1 \leq l \leq k\}$ (see the detail definition in Section 2) are invariant cones along the heat equation.

We first recall some results concerning partial convexity. For elliptic case, Han-Ma-Wu [9] obtained a constant rank theorem for the k-convex solutions to semilinear elliptic partial differential equations and obtained an existence theorem for k-convex starshaped hypersurface with prescribed mean curvature in R^{n+1}. For parabolic case, a famous result is that in 1976, Brascamp-Lieb [10] established the logarithmic concavity of the fundamental solution of diffusion equation with convex potential in bounded convex domain in R^n. As a consequence, they proved the logarithmic concavity of the first eigenfunction of Laplacian equation in convex domain. This logarithmic concavity property is reproved by using an ingenious P-function and deformation method in a the paper [11]. In geometry, the assumption on the curvature of surface, such as positive Ricci curvature or positive curvature operator in some sense can be interpreted as partial convexity conditions. For parabolic case, Hu-Ma [12] obtained a constant rank theorem of the space-time Hessian for the space-time convex solution to standard heat equation. For geometric evolution equation, invariant cones play important roles and hence geometric quantities satisfying partial convexity properties can be used to construct invariant cones. For example, Huisken-Sinestrari [13] classified the compact 2-convex hypersurfaces in R^n using the technique of mean curvature flow. They proved that if $F_0 : M^n \to R^{n+1}$ be a smooth immersion of a closed n- dimensional hypersurface, with $n \geq 3$ and if $M_0 = F_0(M)$ is two convex, i.e., $\lambda_1 + \lambda_2 > 0$ everywhere on M_0; then there exists a mean curvature flow with surgeries starting from M_0 which terminates after a finite number of steps. As corollary, they classified all closed hypersurface with two positive curvature operator. For Ricci flow, there are plenty of such results. For example, Hamilton [14] ([15]) proved that if a compact 3-manifold (4-manifold) M^n admits a Riemannian metric g_0 with positive Ricci curvature (positive curvature operator), then this metric can be deformed to a metric g of constant positive sectional curvature. In addition to these, Chen [16] and Böhm-Wilking [17] studied the classification of compact Riemannian manifolds with 2-positive curvature operator via Ricci flow. They proved that if (M^n, g) has 2-positive curvature operator, then the normalized Ricci flow evolves the initial metric g to a constant curvature limit metric. In their proof, they constructed a pinching family with initial cone being the cone of 2-positive curvature operator. With the existence of such pinching family, they could prove the convergence of the normalized Ricci flow to a constant curvature limit metric.

As significance and broad applications of the partial convexity as illustrated in the above, the partial convexity property of solution to differential equations is well worth studying and hence we consider this subject in this paper. We first consider a model of smooth solution to the heat equation in R^n:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u \quad in \quad R^n \times [0,T), \\ u\Big|_{t=0} = u_0(x) \quad on \quad R^n. \end{cases} \tag{1}$$

We prove some partial convexity results for solutions of heat equation (1). The first result is the following result concerning the preservation of the partial convexity .

Theorem 1. Let $u(x,t)$ be a smooth solution to the heat equation (1) in $R^n \times [0,T)$. We denote by D^2u the Hessian matrix $\{u_{ij}\}$ of the solution u and assume $D^2u \in C_{exp}$. If the initial data u_0 is k convex, i.e., the sum of smallest k eigenvalues of the matrix D^2u_0 is nonnegative (positive), then k convexity will be preserved for solutions by the heat equation, i.e., for any $t > 0$, the sum of smallest k eigenvalues of $D^2u(x,t)$ is nonnegative (positive).

Since the heat equation preserve nonnegativeness of the sum of smallest k eigenvalues of solutions, we may naturally ask whether it preserves the convex cones Γ_k. For the case of Euclidean space and the cone Γ_2, we have the following result.

Theorem 2. Let u be a smooth solution to the heat equation (1) in $R^n \times [0,T)$ and denote by D^2u the Hessian matrix $\{u_{ij}\}$ of solution u. We also assume $D^2u \in C_{exp}$. If

$$\sigma_i(D^2u) \geq (>) 0 \quad \text{at} \quad t = 0 \quad \text{for} \quad i = 1,2, \tag{2}$$

then for any $t > 0$, we have

$$\sigma_i(D^2u) \geq (>) 0 \quad \text{for} \quad i = 1,2. \tag{3}$$

The paper is organized as follows. We first recall the definitions and some fundamental facts concerning the elementary symmetric functions σ_k in Section 2. In section 3, we prove Theorem 1 and Theorem 2 that partial convexity for the smooth solutions of heat equation is preserved for the case $k = 2$ on n-dimensional Euclidean space R^n.

