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RESEARCH ARTICLE

EFFECT OF MHD ON UNSTEADY HELICAL FLOWS OF A GENERALIZED OLDROYD-B FLUID

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ABSTRACT

This paper present a research for the magetohydrodynamic (MHD)helical flows of an incompressible generalized Oldoyd-B fluid in concentric cylinders and circular cylinder. the fractional calculus approach is introduced to establish the constitutive relation ship of the Oldroyd-B fluid . The exact analytical solution are obtianed by means of integral transform method Laplace transform and finite Hankel transform.The solution of velocity fields and the shear stresses of unsteady magetohydrodynamic (MHD)helical flows of an Oldoyd-B fluid in an annular pipe are obtained under series form in terms of Mittag –leffler function,satisfy all imposed initial and boundary condition , Finally ,some characteristic of the motion as well as the influence of material parameters on the velocity and shear stress are analyzed by graphical illustrations.

1. INTRODUCTION

In recent years , the flow of non-Netonian fluids has received much attention for their increasing industrial and technological application . An important classof non-Netonian fluids is viscolastic fluids which exhibits elastic and viscous properties .Mathematically such fluids have a non-linear relation ship between the shear stress and the rate of strain at a point. The starting point of the fractional derivative model of non-Newtonian fluid is usually a classical differential equation which is modified by replacing the time derivative of an integer order by co-called Riemann –Lioville fractional calculus operators .The generalization allows one to define precisely non-integer order integrals or derivatives.

The Oldroyd-B fluid is a special non-Newtonian fluids and its transport behavior cannot be properly described by the typical relation between shear rate and shear stress in a simple shear flow. For this reason many models of constitutive equations have been proposed for this fluids in the last time .

The Oldroyd –B fluid model [11], which takes into account elastic and memory effects exhibited by most polymeric and biological liquids, has been used quite widely [4] .Existence

,uniqueness and stability results for some shearing motions of such a fluid have been obtained in[13] . the exact solution for the flow of an Oldroyd –B fluid was established by Waters and Kings [14], Rajagopal and Bhatnager [12],Fetecau [3], and Fetecau [2], other analytical results were given by Georgious [8] for small one- dimensional perturbations and for the limiting case of zero Reynold number unsteady(unidirectional and rotating)transient flows of an Oldroyd –B fluid in an annular are obtained by Tong[7] .the general case of helical flow of Oldroyd –B fluid due to combine action of rotating cylinders(with constant angular velocities) and a constant axial pressure gradient has consider byWood [16].The velocity fields and the associated tangential stresses corresponding to helical flows of Oldroyd –B fluids using forms of series in term of Bessel functions are given by Fetecau et al [1].

Recently , the velocity field ,shear stress and vortex sheet of a generalized second –order fluid with fractional , fractional derivative using to the constitutive relationship models of Maxwell viscoelastic fluid and second order ,and some unsteady flows of a viscoelastic fluid and of second order fluids between two parallel plates are examined by Mingyo and wenchang [17].Unidirectional flows of a viscoelastics fluid with the fractional Maxwell model helical flows of a generalized Oldroyd-B fluid with fractional calculus between two infinite coaxial circular cylinders are investigated by Dengke [6].

In this paper ,we study the effect of MHD on the helical flows of a generalized Oldroyd-B fluid with fractional calculus between two infinite coaxial circular cylinders. The velocity fields and the resulting shear stresses are determined by means of Laplace and finite Hankel transform and are presented under integral and series forms in the Mittag –leffler function.

2. Basic governing equations of Helcial flow between concentric cylinders1

We consider here an unsteady helical flow between two infinite coaxial cylinders located at $r = R_1$ and $r = R_2$ $(R_1 < R_2)$ in the cylindrical coordinates (r, θ, z) , the helcial velocity is given by

$$\boldsymbol{V} = r\, v(r,t) e_\theta + w(r,t) e_z \tag{1}$$

is called helical , because its streamlines are helical and e_θ and e_z are the unit vectors in the θ and $z-$ directions , respectively . Since the velocity field is independent of θ and z and the constraint of incompressibility is automatically satisfied .

The constitutive equation of generalized Oldroyd-B (G Oldroyd-B) fluid has the form [1]

$$\boldsymbol{S} + \lambda_1^\alpha \frac{\delta^\alpha \boldsymbol{S}}{\delta t^\alpha} = \mu \left(1 + \lambda_2^\beta \frac{\delta^\beta}{\delta t^\beta}\right) \boldsymbol{A}_1 \tag{2}$$

Where $\boldsymbol{S}$ is the extra stress tensor , μ is the dynamic viscosity , λ_1 and λ_2 are material time constants referred to, the characteristic relaxation and characteristic retardation times , respectively . it is assumed that $\lambda_1 \geq \lambda_2 \geq 0$. $\boldsymbol{A}_1 = L + L^T$ is the first Rivlin –Erickscn tensor with L the velocity gradient , α and β are fractional calculus parameters such that $0 \leq \alpha \leq \beta \leq 1$ and the fractional operator $\frac{\delta^\alpha \boldsymbol{S}}{\delta t^\alpha}$ on any tensor $\boldsymbol{S}$ is defined by

$$\frac{\delta^\alpha \boldsymbol{S}}{\delta t^\alpha} = D_t^\alpha \boldsymbol{S} + \boldsymbol{V}.\nabla \boldsymbol{S} - \boldsymbol{S}L - L^T \boldsymbol{S} \tag{3}$$

The operator D_t^α based on Caputo's fractional differential of order α is defined as

$$D_t^\alpha [y(t)] = \frac{1}{\Gamma(n-a)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau , \quad n-1 < \alpha \leq n \tag{4}$$

where $\Gamma(.)$ denotes the Gamma function. This model can be reduced to the ordinary Maxwell fluid model when $\alpha = 1$,to a generalized the Maxwell fluid model when $\beta = 1$ and to an ordinary Oldroyd-B model $\beta = \alpha = 1$.

$$v(r,0) = w(r,0) = 0 \text{ and } \boldsymbol{S}(r,0) = 0 \tag{5}$$

Here we assume that the generalized Oldroyd-B fluid is incompressible then

$$\nabla.\boldsymbol{V} = 0 \tag{6}$$

Substituting Eq.(1) into Eq.(2) and Eq.(3) and taking into account (5), we find that $\boldsymbol{S}_{rr} = 0$, $\boldsymbol{\tau}(r,t) = \boldsymbol{S}_{r\theta}(r,t)$ is the shear stress and

$$(1 + \lambda_1^\alpha D_t^\alpha)\boldsymbol{S}_{r\theta} = \mu\left(1 + \lambda_2^\beta D_t^\beta\right)\left(r\frac{\partial \mathrm{v}}{\partial r}\right) \tag{7}$$

$$(1 + \lambda_1^\alpha D_t^\alpha)\boldsymbol{S}_{rz} = \mu\left(1 + \lambda_2^\beta D_t^\beta\right)\left(\frac{\partial w}{\partial r}\right) \tag{8}$$

$$(1 + \lambda_1^\alpha D_t^\alpha)\boldsymbol{S}_{\theta z} = \lambda_1^\alpha\left[\left(r\frac{\partial \mathrm{v}}{\partial r}\right)\boldsymbol{S}_{rz} + \frac{\partial w}{\partial r}\boldsymbol{S}_{r\theta}\right] - 2\mu\,\lambda_2^\beta\frac{\partial w}{\partial r}\left(r\frac{\partial \mathrm{v}}{\partial r}\right) \tag{9}$$

$$(1 + \lambda_1^\alpha D_t^\alpha)\boldsymbol{S}_{\theta\theta} = 2\lambda_1^\alpha\left(r\frac{\partial \mathrm{v}}{\partial r}\right)\boldsymbol{S}_{r\theta} - 2\mu\,\lambda_2^\beta\left(r\frac{\partial \mathrm{v}}{\partial r}\right)^2 \tag{10}$$

$$(1 + \lambda_1^\alpha D_t^\alpha)\boldsymbol{S}_{zz} = 2\lambda_1^\alpha\frac{\partial w}{\partial r}\boldsymbol{S}_{rz} - 2\mu\,\lambda_2^\beta\left(\frac{\partial w}{\partial r}\right)^2 \tag{11}$$

3. Momentum and continuity equation

We will write the formula of the momentum equation which governing the magnetohydrodynamic as fallows :

$$\rho\frac{d\boldsymbol{V}}{dt} = -\nabla p + \nabla.\boldsymbol{S} + \boldsymbol{J}\times\boldsymbol{B} \tag{12}$$

Where ρ is the density of the fluid , $\boldsymbol{J}$ is the current density and $\boldsymbol{B} = [0,\beta_0,0]$ is the total magnetic field .

