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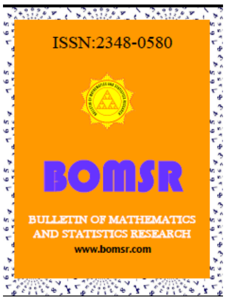
RESEARCH ARTICLE

HOMOLOGY GROUPS, BETTI NUMBERS, EULER CHARACTERISTICS AND THE GENUS OF A COMPLEX AND ITS FOLDING

E.EL-Kholy & N. El-Sharkawey

Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt



ABSTRACT

In this paper, we clarified the relations between homology groups, Betti numbers, Euler characteristic and the genus of a regular CW-complex M and its image *f* (*M*) if the map f is a cellular folding and a neat cellular folding. Finally, we obtained the corresponding relations between M and (g*o**f*)(*M*), if *f* and *g* are both cellular foldings or *f* a is cellular while *g* is a neat cellular folding.

Key words: CW-complex, Cellular folding, Homology groups, Betti numbers, Euler characteristic, genus.

1. INTRODUCTION

1.1 A pair (χ , ζ) consisting of a Hausdorff space χ and a cell – decomposition ζ of χ is called a ***CW – Complex*** if the following three axioms are satisfied: -

1- (Characteistic Maps) : For each n-cell e ϵ ζ there is a continous map ϕₑ : D_n → χ restricting to a homeomophism $\phi_{e|int\,(D_n)}$: int(D_n) → e and taking S^{n-1} into X^{n-1}.
2- (Closure Finiteness): For any cell e ϵ ζ the closure $\bar{e}$ intersects only a finite numbe of othe cells in ζ .
3- (Weak Topology): A subset A ⊆ is closed iff A ⋂ $\bar{e}$ is closed in X for e ϵ ζ , [3].

Let M be a directed complex. An (integral) ***k- chain*** C in M is a: σ_1, : σ_2, : σ_3, ….., σ_n sum c = $a_1\sigma_1 + a_2\sigma_2$+…… …. +$a_n\sigma_n$, whee are k – cells in M and a_1, : a_2, : a_3, ….., a_n ae integers. Define 0: σ = ϕ. The k – chains in a diected complex M forms a group, this group is called the group of all k – chains and is denoted ny C_k (M) for k = 0,1, …….. , dim (M).

1.2 If C is a k – chain in a directed complex K and ∂(C) = ϕ, then C is a k – cycle . the set of all k – cycles in M is Z_k (M) ⊆ C_k (M).

If C is a K – cycle in a directed complex M such that thee exists a (k + 1)- chain D with ∂(D) = C, then C is a k – boundary. The set of all k – boundaries in M is is B_k (M) ⊆ C_k (M).

The k – chains C_1 and C_2 are ***homologous*** is $C_1 \sim C_2$ ϵ B_k (k), i.e., if C_1 - C_2 = ∂(D) for some (k + 1) – chain D.

Now, the kth ***homology group,*** of M is H_k (M) = Z_k (M)/ B_k (M), the goup of equivalence classes of elements of Z_k (M) with the homology relation. In other words, H_k (M) is z_k (M) with homology used instead of equality, [4] .

1.3 There is an obvious relation between the Euler Characteristic $\varkappa$(M) and the chain groups, since C_k (M) is generated by the k – cells. Thus, the rank rk(C_k (M)) , is the number of k – cells. For a 2 – complex M, $\varkappa$(M) = v- e + f = rk(C_0 (M)) - rk (C_1 (M)) + rk(C_2 (M)).

In general, for an n – complex M,

$\varkappa$(M) = rk(C_0 (M)) - rk (C_1 (M)) + rk(C_2 (M)) -……….+ (-1)^n rk(C_n (K)), [4].

The ***Betti number*** , β(G), of a complex M are β(G) = rk(H_K (M)). But H_K (M))= Z_k (K)/ B_k (M), so β_k = z_k - b_k. Thus for an n – complex M, we have $\varkappa$(M) = C_0 - C_1 (M)) + rk(C_2 (M)) -……….+ (-1)^n β_n

Not that the **genus** g of a compact suface M is given by

$$g(M) = \begin{cases} \frac{1}{2}(2-x) & if\ M\ is\ orientable \\ 2-x & if\ M\ is\ non\ orientable \end{cases}$$

1.4. Let M and N be cellular complexes and f : |M | → | N | a continuous map. Then f: M→ N is a

(i) for each cell $\sigma \in M, f(\sigma)$ is a cell in N,

(ii) dim(f(σ)) ≤ dim (σ), [4].

