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APPROXIMATE FIXED POINTS IN DISLOCATED QUASI-METRIC SPACE

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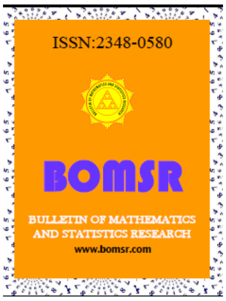

ABSTRACT

In this Paper we have proved Approximate Fixed Point Theorems in dislocated quasi-metric space.

Key-words: Dislocated metric space, dislocated quasi-metric space, approximate fixed point, dq- Cauchy sequence.

1 INTRODUCTION

The notion of dislocated metric space was introduced by Hitzler and Seda [3,4]. These metrics play very important role in many branches of science involving mathematics especially in logic programming and electronic engineering [3]. Fixed point theory is used in solving many problems in applied mathematics. It is observed that in many real situations an approximate solution is more than sufficient and problem can be solved by considering nearly fixed point instead of existence of fixed point. So the concept of ε-fixed point or approximate fixed point was introduced and that of function with approximate fixed point property. Motivated by the work of Madalina Berinde [3] we try to prove some results in dislocated quasi metric space

2 Preliminaries

In this section we give some definitions and Lemmas.

Definition 2.1 [1] Let X be a non-empty set d: $X \times X \to R^+$ be a function, called a distance function if for all x,y,z$\in$X, satisfies:

d_1 : $d(x,x) = 0$

d_2 : $d(x,y) = d(y,x) = 0 \Rightarrow x = y$

d_3 : $d(x,y) = d(y,x)$

d_4 : $d(x,y) \leq d(x,z) + d(z,y)$

If d satisfies the condition $d_1 - d_4$ then d is called a metric on X.

If it satisfies the condition d_1, d_2 and d_4, it is called quasi-metric space.

If d satisfies condition d_2 and d_4, it is called dislocated-quasi metric (or simply dq-metric) on X and a non empty set X with dq-metric i. e. (X,d) is called a dislocated quasi-metric space

Definition 2.2 [4] A sequence $< x_n >_{n\in N}$ in dq-metric space (X,d) is called Cauchy if for all $\varepsilon > 0, \exists\, n_0 \in N$ such that $\forall\, m, n \geq n_0,\ d(x_m, x_n) < \varepsilon$ or $d(x_n, x_m) < \varepsilon$

Definition 2.3 [4] A sequence $< x_n >_{n\in N}$ dislocated quasi converges or dq-converges to x if $\lim_{n\to\infty} d(x_n, x) = \lim_{x\to\infty} d(x, x_n) = 0$

Definition 2.4 [4] A dq-metric space (X, d) is complete if every Cauchy sequences in dq-metric space (X,d) converges to a point in (X,d).

Definition 2.5 [2] Let f : $X \to X$, $\varepsilon > 0$, $x_0 \in X$ then x_0 is an ε-fixed point (or approximate fixed point) of f if $d(f(x_0), x_0) < 0$ or $d(x_0, f(x_0)) < \varepsilon$ For given $\varepsilon > 0$ we write $F_\varepsilon(f) = \{ x \in X : x$ is an ε- fixed point of f$\}$

Definition 2.6 [2] Let f : $X \to X$ Then we say that f has the approximate fixed point property (a.f.p.p.) if $F_\varepsilon(f) \neq \emptyset$

Definition 2.7 [2] Let (X,d) be a dq-metric space and f : $X \to X$ Then we say f is asymptotic regular if $d(f^n(x), f^{n+1}(x)) \to 0$ or $d(f^{n+1}(x), f^n(x)) \to 0$ as $n \to \infty, \forall\, x \in X$

Lemma 2.8 [2] Let (X,d) be a dq metric space f : $X \to X$ such that f is asymptotic regular then f has the approximate fixed property.

Proof : Let $x_0 \in X$

Then $d(f^n(x_0), f^{n+1}(x_0)) \to 0$ as $n \to \infty$

$\Rightarrow$ Given $\varepsilon > 0\ \exists\, n_0 \in N$ such that $\forall\, n \geq n_0$

$d(f^n(x_0), f^{n+1}(x_0)) < \varepsilon$

or $d(f^n(x_0), f(f^n(x_0))) < \varepsilon\ \ \forall\, n \geq n_0$

Put $f^n(x_0) = y_0$

$\Rightarrow \forall\, \varepsilon > 0\ \exists\, y_0 \in X$ such that $d(y_0, f(y_0)) < \varepsilon$

Similarly $d(f(y_0), y_0) < \varepsilon$

$\Rightarrow \forall\, \varepsilon$ - fixed point of f in X which is y_0 or f has the approximate fixed point property.