In the absence of body forces and a pressure gradient, the equation of motion reduce to the relevant equations

$$\rho\frac{\partial w}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\boldsymbol{S}_{rz} - \sigma\beta_0^2\, w \tag{13}$$

$$\rho r\frac{\partial \mathrm{v}}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\boldsymbol{S}_{r\theta} - \sigma\beta_0^2 r\mathrm{v} \tag{14}$$

$$\frac{\partial p}{\partial r} + \frac{\boldsymbol{S}_{\theta\theta}}{r} = \rho r\,\mathrm{v}^2 \tag{15}$$

Where σ is the electric conductivity .

Eliminating $\boldsymbol{S}_{rz}$ and $\boldsymbol{S}_{r\theta}$ among Eqs(7),(8),(13) and(14) ,we attain to the governing equations are

$$\rho\,\frac{\partial w}{\partial t} = \mu\frac{\left(1 + \lambda_2^\beta D_t^\beta\right)}{(1 + \lambda_1^\alpha D_t^\alpha)}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) - \sigma\beta_0^2\, w \tag{16}$$

$$\rho\, r\frac{\partial \mathrm{v}}{\partial t} = \mu\frac{\left(1 + \lambda_2^\beta D_t^\beta\right)}{(1 + \lambda_1^\alpha D_t^\alpha)}\left(r\frac{\partial^2 \mathrm{v}}{\partial r^2} + \frac{1}{r}r\frac{\partial \mathrm{v}}{\partial t}\right) - \sigma\beta_0^2 r\,\mathrm{v} \tag{17}$$

Multiply above two equations by $(1 + \lambda_1^\alpha D_t^\alpha)$,we get

$$\rho(1 + \lambda_1^\alpha D_t^\alpha)\,\frac{\partial w}{\partial t} = \mu\left(1 + \lambda_2^\beta D_t^\beta\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) - \sigma\beta_0^2(1 + \lambda_1^\alpha D_t^\alpha)\, w \tag{18}$$

$$\rho\, r(1 + \lambda_1^\alpha D_t^\alpha)\frac{\partial \mathrm{v}}{\partial t} = \mu\left(1 + \lambda_2^\beta D_t^\beta\right)\left(r\frac{\partial^2 \mathrm{v}}{\partial r^2} + \frac{1}{r}r\frac{\partial \mathrm{v}}{\partial t}\right) - \sigma\beta_0^2 r(1 + \lambda_1^\alpha D_t^\alpha)\,\mathrm{v} \tag{19}$$

Divide Eq(18) by ρ and divide Eq(19) by $\rho\, r$,we get

$$(1 + \lambda_1^\alpha D_t^\alpha)\,\frac{\partial w}{\partial t} = \upsilon\left(1 + \lambda_2^\beta D_t^\beta\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) - \frac{\sigma\beta_0^2}{\rho}(1 + \lambda_1^\alpha D_t^\alpha)\, w \tag{20}$$

$$(1 + \lambda_1^\alpha D_t^\alpha)\frac{\partial \mathrm{v}}{\partial t} = \upsilon\left(1 + \lambda_2^\beta D_t^\beta\right)\left(\frac{\partial^2 \mathrm{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \mathrm{v}}{\partial t}\right) - \frac{\sigma\beta_0^2}{\rho}(1 + \lambda_1^\alpha D_t^\alpha)\,\mathrm{v} \tag{21}$$

Where $\upsilon = \frac{\mu}{\rho}$ is the kinematic viscosity of the fluid.

We will consider an incompressible generalized Oldroyd –B fluid at rest in annular region between two straight circular cylinders of radius R_1 and R_2 $(R_1 < R_2)$.

At time $t = 0$,we note the two cylinders begin to rotate about $r = 0$ with angular velocities of constant accelerations Ω_1 and Ω_2 and to slide along the same axis with linear velocities of accelerations U_1 and U_2

The boundary conditions are expressed by

$$w(R_1, t) = U_1 t\,, \quad w(R_2, t) = U_2 t\,, \;\; t > 0 \tag{22}$$

And

$$\mathrm{v}(R_1, t) = \Omega_1 t\,, \quad \mathrm{v}(R_2, t) = \Omega_2 t\,, \;\; t > 0 \tag{23}$$

The initial conditions are expressed by

$$w(r, 0) = \mathrm{v}(r, 0) = 0 \tag{24}$$

$$\partial_t w(r, 0) = \partial_t \mathrm{v}(r, 0) = 0 \tag{25}$$

3.1.Calculation of the velocity field

Making the change of unknown function

$$\mathrm{v}(r, t) = \frac{u(r, t)}{r} \tag{26}$$

substitute the value of velocity $\mathrm{v}(r, t)$ in Eq. (21) with initial and boundary conditions ,we get

$$(1 + \lambda_1^\alpha D_t^\alpha)\frac{1}{r}\frac{\partial \mathrm{u}}{\partial t} = \frac{\upsilon}{r}\left(1 + \lambda_2^\beta D_t^\beta\right)\left(\frac{\partial^2 \mathrm{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \mathrm{u}}{\partial t} - \frac{u}{r^2}\right) - \frac{\sigma\beta_0^2}{\rho r}(1 + \lambda_1^\alpha D_t^\alpha)\,\mathrm{u} \tag{27}$$

Multiply above equation by *r* ,we get

$$(1 + \lambda_1^\alpha D_t^\alpha)\,\frac{\partial \mathrm{u}}{\partial t} = \upsilon\left(1 + \lambda_2^\beta D_t^\beta\right)\left(\frac{\partial^2 \mathrm{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \mathrm{u}}{\partial t} - \frac{u}{r^2}\right) - \frac{\sigma\beta_0^2}{\rho}(1 + \lambda_1^\alpha D_t^\alpha)\,\mathrm{u} \tag{28}$$

And

$$u(R_1, t) = R_1\Omega_1 t, \quad u(R_2, t) = R_2\Omega_2 t\,, \;\; t > 0 \tag{29}$$

$$u(r, 0) = \partial_t \mathrm{u}(r, 0) = 0 \tag{30}$$

To obtain the exact analytical solution of the above problems Eq.(20) and Eq.(28) , and using initial conditions(24),(25) and (30) ,we first apply Laplace transform of fractional derivatives , with respect to t ,we get

$$s(1 + \lambda_1^\alpha s^\alpha)\overline{w} \;= \upsilon\left(1 + \lambda_2^\beta s^\beta\right)\left(\frac{\partial^2 \overline{w}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{w}}{\partial r}\right) - \frac{\sigma\beta_0^2}{\rho}(1 + \lambda_1^\alpha s^\alpha)\,\overline{w} \tag{31}$$

$$\overline{w}(R_1, s) = \frac{U_1}{s^2}\,, \quad \overline{w}(R_2, s) = \frac{U_2}{s^2} \tag{32}$$

$$s(1 + \lambda_1^\alpha s^\alpha)\overline{u} \;= \upsilon\left(1 + \lambda_2^\beta s^\beta\right)\left(\frac{\partial^2 \overline{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{u}}{\partial r} - \frac{\overline{u}}{r^2}\right) - \frac{\sigma\beta_0^2}{\rho}(1 + \lambda_1^\alpha s^\alpha)\,\overline{u} \tag{33}$$

$$\overline{u}(R_1, s) = \frac{R_1\Omega_1}{s^2}\,, \quad \overline{u}(R_2, s) = \frac{R_2\Omega_2}{s^2} \tag{34}$$

We use the finite Hankel transform with respect to *r* [5],defined as follows

$$\overline{\overline{w}} = \int_{R_1}^{R_2} r\,\overline{w}(r, s)\psi_1(s_{1n} r)dr \tag{35}$$

$$\overline{\overline{u}} = \int_{R_1}^{R_2} r\,\overline{u}(r, s)\psi_2(s_{2n} r)dr \tag{36}$$

And the inverse Hankel transform are

$$\bar{w}(r,s)=\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{s_{1n}^2 J_0^2(s_{1n}R_1)\bar{\bar{w}}(r,s)\psi_1(s_{1n}r)}{J_0^2(s_{1n}R_1)-J_0^2(s_{1n}R_2)} \tag{37}$$

$$\bar{u}(r,s)=\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{s_{2n}^2 J_1^2(s_{2n}R_1)\bar{\bar{u}}(r,s)\psi_2(s_{2n}r)}{J_1^2(s_{2n}R_1)-J_1^2(s_{2n}R_2)} \tag{38}$$