The notion of cellular foldings are invented by E.EL –Kholy and H.A.AL.Khurasani [1].

Let M and N be regular CW – complexes. A cellula map f : M → N is a **cellular folding** iff

(i) For each i-cell $\sigma^i \in M, f(\sigma^i)$ is an i-cell in N i.e., f maps i-cells to i-cells,

(ii) If $\bar{\sigma}$ contains n vertices, then $\overline{f(\sigma)}$ must contains n distinct vertices [2].

2. The relations between homology groups, Betti numbers, Eule characteristics and the genus of a complex and its Image unde a cellular folding:

Fom now on by a complex we mean a regula C – complex.

Theorem (2.1): Let M and N be complexes of the same dimension n. if f ϵ C(M,N), such that f ϵ C(M,N), such that f(M) = N ≠ M then:

1) H_P(M) ≅ H_P(N) $\oplus\ kerf_p^\star$

2) β_P(M) ≅ β_P(N) $+ rk\ (kerf_p^\star\ )$

3) x (M) ≅ x(N) + $\sum_{p=0}^{n}(-)^p rk(\ker f_p^*)$.

4) g(M) =

$$\begin{cases} g(N) - \frac{1}{2}\sum_{p=0}^{n}(-)^p rk(\ker f_p^*), & if\ M\ is\ orientable \\ g(N) - \sum_{p=0}^{n}(-)^p rk(\ker f_p^*), & if\ M\ is\ non\ orientable \end{cases}$$

Where f_p^*: H_P(M) → H_P(N) is the induced homomorphism.

Proof:

1) Consider the induced homomorphism f_p^*: H_P(M) → H_P(N), there is a shot exact sequence

$$0 \rightarrow kerf_p^* \xrightarrow{i^*} H_P(M) \xrightarrow{f_p^*} \mathrm{Im} f_p^* \rightarrow 0,$$

Where $i^\star$ is the induced homomorphism by the inclusion.

Since f is surjective, we have Im f_p^* ≅ H_P(N), hence the aboe sequence with take the form,

$$0 \rightarrow kerf_p^* \xrightarrow{i^*} H_P(M) \xrightarrow{f_p^*} H_P(N) \rightarrow 0.$$

This sequence can be spilt by the homomorphism

h: H_P(N) → H_P(M) such that h(σ) = f^{x-1}(σ) fo all σ ϵ H_P(N) and hence we have the result.

From (1), we have H_P(M) ≅ H_P(N) $\oplus$ $kerf_p^*$. Thus

rk (H_P(M)) = rkH_P(N) $\oplus$ $kerf_p^*$ = rk (H_P(N)) + rk ($kerf_p^*$).

Therefore, β_P(M) = β_P(N) $+ rk\ (kerf_p^\star\ )$.

2) From (2), we have β_P(M) = β_P(N) $+ rk\ (kerf_p^\star\ )$, for p = 0,1,…….., n. Then

β_0(M) $\cong \beta_0$(N) $+ rk\ (ker f_0^\star\ )$,
β_1(M) $\cong \beta_1$(N) $+ rk\ (ker f_1^\star\ )$,
.
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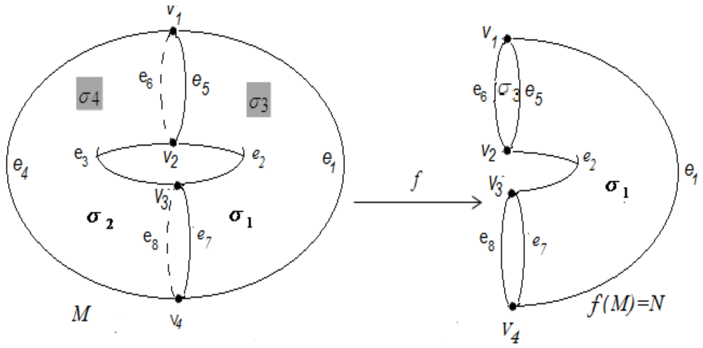
.
β_n(M) $\cong \beta_n$(N) $+ rk\ (ker f_n^\star\ )$,
But x (M) = $\beta_0 - \beta_1 + \beta_2 +$ ....(-)^n β_n = $\sum_{p=0}^{n} \beta_p$(M).
Thus, we have
x (M) = $\beta_0 - \beta_1 + \beta_2 +$ ....(-)^n β_n (N) + $\sum_{p=0}^{n} (-)^P$ rk(ker f_P^*).
Therefore, x (M) = x(N) + $\sum_{p=0}^{n} (-)^P$ rk(ker f_P^*).