2. Preliminary

In this section, we recall the definition and some basic properties of the elementary symmetric functions of $\lambda = (\lambda_1, \cdots, \lambda_n)$.

Definition 1. For any $k = 1, 2, \cdots, n$, we set

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}, \quad \forall \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in R^n.$$

We also set $\sigma_0(\lambda) \equiv 1$ and $\sigma_k(\lambda) = 0$ for $k > n$.

For a symmetric matrix W, we define by letting $\sigma_k(W) = \sigma_k(\lambda(W))$, where $\lambda(W) = (\lambda_1(W), \cdots, \lambda_n(W))$ are the eigenvalues of the symmetric matrix.

In addition, we define

$$\Gamma_k = \{\lambda \in R^n : \sigma_1(\lambda) > 0, \sigma_2(\lambda) > 0, \cdots, \sigma_k(\lambda) > 0\}.$$

Obviously Γ_k contains the positive cone $\Gamma_n = \{\lambda \in R^n : \lambda_1 > 0, \lambda_2 > 0, \cdots, \lambda_n > 0\}$. Γ_k is symmetric in the sense that if $\lambda \in \Gamma_k$, then any permutation of λ also lies in Γ_k.

Let us denote by $\sigma_k(\lambda \mid i)$ the sum of the terms of $\sigma_k(\lambda)$ not containing the factor λ_i. We list some basic properties of elementary symmetric functions which will be frequently used in the following calculation.

Proposition 1. For any $k=0,1,\cdots,n$, $i=1,2,\cdots,n$, and $\lambda\in R^n$, the following identities hold:

$$\frac{\partial\sigma_{k+1}}{\partial\lambda_i}(\lambda)=\sigma_k(\lambda\,|\,i), \tag{4}$$

$$\sigma_{k+1}(\lambda)=\sigma_{k+1}(\lambda\,|\,i)+\lambda_i\sigma_k(\lambda\,|\,i), \tag{5}$$

$$\sum_{i=1}^{n}\sigma_k(\lambda\,|\,i)=(n-k)\sigma_k(\lambda), \tag{6}$$

$$\sum_{i=1}^{n}\lambda_i\sigma_k(\lambda\,|\,i)=(k+1)\sigma_{k+1}(\lambda). \tag{7}$$

Proposition 2. 1. If $\lambda\in\Gamma_k$ for $k\in\{1,2,\cdots,n\}$, then we have $\sigma_h(\lambda\,|\,i)>0$, for any $h\in\{0,1\cdots,k-1\}$ and $i\in\{1,2,\cdots,n\}$.

2. Let $W=\{W_{ij}\}$ be a symmetric matrix such that its eigenvalues belong to Γ_{k-1}, and set $F(W)=\sigma_k\sigma_{k-1}(W)$, then F is concave on Γ_{k-1}, that is

$$\sum_{i,j,k,l=1}^{n}\frac{\partial^2 F}{\partial W_{ij}\partial W_{kl}}(W)\eta_{ij}\eta_{kl}\le 0,\quad \text{for any } \{\eta_{ij}\}\in R^{n\times n}. \tag{8}$$

Proposition 1 is standard which can be directly checked. For the proof of **Proposition 2**, the readers can consult [18] for example.

3. Partial convexity of heat equation in R^n

In this section, we consider partial convexity preservation of heat equation in $n-$ dimensional Euclidean space R^n. We first note that the Euclidean space is noncompact. For rigorous usage of maximum principle, we need to restrict the solution class such that the Hessian of the solution $D^2u\in C_{exp}$, i.e., there exists constant A, such that $|D^2u|\le Aexp(A|x|^2)$. We demand the Hessian of the solution at infinity has no more than exponential growth rate, so that we can construct "barrier function" to guarantee that our auxiliary function constructed in the following theorems can achieve the corresponding minimum value at an interior point. However this is a routine verification by using approximation of the domain and ε perturbation of $exp(-B\frac{|x|^2}{t})$ in the auxiliary function just as in the proof of the existence of the Cauchy problem of heat equation. Therefore we omit the routine process here and just assume the auxiliary functions attain interior minimum value.