Where s_{1n} and s_{2n} are the positive roots of $\psi_1(s_{1n}R_1)=0$ and $\psi_2(s_{2n}R_1)=0$, respectively.
$\psi_1(s_{1n}r)=Y_0(s_{1n}R_2)J_0(s_{1n}r)-J_0(s_{1n}R_2)Y_0(s_{1n}r)$,
$\psi_2(s_{2n}r)=Y_1(s_{2n}R_2)J_1(s_{2n}r)-J_1(s_{2n}R_2)Y_1(s_{2n}r)$
Y_i and J_i are the Bessel functions of the first and second kinds of order zero and one (*i*=0,1) , respectively.
Now applying finite Hankel transform to Eqs.(35) and (36) ,we get

$$\left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{2n}^2\ \upsilon\left(1+\lambda_2^\beta s^\beta\right)\right]\bar{\bar{w}} = \frac{2}{\pi}\ \upsilon\left(1+\lambda_2^\beta s^\beta\right)\frac{U_2\, J_0(s_{2n}R_1)-U_1\, J_0(s_{2n}R_2)}{s^{\,2}\ J_0(s_{2n}R_1)} \tag{39}$$

$$\bar{\bar{w}}=\frac{2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)\ [U_2\, J_0(s_{1n}R_1)-U_1\, J_0(s_{1n}R_2)]}{\pi s^{\,2}\ J_0(s_{1n}R_1)\left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{1n}^2\ \upsilon\left(1+\lambda_2^\beta s^\beta\right)\right]} \tag{40}$$

and

$$\left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{2n}^2\ \upsilon\left(1+\lambda_2^\beta s^\beta\right)\right]\bar{\bar{u}} = \frac{2}{\pi}\ \upsilon\left(1+\lambda_2^\beta s^\beta\right)\frac{R_2\Omega_2\, J_1(s_{2n}R_1)-R_1\Omega_1\, J_1(s_{2n}R_2)}{s^2\ J_1(s_{2n}R_1)} \tag{41}$$

$$\bar{\bar{u}}=\frac{2\,\upsilon\left(1+\lambda_2^\beta s^\beta\right)\ [R_2\Omega_2\, J_1(s_{2n}R_1)-R_1\Omega_1\, J_1(s_{2n}R_2)]}{\pi s^2\ J_1(s_{2n}R_1)\left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{2n}^2\ \upsilon\left(1+\lambda_2^\beta s^\beta\right)\right]} \tag{42}$$

Substitute Eqs.(40) and (42) into Eqs. (37) and (38) respectively ,we obtain

$$\bar{w}(r,s)=\pi\sum_{n=1}^{\infty}\frac{J_0(s_{1n}R_1)\psi_1(s_{1n}r)[U_2\, J_0(s_{1n}R_1)-U_1\, J_0(s_{1n}R_2)]}{J_0^2(s_{1n}R_1)-J_0^2(s_{1n}R_2)}\times\bar{A}_2(\,s_{1n},s) \tag{43}$$

Where

$$\bar{A}_2(\,s_{1n},s)=\frac{s_{1n}^2\ \upsilon\left(1+\lambda_2^\beta s^\beta\right)}{s^2\ \left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{1n}^2\ \upsilon\left(1+\lambda_2^\beta s^\beta\right)\right]} \tag{44}$$

and

$$\bar{u}(r,s)=\pi\sum_{n=1}^{\infty}\frac{J_1(s_{2n}R_1)\psi_2(s_{2n}r)[R_2\Omega_2 J_1(s_{2n}R_1)-R_1\Omega_1 J_1(s_{2n}R_2)]}{J_1^2(s_{2n}R_1)-J_1^2(s_{2n}R_2)}\times\bar{A}_2(\,s_{2n},s) \tag{45}$$

where

$$\bar{A}_2(\,s_{2n},s)=\frac{s_{2n}^2\ \upsilon\left(1+\lambda_2^\beta s^\beta\right)}{s^2\ \left[s(1+\lambda_1^\alpha s^\alpha)+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^\alpha s^\alpha)+s_{2n}^2\ \upsilon\left(1+\lambda_2^\beta s^\beta\right)\right]} \tag{46}$$

Now ,rewrite Eqs.(44) and (46) in series form. And applying inverse Laplace transform .Then substitute the result into Eqs (43) and (45),we get

$$w(r,t)=\pi\sum_{n=1}^{\infty}\frac{J_0(s_{1n}R_1)\psi_1(s_{1n}r)[U_2\, J_0(s_{1n}R_1)-U_1\, J_0(s_{1n}R_2)]}{J_0^2(s_{1n}R_1)-J_0^2(s_{1n}R_2)}\times\left(t-G_1(\,s_{1n},t)\right) \tag{47}$$

Where

$$G_1(s_{1n},t)=\sum_{m=0}^{\infty}(-1)^m\sum_{a+b+c+d=m}^{a,b,c,d\geq 0}\frac{(-1)^a}{a!\,b!\,c!\,d!}\frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{\boldsymbol{b}}(s_{1n}^2\ \upsilon)^{c+d}\left(\lambda_2^{\beta}\right)^d}{(\lambda_1^{\alpha})^{m-b+1}}$$
$$\times\left(\frac{\sigma\beta_0^2}{\rho}t^{(\alpha+1)m+(\alpha-\delta)}\,E_{\alpha+1,((\alpha+1)-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^{\alpha}}\left(1+\frac{\sigma\beta_0^2}{\rho}\right)t^{\alpha+1}\right)\right.$$
$$+\frac{\sigma\beta_0^2}{\rho}\lambda_1^{\alpha}\,t^{(\alpha+1)m-\delta}\,E_{\alpha+1,(1-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^{\alpha}}\left(1+\frac{\sigma\beta_0^2}{\rho}\right)t^{\alpha+1}\right)$$
$$+t^{(\alpha+1)m+(\alpha-\delta-1)}\,E_{\alpha+1,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^{\alpha}}\left(1+\frac{\sigma\beta_0^2}{\rho}\right)t^{\alpha+1}\right)$$
$$\left.+\ \lambda_1^{\alpha}\,t^{(\alpha+1)m-\delta-1}\,E_{\alpha+1\,,-\delta}^{(m)}\left(-\frac{1}{\lambda_1^{\alpha}}\left(1+\frac{\sigma\beta_0^2}{\rho}\right)t^{\alpha+1}\right)\right)\qquad(48)$$

, $\delta=m-2+(\alpha-1)*b+(\beta-1)*d-a-c.$

And

$$u(r,t)=\pi\sum_{n=1}^{\infty}\frac{J_1(s_{2n}R_1)\psi_2(s_{2n}r)[R_2\Omega_2J_1(s_{2n}R_1)-R_1\Omega_1J_1(s_{2n}R_2)]}{J_1^2(s_{2n}R_1)-J_1^2(s_{2n}R_2)}\times\left(t-G_1(s_{2n},t)\right)\qquad(49)$$

Where

$$G_1(s_{2n},t)=\sum_{m=0}^{\infty}(-1)^m\sum_{a+b+c+d=m}^{a,b,c,d\geq 0}\frac{(-1)^a}{a!\,b!\,c!\,d!}\frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{\boldsymbol{b}}(s_{2n}^2\ \upsilon)^{c+d}\left(\lambda_2^{\beta}\right)^d}{(\lambda_1^{\alpha})^{m-b+1}}$$
$$\times\left(\frac{\sigma\beta_0^2}{\rho}t^{(\alpha+1)m+(\alpha-\delta)}\,E_{\alpha+1,((\alpha+1)-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^{\alpha}}\left(1+\frac{\sigma\beta_0^2}{\rho}\right)t^{\alpha+1}\right)\right.$$
$$+\frac{\sigma\beta_0^2}{\rho}\lambda_1^{\alpha}\,t^{(\alpha+1)m-\delta}\,E_{\alpha+1,(1-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^{\alpha}}\left(1+\frac{\sigma\beta_0^2}{\rho}\right)t^{\alpha+1}\right)$$
$$+t^{(\alpha+1)m+(\alpha-\delta-1)}\,E_{\alpha+1,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^{\alpha}}\left(1+\frac{\sigma\beta_0^2}{\rho}\right)t^{\alpha+1}\right)$$
$$\left.+\ \lambda_1^{\alpha}\,t^{(\alpha+1)m-\delta-1}\,E_{\alpha+1\,,-\delta}^{(m)}\left(-\frac{1}{\lambda_1^{\alpha}}\left(1+\frac{\sigma\beta_0^2}{\rho}\right)t^{\alpha+1}\right)\right)\qquad(50)$$

Finally ,we applying the inverse finite Hankel transform to get the velocities ,as follows

$$w(r,t)=\ W(r)t-\pi\sum_{n=1}^{\infty}\frac{J_0(s_{1n}R_1)\psi_1(s_{1n}r)[U_2\,J_0(s_{1n}R_1)-U_1\,J_0(s_{1n}R_2)]}{J_0^2(s_{1n}R_1)-J_0^2(s_{1n}R_2)}\times G_1(s_{1n},t)\qquad(51)$$

Where $W(r)=\left[U_2+\left(\frac{In\left(\frac{r}{R_2}\right)}{In\left(\frac{R_2}{R_1}\right)}\right)*(U_2-U_1)\right].$

And

$$u(r,t) = U(r)t - \pi \sum_{n=1}^{\infty} \frac{J_1(s_{2n}R_1)\psi_2(s_{2n}r)[R_2\Omega_2 J_1(s_{2n}R_1) - R_1\Omega_1 J_1(s_{2n}R_2)]}{J_1^2(s_{2n}R_1) - J_1^2(s_{2n}R_2)} \times G_1(\, s_{2n}, t) \tag{52}$$

Where $U(r) = \left[r\Omega_2 + \left(\frac{R_1^2(r^2-R_2^2)}{r(R_2^2-R_1^2)}\right) * (\Omega_2 - \Omega_1)\right]$.