3) If M is orientable, then g = $\frac{1}{2}$ (2 – x). Thus X = 2 – 2g.
Now, 2 – 2g (M) = x(M) + $\sum_{p=0}^{n} (-)^P$ rk(ker f_P^*)
= 2 – 2g(N) + $\sum_{p=0}^{n} (-)^P$ rk(ker f_P^*).
and hence g(M) = g(N) - $\frac{1}{2}\sum_{p=0}^{n} (-)^P$ rk(ker f_P^*).
If M is non oientable, then g = (2 – x). Then x= 2 – g.
Now, 2 – g(M) = 2 – g(N) + $\sum_{p=0}^{n} (-)^P$ rk(ker f_P^*) and
hence g (M) = g (N) - $\sum_{p=0}^{n} (-)^P$ rk(ker f_P^*).

Example(2.2): Let M be a complex, | M| is a tours, with cellular subdivisions shown in Fig.(1). Let f be the cellula folding defined by f(v_i) = v_i, i = 1,2,3,, f(e_3, e_4, e_6, e_8) = (e_2, e_1, e_5, e_7) And $f(\sigma_2, \sigma_4) = f\ (\sigma_1, \sigma_3)$, where the omitted cells through this paper will be mapped to themselves. The image f(M) = N, [N] is cylinder, is a complex with cellula subdivisions shown in Fig.(1).



Note that : H_0(M) $\cong$ Z H_0(N) $\cong$ Z, ker $f_0^* \cong$ 0
H_1(M) $\cong$ Z $\oplus Z$ H_1(N) $\cong$ Z, ker $f_1^* \cong$ Z
H_2(M) $\cong$ Z H_2(N) $\cong$ 0 , ker $f_2^* \cong Z$.
Hence the first relations are satisfied.
Also, the Betti Numbers of M are β_2 = 2, β_1 = 4, β_0 = 2, the Betti numbes of N are :
β_2 = 1, β_1 = 3, β_0 = 2 and the ranks of ker f_p^* are:
rk ker $f_2^* = 1$, rk ker $f_1^* = 1$, rk ker $f_0^* = 0$.
Also, the Euler characteristic of M is :
X (M) = (β_0 - β_1+ β_2) (M) = 2 – 4 + 2 = 0,
the Euler characteristic of N is :
X (N) = (β_0 - β_1+ β_2) (N) = 2 – 3 + 1 = 0,
And $\sum_{p=0}^{2} (-)^P$ rk(ker f_0^*) = rk(ker f_1^*) + rk(ker f_2^*) = 0 – 1 + 1 = 0.
Since M and N are orientables, the g(M) =
then g (M) = $\frac{1}{2}$ (2 – x) = $\frac{1}{2}$ (2- 0) = 1, g (N) = $\frac{1}{2}$ (2- x) = $\frac{1}{2}$ (2- 0)= 1 and
$\sum_{p=0}^{2} (-)^P$ rk(ker f_P^*) = rk(ker f_0^*) - rk(ker f_1^*) + rk(ker f_2^*) = 0 – 1 + 1 = 0

Thus the relations of Theoem (2.1) are satisfied.

3 The relations between homology goups, Betti numbes, Eulers characteristics and the genus of a complex and its image under neat cellula folding:

Theorem (3.1): Let M and N be complexes of the same dimension n. if $f \in \mathcal{N}(M, N)$, such that f (M) = N $\neq M. then$

1) H_P(M) $\cong kerf_p^\star$, p ≥ 1
2) β_P(M) $= rk\ (kerf_p^\star)$
3) x (M) = 1 + $\sum_{p=1}^{n}(-)^p rk(\ker f_p^*)$.
4) g(M) =

$$\begin{cases} \frac{1}{2}[1 - \sum_{p=0}^{n}(-)^p rk(\ker f_p^*), & if\ M\ is\ orientable \\ 1 - \sum_{p=0}^{n}(-)^p rk(\ker f_p^*), & if\ M\ is\ non\ orientable \end{cases}$$

Where f_p^*: H_P(M) → H_P(N) is the induced homomorphism.