In the following we give a proof of Theorem 1 which states that the nonnegativeness of the sum of the smallest k eigenvalues of D^2u is preserved along the heat equation in R^n.

Proof of Theorem 1. The proof is standard. We need to make some simplifications. By perturbation argument, we may assume the sum of the smallest k eigenvalues of Hessian for D^2u_0 of the initial data u_0 is positive, otherwise we may consider $u(x,t)+\varepsilon(|x|^2\,2n-t)$ instead and let $\varepsilon\to 0$.

We may assume the sum of the smallest k eigenvalues of $D^2u(x,t)$ vanishes at some space-time point, otherwise Theorem 1 naturally holds. We may assume t_0 be the first vanishing time and assume the vanishing point be attained at (x_0,t_0). we can also rotate the coordinates such that the matrix D^2u is diagonal and its eigenvalues satisfy $u_{nn}\ge u_{n-1n-1}\ge\cdots\ge u_{11}$ at this point.

Therefore at the space-time point (x_0,t_0) , we have

$$\begin{aligned}&\frac{\partial}{\partial t}(u_{11}+\cdots+u_{kk})-\Delta(u_{11}+\cdots+u_{kk})\\&=(\frac{\partial u}{\partial t}-\Delta u)_{11}+\cdots+(\frac{\partial u}{\partial t}-\Delta u)_{kk}=0,\end{aligned} \tag{9}$$

where we have used the heat equation (1) in the last equality of (9).

From the above calculation and using the maximum principle we know that the sum of the smallest k eigenvalues of D^2u is non-negative along the heat equation (1). From the strict parabolic maximum principle, the case of strict inequality in **Theorem 1** follows.

Remark 1. In the above proof, we follow a method of Lions et al. [19], where they used a special form of parabolic maximum principle to find necessary and sufficient conditions on preservation of convexity along parabolic equations.

In the following, we will use the maximum principle to prove Theorem 2.

Proof of Theorem 2. For convenience, we divide the proof into 4 steps.

Step 1: For $i=1$, we take second derivatives of equation (1) with respect to the variables x_i and x_j to get

$$\frac{\partial u_{ij}}{\partial t}=\Delta u_{ij}.$$

By taking trace, we see the above equation leads to

$$\frac{\partial\sigma_1(D^2u)}{\partial t}=\Delta\sigma_1(D^2u). \tag{10}$$

From (10) and the strong maximum principle, we obtain $\sigma_1(D^2u)(t)>0$ for any $t>0$ provided $\sigma_1(D^2u_0)\geq 0$ but not identically equal to 0.

Step 2: For $i=2$, we first make some simplifications. From Step 1, we may assume $\sigma_1(D^2u)>0$. We may also assume

$$\sigma_2(D^2u_0)>0, \tag{11}$$

otherwise we may consider $u(x,t)+\varepsilon(|x|^2\ 2n-t)$ instead and let $\varepsilon\to 0$. Let

$$t_0=\inf\{t\geq 0:\inf_{x\in R^n}\sigma_2(D^2u)(x)=0\}. \tag{12}$$

We may assume that $0<t_0<\infty$ because of (11). By appropriate perturbation arguments which are somewhat classical in the use of maximum principle, we may assume without loss of generality that there exist $x_o\in R^n$ such that $\sigma_2(D^2u)(x_0,t_0)=0$ and it is then enough to show that

$$\frac{\partial}{\partial t}\{\sigma_2(D^2u)(x_0,t_0)\}\geq 0. \tag{13}$$

Therefore at the point (x_0,t_0) , we have

$$\sigma_2(D^2u)(x_0,t_0)=0; \tag{14}$$

$$\nabla\sigma_2(D^2u)(x_0,t_0)=0; \tag{15}$$

$$\Delta\sigma_2(D^2u)(x_0,t_0)\geq 0. \tag{16}$$

In the following we denote the eigenvalues of the matrix $D^2u(x,t)$ by $\lambda=\lambda(D^2u)$. By using (14),
we have for $1\le k\le n$,

$$\sum_{i=1}^{n}\sigma_1(\lambda\,|\,i)u_{iik}=0, \tag{17}$$

and

$$\Delta\sigma_2(D^2u)=\sum_{1\le i,k\le n}\sigma_1(\lambda\,|\,i)u_{iikk}+\sum_{1\le k\le n}\sum_{i\ne j}u_{iik}u_{jjk}-\sum_{1\le k\le n}\sum_{i\ne j}u_{ijk}^2. \tag{18}$$