And the associated tangential tension fields have the forms

$$\boldsymbol{S}_{rz} = \rho\pi \left(\sum_{n=1}^{\infty} \frac{J_0(s_{1n}R_1)\psi_3(s_{1n}r)[U_2\, J_0(s_{1n}R_1) - U_1\, J_0(s_{1n}R_2)]}{s_{1n}(J_0^2(s_{1n}R_1) - J_0^2(s_{1n}R_2))} \times G_2(\, s_{1n}, t) \right) \tag{53}$$

where

$$G_2(\, s_{1n}, t) = \sum_{m=0}^{\infty} (-1)^m \sum_{a+b+c+d=m}^{a,b,c,d\geq 0} \frac{(-1)^a}{a!\, b!\, c!\, d!} \frac{(s_{1n}^2\ \upsilon)^{2+c+d} \left(\lambda_2^\beta\right)^d}{(\lambda_1^\alpha)^{m-b+1}} * \Bigg[t^{(\alpha+1)m+(2\alpha-\delta)-1} E_{\alpha+1,(\alpha+1-\delta)}^{(m)} \left(-\frac{1}{\lambda_1^\alpha}\left(1 + \frac{\sigma\beta_0^2}{\rho}\right) t^{\alpha+1} \right) + 2\lambda_2^\beta t^{(\alpha+1)m+(2\alpha-(\delta+\beta))-1}\, E_{\alpha+1,\alpha+1-(\delta+\beta)}^{(m)} \left(-\frac{1}{\lambda_1^\alpha}\left(1 + \frac{\sigma\beta_0^2}{\rho}\right) t^{\alpha+1} \right) + \lambda_2^{2\beta} t^{(\alpha+1)m+(2\alpha-(\delta+2\beta))-1} E_{\alpha,\alpha+1-(\delta+2\beta)}^{(m)} \left(-\frac{1}{\lambda_1^\alpha}\left(1 + \frac{\sigma\beta_0^2}{\rho}\right) t^{\alpha+1} \right) \Bigg] E_{\alpha,\alpha}(-\lambda_1^\alpha t^\alpha)$$

$$\boldsymbol{S}_{r\theta} = \rho\pi \left(\sum_{n=1}^{\infty} \frac{J_1(s_{2n}R_1)\big(rs_{2n}\psi_3(s_{2n}r) - \psi_2(s_{2n}r)\big)[R_2\Omega_2 J_1(s_{2n}R_1) - R_1\Omega_1 J_1(s_{2n}R_2)]}{rs_{2n}^2\, (J_1^2(s_{2n}R_1) - J_1^2(s_{2n}R_2))} \times G_2(\, s_{2n}, t) \right) \tag{54}$$

Where

$$G_2(\, s_{2n}, t) = \sum_{m=0}^{\infty} (-1)^m \sum_{a+b+c+d=m}^{a,b,c,d\geq 0} \frac{(-1)^a}{a!\, b!\, c!\, d!} \frac{(s_{2n}^2\ \upsilon)^{2+c+d} \left(\lambda_2^\beta\right)^d}{(\lambda_1^\alpha)^{m-b+1}} * \Bigg[t^{(\alpha+1)m+(2\alpha-\delta)-1} E_{\alpha+1,(\alpha+1-\delta)}^{(m)} \left(-\frac{1}{\lambda_1^\alpha}\left(1 + \frac{\sigma\beta_0^2}{\rho}\right) t^{\alpha+1} \right) + 2\lambda_2^\beta t^{(\alpha+1)m+(2\alpha-(\delta+\beta))-1}\, E_{\alpha+1,\alpha+1-(\delta+\beta)}^{(m)} \left(-\frac{1}{\lambda_1^\alpha}\left(1 + \frac{\sigma\beta_0^2}{\rho}\right) t^{\alpha+1} \right) + \lambda_2^{2\beta} t^{(\alpha+1)m+(2\alpha-(\delta+2\beta))-1} E_{\alpha,\alpha+1-(\delta+2\beta)}^{(m)} \left(-\frac{1}{\lambda_1^\alpha}\left(1 + \frac{\sigma\beta_0^2}{\rho}\right) t^{\alpha+1} \right) \Bigg] E_{\alpha,\alpha}(-\lambda_1^\alpha t^\alpha)$$

6.Helical flow through circular cylinder II

Taking the limit of Eqs.(35) and (36), when $R_1 \to 0$ and $R_2 \to R$, we find the Hankel transform

$$\bar{\bar{w}} = \int_0^R rJ_0(s_{3n}r)\, \bar{w}(r,s)dr \tag{55}$$

Where s_{3n} is positive root of $J_0(s_{3n}R) = 0$, and

$$\bar{\bar{u}} = \int_0^R rJ_1(s_{4n}r)\, \bar{u}(r,s)dr \tag{56}$$

Where s_{4n} is positive root of $J_1(s_{4n}R) = 0$, and the inverse Hankel transform are

$$\bar{w}(r,s) = \frac{2}{R^2} \sum_{n=1}^{\infty} \frac{\bar{\bar{w}}(s_{3n}, s) J_0(s_{3n}r)}{J_1^2(s_{3n}R)} \tag{57}$$

and

$$\bar{u}(r,s)=\frac{2}{R^2}\sum_{n=1}^{\infty}\frac{\bar{\bar{u}}(s_{4n},s)J_1(s_{4n}r)}{J_2^2(s_{4n}R)} \tag{58}$$

with

Corresponding to the helical flow through an infinite circular cylinder II . the boundary conditions must be changed by

$$|w(0,t)|<\infty\,,\;\; w(R,t)=Ut\;\;,\;\; |u(0,t)|<\infty\;\;,\quad u(R,t)=R\Omega t \tag{59}$$

Now apply Laplace transform to the boundary conditions, with respect to t , we get

$$|w(0,s)|<\infty\,,\; w(R,s)=\frac{U}{s^2}\;\;,\;\; |u(0,s)|<\infty\;\;,\quad u(R,s)=\frac{R\Omega}{s^2} \tag{60}$$

Now applying finite Hankel transform to Eqs.(31) and (33) ,we get

$$\left[s(1+\lambda_1^{\alpha}s^{\alpha})+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^{\alpha}s^{\alpha})+s_{3n}^2\ \upsilon\left(1+\lambda_2^{\beta}s^{\beta}\right)\right]\bar{\bar{w}}$$

$$=\ \upsilon\left(1+\lambda_2^{\beta}s^{\beta}\right)R\ s_{3n}\ \frac{U}{s^2}\ J_1(s_{3n}\ R) \tag{61}$$

then

$$\bar{\bar{w}}=\frac{\upsilon\left(1+\lambda_2^{\beta}s^{\beta}\right)\ R\ s_{3n}\ U\ J_1(s_{3n}\ R)}{s^2\ \left[s(1+\lambda_1^{\alpha}s^{\alpha})+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^{\alpha}s^{\alpha})+s_{3n}^2\ \upsilon\left(1+\lambda_2^{\beta}s^{\beta}\right)\right]} \tag{62}$$

and

$$\left[s(1+\lambda_1^{\alpha}s^{\alpha})+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^{\alpha}s^{\alpha})+s_{4n}^2\ \upsilon\left(1+\lambda_2^{\beta}s^{\beta}\right)\right]\bar{\bar{u}}$$

$$=-\frac{R^2\Omega}{s^2}s_{4n}\ \upsilon\left(1+\lambda_2^{\beta}s^{\beta}\right)\ J_2(s_{4n}\ R) \tag{63}$$

Then

$$\bar{\bar{u}}=-\frac{R^2\Omega s_{4n}\ \upsilon\left(1+\lambda_2^{\beta}s^{\beta}\right)\ J_2(s_{4n}\ R)}{s^2\left[s(1+\lambda_1^{\alpha}s^{\alpha})+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^{\alpha}s^{\alpha})+s_{4n}^2\ \upsilon\left(1+\lambda_2^{\beta}s^{\beta}\right)\right]} \tag{64}$$