Proof:

1) Consider the induced homomorphism f_p^*: H_P(M) → H_P(N),

Again, there is a short exact sequence

$$0 \to kerf_p^* \xrightarrow{i^*} H_P(M) \xrightarrow{f_p^*} \mathrm{Im} f_p^* \to 0,$$

Where $i^\star$ is the induced homomorphism by the inclusion.

Since f is surjective, we have

Im $f_p^* \cong H_P$(N), but H_P(N)$\cong$ 0

for a neat cellular folding, hence the above sequence will take the form,

$0 \to kerf_p^* \xrightarrow{i^*} H_P(M) \to 0$, and the proof of (1)

Will then follows fom the exactness of the sequence.

2) Form (1), we have H_P(M) $\cong$ $kerf_p^*$, p $\geq$ 1. Thus

rkH_P(M) $\cong$ $rk(\ kerf_p^*)$ and hence β_P(M) $\cong$ $rk(\ kerf_p^*)$.

3) From (2) , we have

β_P(M) = $rk\ (kerf_p^\star)$, for p = 1,2, …….., n. Then

β_1(M) = $rk\ (kerf_1^\star)$,

β_2(M) = $rk\ (kerf_2^\star)$,

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.

β_n(M) = $rk\ (kerf_n^\star)$,

Thus, we have

$\beta_0 - \beta_1 + \beta_2 +$ ….(-)^n β_n (N) = $\beta_0(L)$ + $\sum_{p=0}^{n}(-)^P$ rk(ker f_P^*).

and hence x (M) = 1 + $\sum_{p=1}^{n}(-)^P$ rk(ker f_P^*).

4) If M is orientable, then X = 2 – 2g.

and hence g(M) = $\frac{1}{2}\sum_{p=1}^{n}(-)^P$ rk(ker f_P^*).

If M is non oientable, then X = (2 – g).

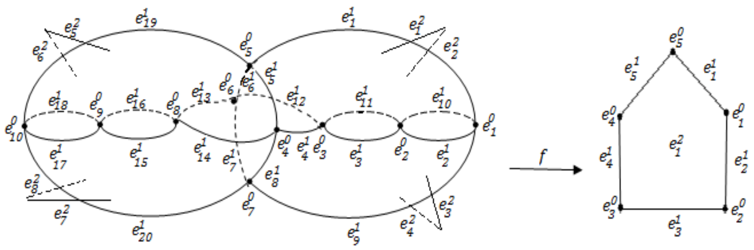
Now, 2 – g(M) = 1 + $\sum_{p=1}^{n}(-)^P$ rk(ker f_P^*) and

hence g (M) = 1) - $\sum_{p=1}^{n}(-)^P$ rk(ker f_P^*).

Example (3.2): Let M be a complex such that | M | = T # T, the double tours, with the cellular subdivisions consisting of ten 0 – cells, twenty 1 – cells and eight 2 – cells, see Fig (2).

Let f : M → M be the neat cellula foldings defined by:

$f\,(e_6^0,\ldots\ldots, e_{10}^0) = (e_4^0, e_5^0, e_3^0, e_2^0, e_1^0)$, $f\,(e_6^1,\ldots\ldots, e_{20}^1)$
$= ((e_5^1, e_5^1, e_5^1, e_1^1, e_2^1, e_2^1, e_3^1 e_4^1, e_4^1, e_4^1, e_3^1, e_2^1, e_2^1, e_1^1, , e_1^1)$ and $f\,(e_i^2) = e_1^2$, I = 1,2,……., 8. The image f (M) = N, |N| where is a disc with cellular subdivision consists of five 0 – cells and one 2- cell.



M f(M)

Fig. (2)

The relations of Theorem (3.1) can be easily be checked.