We can choose coordinates on a neighborhood of (x_0,t_0), such that the Hessian matrix $\{u_{ij}\}$ are diagonal at (x_0,t_0) and the eigenvalues satisfy

$$u_{11}\ge u_{22}\ge u_{33}\ge\cdots\ge u_{nn}. \tag{19}$$

We claim that $\sigma_1(\lambda\,|\,n)>0$. If this were not true, suppose $\sigma_1(\lambda\,|\,n)\le 0$, we get from (19) that $u_{nn}\le 0$. Therefore $\sigma_1(D^2u)=\sigma_1(\lambda\,|\,n)+u_{nn}\le 0$, a contradiction to Step 1.
From (17), we get

$$u_{nnk}=-\frac{1}{\sigma_1(\lambda\,|\,n)}\sum_{1\le i\le n-1}\sigma_1(\lambda\,|\,i)u_{iik}. \tag{20}$$

Combining (18), heat equation (1) and the normal coordinates we chose previously, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\sigma_2(D^2u)&=\frac{\partial\sigma_2(D^2u)}{\partial u_{ij}}\cdot\frac{\partial u_{ij}}{\partial t}=\sum_{1\le i\le n}\sigma_1(\lambda\,|\,i)\cdot(\frac{\partial u}{\partial t})_{ii}\\&=\Delta\sigma_2(D^2u)+\sum_{1\le k\le n}\sum_{i\ne j}u_{ijk}^2-\sum_{1\le k\le n}\sum_{i\ne j}u_{iik}u_{jjk}.\end{aligned} \tag{21}$$

In order to verify the inequality (13), we need to verify that the last term in (21)

$$-\sum_{i\ne j}u_{iik}u_{jjk}\ge 0.$$

Substituting the expression (20) into above, we conclude that

$$\begin{aligned}-\sum_{i\ne j}u_{iik}u_{jjk}&=-2\sum_{1\le i<j\le n-1}u_{iik}u_{jjk}-2(\sum_{1\le i\le n-1}u_{iik})u_{nnk}\\&=-2\sum_{1\le i<j\le n-1}u_{iik}u_{jjk}+\frac{2}{\sigma_1(\lambda\,|\,n)}(\sum_{1\le i\le n-1}u_{iik})(\sum_{1\le j\le n-1}\sigma_1(\lambda|j)u_{jjk})\\&=\frac{1}{\sigma_1(\lambda\,|\,n)}(u_{11k},\cdots,u_{n-1n-1k})\cdot A\cdot(u_{11k},\cdots,u_{n-1n-1k})^T,\end{aligned} \tag{22}$$

where the $(n-1)\times(n-1)$ matrix A is

$$\begin{pmatrix}2a_1 & a_1+a_2-a_n & \cdots & a_1+a_{n-1}-a_n\\ a_2+a_1-a_n & 2a_2 & \cdots & a_2+a_{n-1}-a_n\\ \cdots & \cdots & \cdots & \cdots\\ a_{n-1}+a_1-a_n & a_{n-1}+a_2-a_n & \cdots & 2a_{n-1}\end{pmatrix},$$

where and in the following, for convenience of notations, we denote by $a_i=\sigma_1(\lambda\,|\,i)$ for $1\le i\le n$.
It is sufficient to verify that the matrix A is nonnegative definite.
Let us claim that the eigenvalues of the matrix A are:

$$\sigma_1(\lambda \mid n)\text{with}(n-3)\text{multiplicities}, \quad 0, \quad \text{and} \quad \frac{n-1}{\sigma_1(\lambda \mid n)} \cdot (\sum_{1 \le i \le n-1} \lambda_i^2), \tag{23}$$

from which we conclude the nonnegative definiteness of the matrix A.

Step 3: (Proof of the above claim (23).) The matrix A can be written as $A=[A-\sigma_1(\lambda \mid n)I]+\sigma_1(\lambda \mid n)I$, where I is the $(n-1)\times(n-1)$ identity matrix. We therefore turn to consider the eigenvalues of the matrix $A-\sigma_1(\lambda \mid n)I$.