Substitute Eqs.(62) and(64) into Eqs(57) and (58) respectively ,we obtain

$$\bar{w}(r,s)=\frac{2U}{R}\sum_{n=1}^{\infty}\frac{J_0(s_{3n}r)}{s_{3n}\ J_1(s_{3n}\ R)}\ \bar{A}_2(\ s_{3n},s) \tag{65}$$

where

$$\bar{A}_2(\ s_{3n},s)=\frac{s_{3n}^2\ \upsilon\left(1+\lambda_2^{\beta}s^{\beta}\right)}{s^2\left[s(1+\lambda_1^{\alpha}s^{\alpha})+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^{\alpha}s^{\alpha})+s_{3n}^2\ \upsilon\left(1+\lambda_2^{\beta}s^{\beta}\right)\right]} \tag{66}$$

and

$$\bar{u}(r,s)=2\Omega\sum_{n=1}^{\infty}\frac{J_1(s_{4n}r)}{s_{4n}\ J_0(s_{4n}\ R)}\ \bar{A}_2(\ s_{4n},s) \tag{67}$$

where

$$\bar{A}_2(\ s_{4n},s)=\frac{s_{4n}^2\ \upsilon\left(1+\lambda_2^{\beta}s^{\beta}\right)}{s^2\left[s(1+\lambda_1^{\alpha}s^{\alpha})+\frac{\sigma\beta_0^2}{\rho}(1+\lambda_1^{\alpha}s^{\alpha})+s_{4n}^2\ \upsilon\left(1+\lambda_2^{\beta}s^{\beta}\right)\right]} \tag{68}$$

Now ,rewrite Eqs.(66) and (68) in series form. And applying inverse Laplace transform .Then substitute the result into Eqs (65) and (67),we get

$$w(r,t)=\frac{2U}{R}\sum_{n=1}^{\infty}\frac{J_0(s_{3n}r)}{s_{3n}\,J_1(s_{3n}\,R)}\times\left(t-G_1(\,s_{3n},t)\right) \tag{69}$$

Where

$$G_1(\,s_{3n},t)=\sum_{m=0}^{\infty}(-1)^m\sum_{a+b+c+d=m}^{a,b,c,d\geq 0}\frac{(-1)^a}{a!\,b!\,c!\,d!}\frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{\boldsymbol{b}}(s_{3n}^2\;\upsilon)^{c+d}\left(\lambda_2^{\beta}\right)^d}{(\lambda_1^{\alpha})^{m-b+1}}$$
$$\times\left(\frac{\sigma\beta_0^2}{\rho}t^{(\alpha+1)m+(\alpha-\delta)}\,E_{\alpha+1,((\alpha+1)-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^{\alpha}}\left(1+\frac{\sigma\beta_0^2}{\rho}\right)t^{\alpha+1}\right)\right.$$
$$+\frac{\sigma\beta_0^2}{\rho}\,\lambda_1^{\alpha}\,t^{(\alpha+1)m-\delta}\,E_{\alpha+1,(1-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^{\alpha}}\left(1+\frac{\sigma\beta_0^2}{\rho}\right)t^{\alpha+1}\right)$$
$$+\,t^{(\alpha+1)m+(\alpha-\delta-1)}\,E_{\alpha+1,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^{\alpha}}\left(1+\frac{\sigma\beta_0^2}{\rho}\right)t^{\alpha+1}\right)$$
$$\left.+\;\lambda_1^{\alpha}\,t^{(\alpha+1)m-\delta-1}\,E_{\alpha+1\,,-\delta}^{(m)}\left(-\frac{1}{\lambda_1^{\alpha}}\left(1+\frac{\sigma\beta_0^2}{\rho}\right)t^{\alpha+1}\right)\right) \tag{70}$$

$,\delta=m-2+(\alpha-1)*b+(\,\beta-1)*d-a-c.$

And

$$u(r,t)=2\Omega\sum_{n=1}^{\infty}\frac{J_1(s_{4n}r)}{s_{4n}\,J_0(s_{4n}\,R)}\times\left(t-G_1(\,s_{4n},t)\right) \tag{71}$$

Where

$$G_1(\,s_{4n},t)=\sum_{m=0}^{\infty}(-1)^m\sum_{a+b+c+d=m}^{a,b,c,d\geq 0}\frac{(-1)^a}{a!\,b!\,c!\,d!}\frac{\left(\frac{\sigma\beta_0^2}{\rho}\right)^{\boldsymbol{b}}(s_{4n}^2\;\upsilon)^{c+d}\left(\lambda_2^{\beta}\right)^d}{(\lambda_1^{\alpha})^{m-b+1}}$$
$$\times\left(\frac{\sigma\beta_0^2}{\rho}t^{(\alpha+1)m+(\alpha-\delta)}\,E_{\alpha+1,((\alpha+1)-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^{\alpha}}\left(1+\frac{\sigma\beta_0^2}{\rho}\right)t^{\alpha+1}\right)\right.$$
$$+\frac{\sigma\beta_0^2}{\rho}\,\lambda_1^{\alpha}\,t^{(\alpha+1)m-\delta}\,E_{\alpha+1,(1-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^{\alpha}}\left(1+\frac{\sigma\beta_0^2}{\rho}\right)t^{\alpha+1}\right)$$
$$+\,t^{(\alpha+1)m+(\alpha-\delta-1)}\,E_{\alpha+1,(\alpha-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^{\alpha}}\left(1+\frac{\sigma\beta_0^2}{\rho}\right)t^{\alpha+1}\right)$$
$$\left.+\;\lambda_1^{\alpha}\,t^{(\alpha+1)m-\delta-1}\,E_{\alpha+1\,,-\delta}^{(m)}\left(-\frac{1}{\lambda_1^{\alpha}}\left(1+\frac{\sigma\beta_0^2}{\rho}\right)t^{\alpha+1}\right)\right) \tag{72}$$

Finally ,we applying the inverse finite Hankel transform to get the velocities ,as follows

$$w(r,t)=Ut-\frac{2U}{R}\sum_{n=1}^{\infty}\frac{J_0(s_{3n}r)}{s_{3n}\,J_1(s_{3n}\,R)}\times G_1(\,s_{3n},t) \tag{73}$$

and

$$u(r,t)=-\,2\Omega t+2\Omega\sum_{n=1}^{\infty}\frac{J_1(s_{4n}r)}{s_{4n}\,J_0(s_{4n}\,R)}\;\times G_1(\,s_{4n},t) \tag{74}$$

And the associated tangential tension fields have the forms

$$\boldsymbol{S}_{rz}=-\frac{2\,\rho U}{R}\sum_{n=1}^{\infty}\frac{J_1(s_{3n}r)}{s_{3n}^2\;J_1(s_{3n}\,R)}\times G_2(\,s_{3n},t) \tag{75}$$

where

$$G_2(s_{3n},t)=\sum_{m=0}^{\infty}(-1)^m\sum_{a+b+c+d=m}^{a,b,c,d\geq 0}\frac{(-1)^a}{a!\,b!\,c!\,d!}\frac{(s_{3n}^2\ \upsilon)^{2+c+d}\left(\lambda_2^{\beta}\right)^d}{(\lambda_1^{\alpha})^{m-b+1}}$$
$$*\left[t^{(\alpha+1)m+(2\alpha-\delta)-1}E_{\alpha+1,(\alpha+1-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^{\alpha}}\left(1+\frac{\sigma\beta_0^2}{\rho}\right)t^{\alpha+1}\right)\right.$$
$$+2\lambda_2^{\beta}t^{(\alpha+1)m+(2\alpha-(\delta+\beta))-1}\,E_{\alpha+1,\alpha+1-(\delta+\beta)}^{(m)}\left(-\frac{1}{\lambda_1^{\alpha}}\left(1+\frac{\sigma\beta_0^2}{\rho}\right)t^{\alpha+1}\right)$$
$$\left.+\lambda_2^{2\beta}t^{(\alpha+1)m+(2\alpha-(\delta+2\beta))-1}E_{\alpha,\alpha+1-(\delta+2\beta)}^{(m)}\left(-\frac{1}{\lambda_1^{\alpha}}\left(1+\frac{\sigma\beta_0^2}{\rho}\right)t^{\alpha+1}\right)\right]E_{\alpha,\alpha}(-\lambda_1^{\alpha}t^{\alpha})$$

and

$$\boldsymbol{S}_{r\theta}=2\rho\Omega\sum_{n=1}^{\infty}\frac{\left[2\left(\frac{J_1(s_{4n}r)}{r}\right)-s_{4n}\,J_0(s_{4n}r)\right]}{s_{4n}^3\ J_0(s_{4n}\,R)}\ \times G_2(s_{4n},t) \tag{76}$$

where

$$G_2(s_{4n},t)=\sum_{m=0}^{\infty}(-1)^m\sum_{a+b+c+d=m}^{a,b,c,d\geq 0}\frac{(-1)^a}{a!\,b!\,c!\,d!}\frac{(s_{4n}^2\ \upsilon)^{2+c+d}\left(\lambda_2^{\beta}\right)^d}{(\lambda_1^{\alpha})^{m-b+1}}$$
$$*\left[t^{(\alpha+1)m+(2\alpha-\delta)-1}E_{\alpha+1,(\alpha+1-\delta)}^{(m)}\left(-\frac{1}{\lambda_1^{\alpha}}\left(1+\frac{\sigma\beta_0^2}{\rho}\right)t^{\alpha+1}\right)\right.$$
$$+2\lambda_2^{\beta}t^{(\alpha+1)m+(2\alpha-(\delta+\beta))-1}\,E_{\alpha+1,\alpha+1-(\delta+\beta)}^{(m)}\left(-\frac{1}{\lambda_1^{\alpha}}\left(1+\frac{\sigma\beta_0^2}{\rho}\right)t^{\alpha+1}\right)$$
$$\left.+\lambda_2^{2\beta}t^{(\alpha+1)m+(2\alpha-(\delta+2\beta))-1}E_{\alpha,\alpha+1-(\delta+2\beta)}^{(m)}\left(-\frac{1}{\lambda_1^{\alpha}}\left(1+\frac{\sigma\beta_0^2}{\rho}\right)t^{\alpha+1}\right)\right]E_{\alpha,\alpha}(-\lambda_1^{\alpha}t^{\alpha})$$