4 - The relations between homology groups, Betti numbers, Euler characteristics and the genus of a complex and its image under composition of cellular foldings:

First if each of foldings maps is a cellula foldings:

Theorem (4.1): Let M, N and L be complexes of the same dimension n. If $f \in c\ (M, N)$ *such that f(M)* = N $\neq M$ *and* g ϵ C (N, L) such that g (N) = L $\neq N$. Then (g o f) ϵ C (M ,L) and the following conditions are satisfied:

1. H_P(M) ≅ H_P(L) ⊕ $\ker(\mathrm{g\ o\ f})_p^*$
2. β_P(M) ≅ β_P(L) $+ rk$ (ker (g o f $)_p^*$)
3. x (M) ≅ x(L) + $\sum_{p=0}^{n}(-)^p rk(\ker\ (g\ o\ f\ )_p^*)$.
4. g(M) =

$$\begin{cases} g(L\,) - \frac{1}{2}\sum_{p=0}^{n}(-)^p rk(\ker(\,g\ o\ f\,)_p^*), & \textit{in the case of is orientable complexes} \\ g(L) - \sum_{p=0}^{n}(-)^p rk(\ker(\,g\ o\ f)_p^*), & \textit{in the case of non orientable complexes} \end{cases}$$

Where $(\,g\ o\ f)_p^* = g_p^*\ o\ f_p^*$: H_P(M) → H_P(N) is the induced homomorphism.

Proof:

1) Consider the induced homomorphism f_p^*: H_P(M) → H_P(N) and g_p^*: H_P(M) → H_P(L),there is a short exact sequence

$$0 \to \ker(\,g\ o\ f\,)_p^* \xrightarrow{i_2^* o\ i_1^*} H_P(\mathrm{M}) \xrightarrow{g_p^* o\ f_p^*} \mathrm{Im}(\,g_p^*\ o\ f_p^*) \to 0,$$

Where i_1^* and $i\,_2^*$ $i$is the induced homomorphism by the inclusion.

Since f and g is surjective, we have Im $(\,g\ o\ f\,)_p^* \cong H_P$(L), hence the above sequence with take the form,

$$0 \to \ker(\,g\ o\ f\,)_p^* \xrightarrow{i_2^* o\ i_1^*} H_P(\mathrm{M}) \xrightarrow{g_p^* o\ f_p^*} H_P(\mathrm{L}) \to 0.$$

This sequence can be spilt by the homomorphism

h: H_P(L) → H_P(M) such that h(σ) = $(\,g^*\ o\ f^*)^{-1}$(σ) fo all σ ϵ H_P(L) and hence we have the result.

H_P(M) ≅ H_P(N) ⊕ $\ker(g\ o\ f)_p^*$

2) From (1), we have H_P(M) ≅ H_P(N) ⊕ $\ker(g\ o\ f)_p^*$. Thus

rk (H_P(M)) = rk[(H_P(L)) + (ker($g\ o\ f)_p^*$)]

rk (H_P(L)) + rk (ker($g\ o\ f)_p^*$)]

∴ β_P(M) = β_P(L) + rk (ker($g\ o\ f)_p^*$)]

3) Since,

β_P(M) = β_P(N) $+ rk\ (kerf_p^{\star}\ )$, for p = 0,1,........, n. Then

We can prove that

4) Once again if M is orientable, then g = 2 – 2 g). Thus X = 2 – 2g, and we have

g (M) = g(L) + $\sum_{p=0}^{n}(-)^P$ rk(ker $(g\ o\ f\ )_P^*$).

Also if L is non orientable, then g = (2 – x). Thusx = 2 – g, and we have

g (M) = g(L) + $\sum_{p=0}^{n}(-)^P$ rk(ker $(g\ o\ f\ )_P^*$).

Example(4.2): Let M be a complex with the cellular subdivisions consisting of eight 0 – cells, Sixteen 1 – cells and eight 2 – cells, see Fig (3).

Let f : M → M be the neat cellula foldings defined by:

$f\,(e_5^0, e_6^0)$ = (e_1^0, e_2^0) , $f\ (e_7^0, \dots \dots \dots .. e_{12}^0)$

= $((e_3^1, e_4^1, e_2^1, e_1^1, e_{15}^1, e_{16}^1)$ and $f\,(e_3^2, \dots\dots\dots, e_6^2)$ = $(e_1^2, e_2^2, e_7^2, e_8^2)$.