To calculate the rank of the matrix $A-\sigma_1(\lambda \mid n)I$, we take elementary transformations. For $2 \le i \le n-1$, we subtract the first row from the i-th row, the matrix $A-\sigma_1(\lambda \mid n)I$ is transformed into

$$\begin{pmatrix} 2a_1-a_n & a_1+a_2-a_n & \cdots & a_1+a_{n-1}-a_n \\ a_2-a_1 & a_2-a_1 & \cdots & a_2-a_1 \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-1}-a_1 & a_{n-1}-a_1 & \cdots & a_{n-1}-a_1 \end{pmatrix},$$

recalling that we denote $\sigma_1(\lambda \mid i)$ by a_i for $1 \le i \le n$.

Therefore Rank$(A-\sigma_1(\lambda \mid n)I)=2$ and $\sigma_1(\lambda \mid n)$ are eigenvalues of the matrix A with $n-1-Rank=n-3$ multiplicities.

In the following calculation, we use Vieta's Formulas in linear algebra to obtain the rest two eigenvalues. Denote by $W=A-\sigma_1(\lambda \mid n)I$, then from **Proposition 1**, we know that the characteristic polynomial of the matrix W is

$$x^{n-1}-\sigma_1(W)x^{n-2}+\sigma_2(W)x^{n-3}+\cdots+(-1)^{n-1}\sigma_{n-1}(W)=0,$$

where

$$\begin{aligned}\sigma_1(W) &= \sum_{i=1}^{n-1}(2\sigma_1(\lambda \mid i)-\sigma_1(\lambda \mid n)) = 2\sum_{i=1}^{n}\sigma_1(\lambda \mid i)-(n+1)\sigma_1(\lambda \mid n) \\ &= 2(n-1)\sigma_1(\lambda)-(n+1)\sigma_1(\lambda \mid n),\end{aligned}$$

$$\begin{aligned}\sigma_2(W) &= \sum_{1 \le i<j \le n-1}\{-(a_i+a_j-a_n)^2+(2a_i-a_n)(2a_j-a_n)\} \\ &= -\sum_{1 \le i<j \le n-1}(a_i-a_j)^2 = -\sum_{1 \le i<j \le n-1}(\lambda_i-\lambda_j)^2 \\ &= -(n-2)\sum_{i=1}^{n-1}\lambda_i^2+2\sigma_2(\lambda|n)\end{aligned}$$

and

$$\sigma_i(W)=0 \qquad \text{for} \qquad 3 \le i \le n-1.$$

Therefore the characteristic polynomial of the matrix W is

$$x^{n-3}\cdot\{x^2-(2(n-1)\sigma_1(\lambda)-(n+1)\sigma_1(\lambda \mid n))\cdot x-(n-2)\sum_{i=1}^{n-1}\lambda_i^2+2\sigma_2(\lambda \mid n)\}=0.$$

Since the Discriminant of the polynomial in the bracket is

$$\Delta=(n-1)^2\cdot(\sigma_1(\lambda \mid n)+2\lambda_n)^2,$$

we obtain the rest two eigenvalues of the matrix W are $x_1 = -\sigma_1(\lambda \mid n)$ and $x_2 = (n-2)\sigma_1(\lambda \mid n) + 2(n-1)\lambda_n$. It is equivalent that the rest two eigenvalues of the matrix $A = A - \sigma_1(\lambda \mid n)I + \sigma_1(\lambda \mid n)I$ are $x_1 + \sigma_1(\lambda \mid n) = 0$ and $x_2 + \sigma_1(\lambda \mid n) = \sum_{i=1}^{n-1} \lambda_i^2 \sigma_1(\lambda \mid n)$.

Therefore the claim (23) is proved.

Step 4: Combining (21) and the claim (23), we conclude that

$$\frac{\partial \sigma_2(D^2u)}{\partial t} \geq \Delta \sigma_2(D^2u) \geq 0$$

and (13) follows. Therefore for any $t > 0$, $\sigma_2(D^2u) \geq 0$ by maximum principle for parabolic equations. Since we have obtained the nonnegativeness of $\sigma_2(D^2u)$, we conclude the case of strict inequality by the strict parabolic maximum principle and finish the proof of **Theorem 2**.

Remark 2. In the above, we prove the case of 2-convexity for the solutions of heat equations. We will use the concavity property of the Hessian operator (8) in **Proposition 2** to study the general case of k-convexity in the future.

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