7. Limiting case

Making the limit of Eqs(51) and (52) when $\alpha\neq 0$, $\beta\neq 0$ and $\left(\frac{\sigma\beta_0^2}{\rho}\right)=M=0$,we can attain the similar solution velocity distribution for unsteady helical flows of a generalized Oldroyd –B fluid , as obtained in [] , thus velocity fields reduces to

$$w(r,t)=W(r)t-\pi\sum_{n=1}^{\infty}\frac{J_0(s_{1n}R_1)\psi_1(s_{1n}r)[U_2\,J_0(s_{1n}R_1)-U_1\,J_0(s_{1n}R_2)]}{J_0^2(s_{1n}R_1)-J_0^2(s_{1n}R_2)}\times G_1(s_{1n},t) \tag{77}$$

Where

$$G(s_{1n},t)=\sum_{m=0}^{\infty}(-1)^m\sum_{c+d=m}^{c,d\geq 0}\frac{(s_{1n}^2\ \upsilon)^{c+d}\left(\lambda_2^{\beta}\right)^d}{c!\,d!\,(\lambda_1^{\alpha})^{m+1}}\left(t^{\alpha m+(\alpha-\delta)-1}\,E_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{t^{\alpha}}{\lambda_1^{\alpha}}\right)+\ \lambda_1^{\alpha}\ t^{\alpha m-\delta-1}\,E_{\alpha\,,-\delta}^{(m)}\left(-\frac{t^{\alpha}}{\lambda_1^{\alpha}}\right)\right)$$

Or

$$G(s_{1n},t)=\sum_{m=0}^{\infty}\frac{(-1)^m}{(m-1)!}\left(\frac{s_{1n}^2\ \upsilon}{\lambda_1^{\alpha}}\right)^m\sum_{k=0}^{m}\binom{m}{k}\left(\lambda_2^{\beta}\right)^k\,t^{(1+\alpha)m-\beta k}E_{\alpha,(\alpha+m+1-\beta k)}^{(m-1)}(-\lambda_1^{-\alpha}t^{\alpha})$$

and

$$u(r,t)=U(r)t-\pi\sum_{n=1}^{\infty}\frac{J_1(s_{2n}R_1)\psi_2(s_{2n}r)[R_2\Omega_2J_1(s_{2n}R_1)-R_1\Omega_1J_1(s_{2n}R_2)]}{J_1^2(s_{2n}R_1)-J_1^2(s_{2n}R_2)}\times G_1(s_{2n},t) \tag{78}$$

Where

$$G(s_{1n},t)=\sum_{m=0}^{\infty}(-1)^m\sum_{c+d=m}^{c,d\geq 0}\frac{(s_{2n}^2\ \upsilon)^{c+d}\left(\lambda_2^{\beta}\right)^d}{c!\,d!\,(\lambda_1^{\alpha})^{m+1}}\left(t^{\alpha m+(\alpha-\delta)-1}\,E_{\alpha,(\alpha-\delta)}^{(m)}\left(-\frac{t^{\alpha}}{\lambda_1^{\alpha}}\right)\right.$$
$$\left.+\ \lambda_1^{\alpha}\,t^{\alpha m-\delta-1}\,E_{\alpha\,,-\delta}^{(m)}\left(-\frac{t^{\alpha}}{\lambda_1^{\alpha}}\right)\right)$$

Or

$$G(s_{1n},t)=\sum_{m=0}^{\infty}\frac{(-1)^m}{(m-1)!}\left(\frac{s_{2n}^2\ \upsilon}{\lambda_1^{\alpha}}\right)^m\sum_{k=0}^{m}\binom{m}{k}\left(\lambda_2^{\beta}\right)^k t^{(1+\alpha)m-\beta k}\,E_{\alpha,(\alpha+m+1-\beta k)}^{(m-1)}(-\lambda_1^{-\alpha}t^{\alpha})$$

7.Numerical result and conclusion

In this paper we established the effect of MHD flow of the unsteady helical flows of an Oldroyd-B fluid in concentric cylinders and circular cylinder. The exact solution of the unsteady helical flows of an Oldroyd-B fluid in an annular for the velocity field u , w and the associated shear stresses τ_1 , τ_2 are obtained by using Hankel transform and Laplace transform for fractional calculus . Moreover , some figures are plotted to show the behavior of various parameters involved in the expression of velocities w ,u (Eqs(51 and 52)), shear stresses $\tau_1=\boldsymbol{S}_{rz}$, $\tau_2=\ \boldsymbol{S}_{r\theta}$ (Eqs(53 and 54)),respectively .

All the result in this section are plotting graph by using MATHEMATICA package .

Fig(1) elucidate the influence of fractional parameter fractional parameter α . the velocity u is decreasing with increase of α . Fig(2) is prepared to show the effect of the of fractional parameter β .the velocity field u is increasing with increase fractional parameter β. Fig(3) provide the graphically illustrations for the effects of relaxation λ_1 . the velocity is increasing with increase relaxation λ_1 . Figs(4-6) are established to show the behavior of the (retardation λ_2 , kinematic viscosity *v* and magnetic parameter M), respectively on the velocity , we can see the velocity u is increasing with increase the parameters retardation λ_2 , kinematic viscosity *v* and magnetic parameter M, respectively.

In Fig(7) the velocity u is plotted for different times .The results indicated that the velocity of u is increased when $r<1.2599$ and the velocity is decreased when $1.2599<r<1.7598$ but it is increased when $r>1.7598$.

Fig(8) elucidate the influence of fractional parameter fractional parameter α . the velocity u is decreasing with increase of α . Fig(9) is prepared to show the effect of the of fractional parameter β .the velocity field u is increasing with increase fractional parameter β. Fig(10) provide the graphically illustrations for the effects of relaxation λ_1 . the velocity is increasing with increase relaxation λ_1 . Figs(11-13) are established to show the behavior of the (retardation λ_2 , kinematic viscosity *v* and magnetic parameter M), respectively on the velocity , we can see the velocity u is increasing with increase the parameters retardation λ_2 , kinematic viscosity *v* and magnetic parameter M, respectively.

In Fig(14) the velocity u is plotted for different times .The results indicated that the velocity of u is increased when $r<1.1599$ and the velocity is decreased when $1.1599<r<1.8575$ but it is increased when $r>1.8575$.

The velocity w

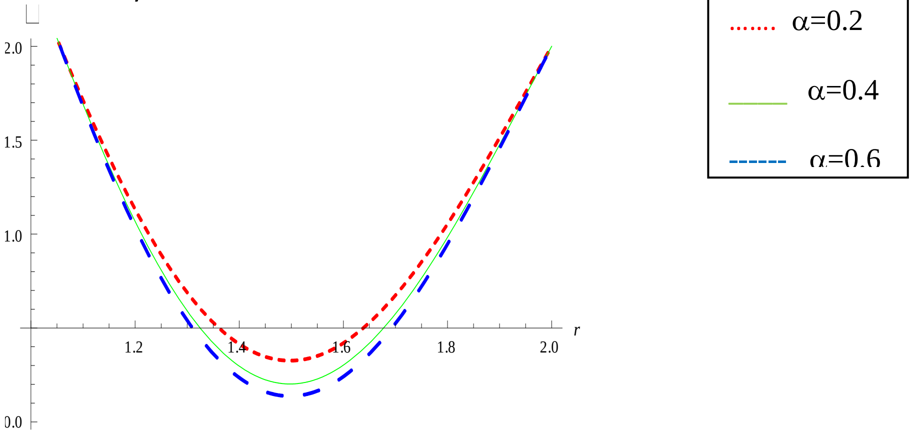

Fig1:- the velocity u for different value α {λ1=15 , λ2=8 , *v*=0.165 , M=0.1, β=0.6, K_1=3.31114 , t=1, R1=1, R2=2, u1=2, u2=2}