The image f(M) = N, where | N | is a cylinder, is a complex with cellula subdivision consists of six 0 – cells, ten 1 – cells and fou 2 – cells. Now, g : N → N be the cellular folding defined by :

$g\ (e_7^0, e_8^0)$ = (e_3^0, e_4^0) , $g\ (e_{13}^0, \dots \dots \dots .. e_{17}^0)$

= $((e_6^1, e_5^1, e_4^1, e_3^1)$ and g (e_7^2, e_8^2) = (e_1^2, e_2^2) . The image g (N) = L, where | L | is a cylinder, is a complex with cellular subdivision consists of four 0 – cells, six 1 – cells, and two 2 – cells.

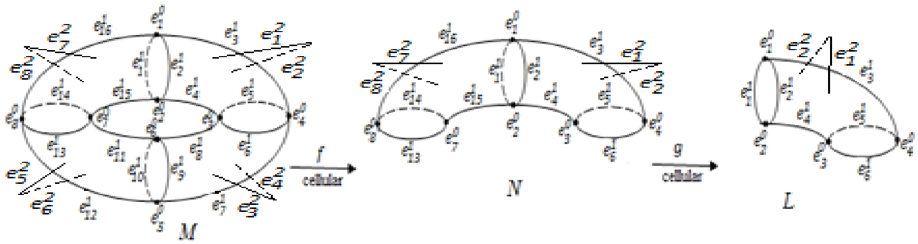


Fig . (3)

It is easy to check that conditions of Theorem (4.1) are satisfied.

Secondly if one the folding maps is cellula and the other is neat it is easy to check that the following relations:.

<u>Theorem (4.3):</u> Let M, N and L be complexes of the same dimension n. If $f \in c\ (\ M, N\ ) and$ g ϵ C (N, L) . Then (g o f) ϵ $\mathcal{N}$ (L , N) and the following conditions are satisfied:

1) H_P(M) ≅ $\ker(\mathrm{g\ o\ f})_p^*$, p ≥ 1
2) β_P(M) = $rk\ (\ker\,(\,\mathrm{g\ o\ f})_p^*\ )$
3) x (M) = 1 + $\sum_{p=1}^{n}(-)^p rk(\ker\ (g\ o\ f\ )_p^*)$.
4) g(M) =

$$\begin{cases} \frac{1}{2}\ [\ 1\ - \sum_{p=0}^{n}(-)^p\ rk(\ker(\ g\ o\ f\ )_p^*), & in\ the\ case\ of\ is\ orientable\ complexes \\ 1 - \sum_{p=0}^{n}(-)^p\ rk(\ker(\ g\ o\ f)_p^*), & in\ the\ case\ of\ non\ orientable\ complexes \end{cases}$$

Example(4.4): Let M be a complex with the cellular subdivisions consisting of four 0 – cells, eight 1 – cells and four 2 – cells, see Fig (4).

Let f : M → M be the neat cellula foldings defined by:

$f\,(e_i^0)$ = e_i^0), i = 1, 2,., 4, f (e_6^1, e_8^1) = (e_3^1, e_4^1) and f (e_3^2, e_4^2) and $f\,(e_3^2, e_4^2)$ = (e_1^2, e_2^2) .

The image f(M) = N, is a cylinder, is a complex with cellular subdivision consists of four 0 – cells, six 1 – cells and 2 – cells. Now, Let g : N → N be the cellular folding defined by :

$g\ (e_i^0)$ = e_i^0, i = 1, 2,., 4, g (e_5^1, e_7^1) = (e_4^1, e_2^1) and g $(e_2^2) = e_1^2$.

The image g (N) = L, has cellular subdivision consists of four 0 – cells, four 1 – cells, and one 2 – cells, see Fig . (4).

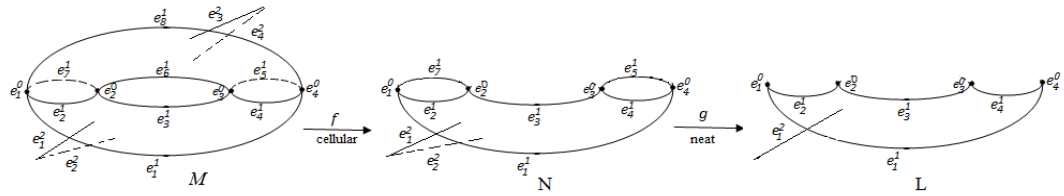


Fig . (4)

The relations of Theoem (4.3) can be easily checked.

Referance

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