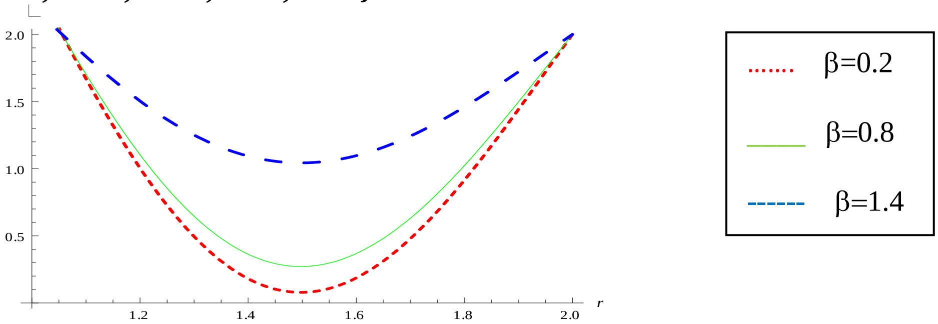

Fig2:- the velocity w for different value β {λ1=15 , λ2=8 , *v*=0.165 , M=0.1, α=0.4, K_1=3.31114 , t=1, R1=1, R2=2, u1=2, u2=2}

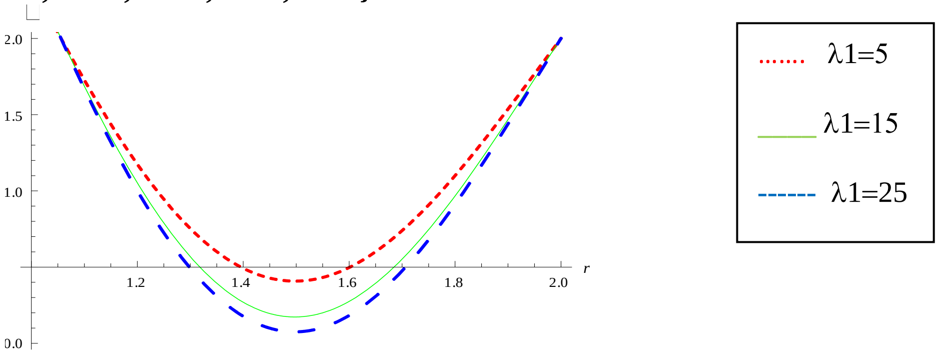

Fig3:- the velocity w for different value λ1 {λ2=8 , *v*=0.165 , M=0.1, β=0.6, α=0.4, K_1=3.31114 , t=1, R1=1, R2=2, u1=2, u2=2}

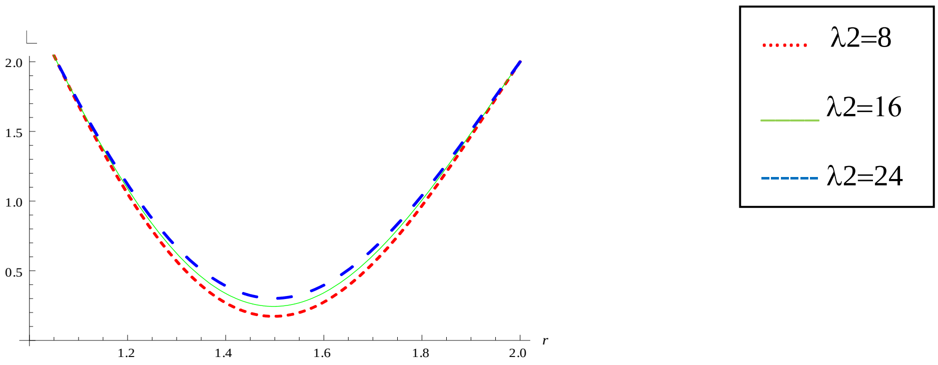

Fig4:- the velocity w for different value λ2 {λ1=15 , *v*=0.165 , M=0.1, α=0.4, K_1=3.31114 , t=1, R1=1, R2=2, u1=2, u2=2}

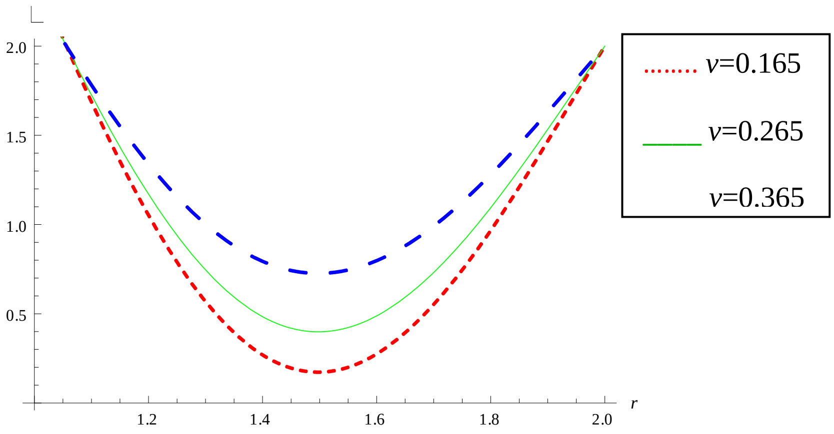

Fig5:- the velocity w for different value *v* {λ1=15 , λ2=8 , M=0.1 , β=0.6, α=0.4, K_1=3.31114 , t=1, R1=1, R2=2, u1=2, u2=2}

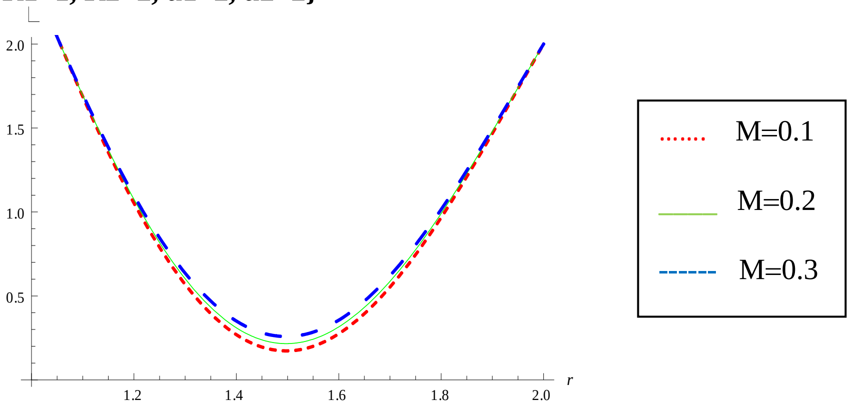

Fig6:- the velocity w for different value M {λ1=15 , λ2=8 , *v*=0.165 , β=0.6, α=0.4, K_1=3.31114 , t=1, R1=1, R2=2, u1=2, u2=2}

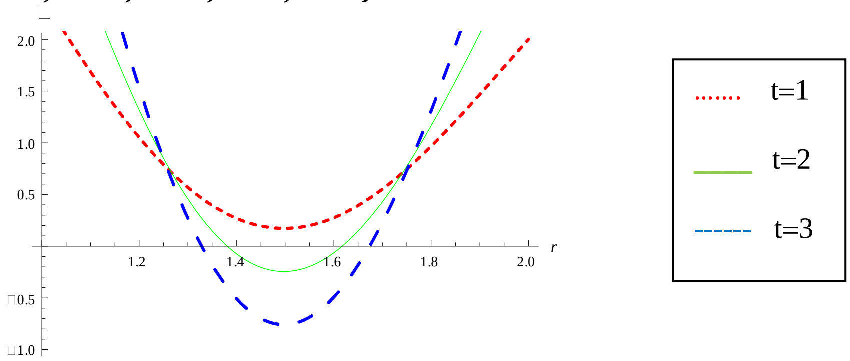

Fig7:- the velocity w for different value t **{λ1=15 , λ2=8 , *v*=0.165 , M=0.1, β=0.6, α=0.4, K_1=3.31114, R1=1, R2=2, u1=2, u2=2}**

The velocity u

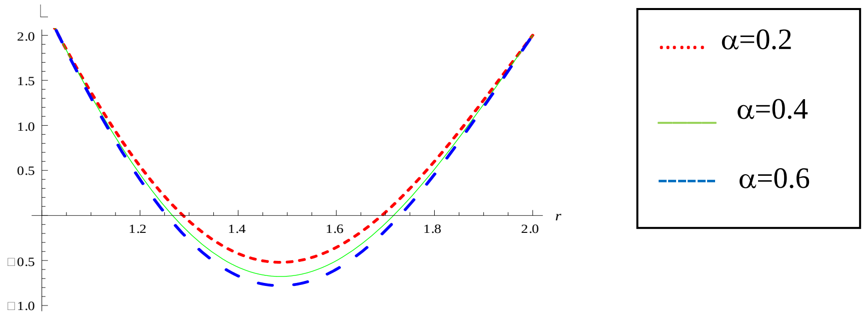

Fig8:- the velocity u for different value α {λ1=15 , λ2=8 , *v*=0.165 , M=0.1, β=0.6, K_1=3.31114 , t=1, R1=1, R2=2, u1=2, u2=1}

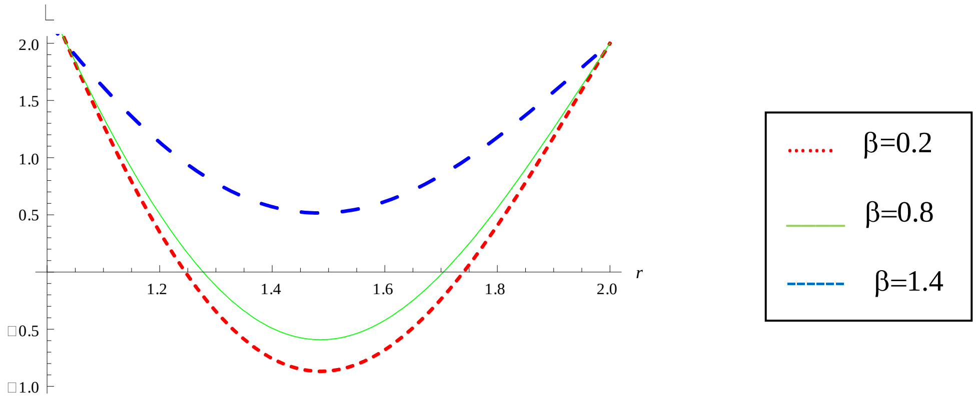

Fig9:- the velocity u for different value β {λ1=15 , λ2=8 , v=0.165 , M=0.1, α=0.4, K_1=3.31114 , t=2, R1=1, R2=2, u1=2, u2=1}

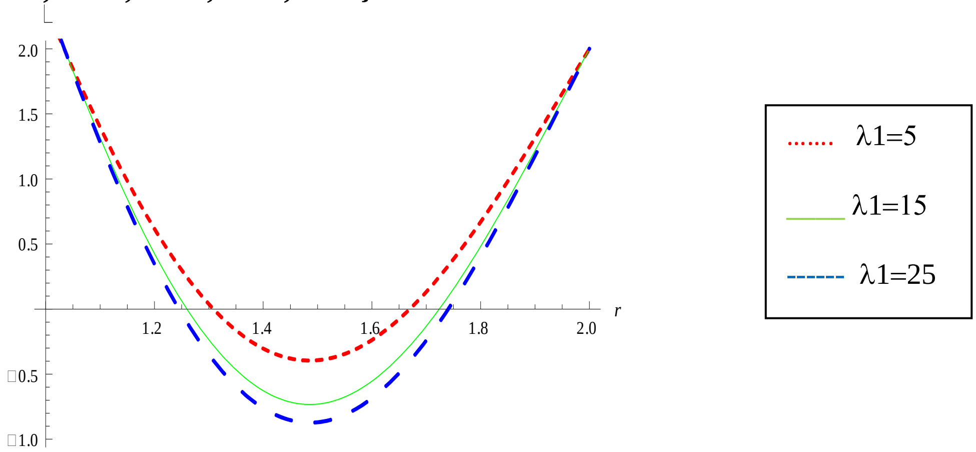

Fig10:- the velocity u for different value λ1 {λ2=8 , v=0.165 , M=0.1, β=0.6, α=0.4, K_1=3.31114 , t=1, R1=1, R2=2, u1=2, u2=1}

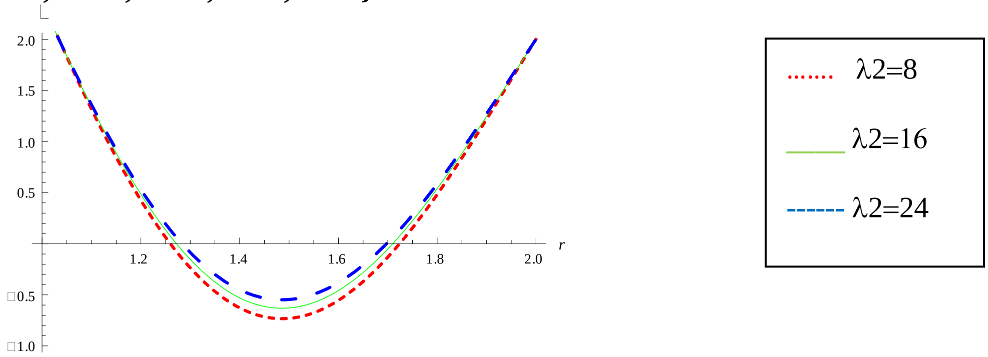

Fig11:- the velocity u for different value λ2 {λ1=15 , v=0.165 , M=0.1, β=0.6 , α=0.4, K_1=3.31114 , t=1, R1=1, R2=2, u1=2, u2=1}

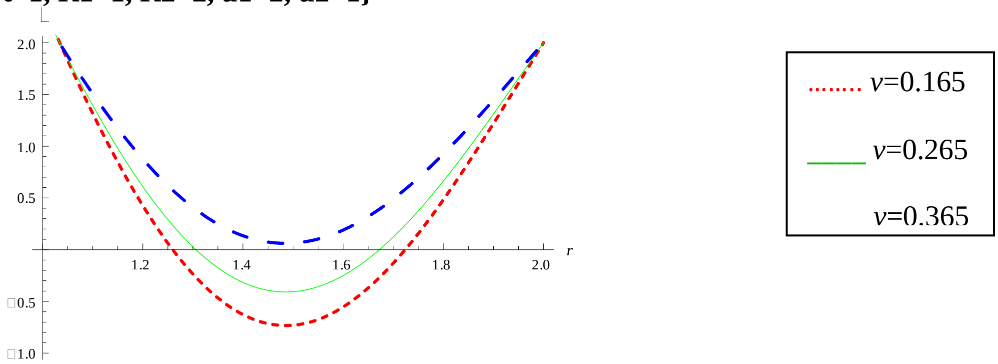

Fig12:- the velocity u for different value v {λ1=15 , λ2=8 , M=0.1 , β=0.6, α=0.4, K_1=3.31114, R1=1, R2=2, u1=2, u2=2, t=1}

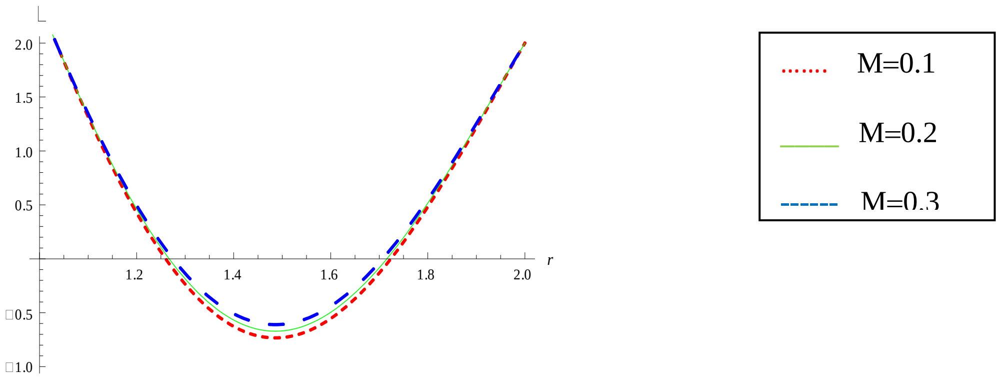

Fig13:- the velocity u for different value M **{λ1=15 , λ2=8 , v=0.165 , β=0.6, α=0.4, K_1=3.31114 , t=1, R1=1, R2=2, u1=2, u2=1}**

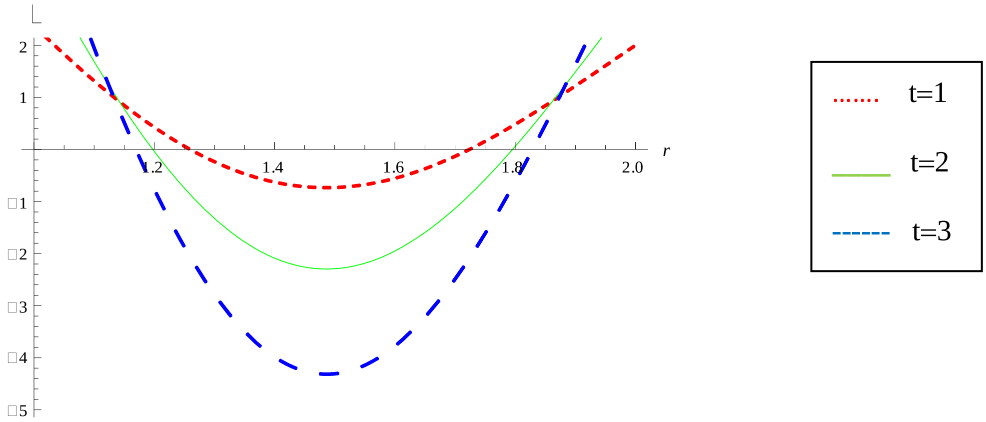

Fig14:- the velocity u for different value t {λ1=15 , λ2=8 , v=0.165 , M=0.1, , β=0.6, α=0.4, K_1=3.31114 , R1=1, R2=2, u1=2, u2=1}

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