Lemma 2.9 [2] Let (X,d) be a dq-metric space and f : $X \to X$

Let (i) $F_\varepsilon(f) \neq \emptyset$, $\varepsilon > 0$

(ii) $\forall\, n > 0\ \exists\, \emptyset(\eta) > 0$ such that

$d(x,y) - d(f(x), f(y)) \leq \eta \ \Rightarrow d(x,y) \leq \emptyset(\eta)\ \ \forall\, x,y \in F_\varepsilon(f)$

or $d(y,x) - d(f(y), f(x)) \leq \eta \ \Rightarrow d(y,x) \leq \emptyset(\eta)\ \ \forall\, x,y \in F_\varepsilon(f)$

Then $\delta\, (F_\varepsilon(f)) \leq \emptyset(2\varepsilon)$, where $\delta\, (F_\varepsilon(f))$ is the diameter of $(F_\varepsilon(f))$.

Proof: Let $\varepsilon > 0$ Let $x,y \in F_\varepsilon(f)$

Then $d(x, f(x)) < \varepsilon$ and $d(y, f(y)) < \varepsilon$

Now $d(x, y) \leq d(x,f(x)) + d(f(x), f(y)) + d(f(y), y)$

$\leq d(f(x),f(y)) + 2\varepsilon$

$\Rightarrow d(x, y) - d(f(x),f(y)) \leq 2\varepsilon$

or $d(x,y) \leq \emptyset(2\varepsilon)$ by (ii)

or $\delta\, (F_\varepsilon(f)) \leq \emptyset(2\varepsilon)$

3. Let (X,d) be a dq-metric space and completeness of X is not required.

Definition 3.1 [2] A mapping f : $X \to X$ is an α-contraction if $\exists\ \alpha \in (0,1)$ such that $d(f(x), f(y)) \leq \alpha\, d(x, y)\ \ \forall\, x,y \in X$

Theorem 3.1 Let (X,d) be a dq-metric space and f : $X \to X$ is an α-contraction. Then $F_\varepsilon(f) \neq \emptyset\ \ \forall\, \varepsilon > 0$

Proof : Let $x \in X$ and $\varepsilon > 0$ Then

$$
\begin{aligned}
d(f^n(x), f^{n+1}(x)) &= d(f(f^{n-1}(x)), f(f^n(x))) \\
&\leq \alpha\, d(f^{n-1}(x), f^n(x)) \\
&\leq \ldots\ldots \leq \alpha^n\, d(x, f(x))
\end{aligned}
$$

And so $d(f^n(x), f^{n+1}(x)) \to 0$ as $n \to \infty$

As $\alpha \in (0, 1)$ $\forall x \in X$ Then by Lemma (2.8)

$F_\varepsilon(f) \neq \emptyset$ $\forall \varepsilon > 0$

4. Main Result

We Prove the following theorems.

Theorem 4.1 - Let (X,d) be a dislocated quasi-metric space and $f : X \to X$ such that f is asymptotically regular if

$$d(f(x), f(y)) \leq \alpha\, d(x,y) + \beta\, \frac{d(x,f(x))\,.\, d(y,f(y))}{d(x,y)}$$

Then $\forall \varepsilon > 0$, $F_\varepsilon(f) \neq \emptyset$ where $\alpha + \beta < 1$

Proof- Let $\varepsilon > 0$ and $x \in X$. So

$$d(f^n(x), f^{n+1}(x)) = d(f(f^{n-1}(x)), f(f^n(x)))$$
$$\leq \propto d(f^{n-1}(x), f^n(x))$$
$$+\beta \quad \frac{d\big(f^{n-1}(x), f^n(x)\big).\, d\big(f^n(x), f^{n+1}(x)\big)}{d\big(f^{n-1}(x), f^n(x)\big)}$$
$$\leq \propto d(f^{n-1}(x), f^n(x)) + \beta\, d(f^n(x), f^{n+1}(x))$$

or $(1-\beta)\, d(f^n(x), f^{n+1}(x)) \leq \propto d(f^{n-1}(x), f^n(x))$

or $d(f^n(x), f^{n+1}(x)) \leq \frac{\propto}{(1-\beta)} d(f^{n-1}(x), f^n(x))$

Now put $\frac{\propto}{1-\beta} = h < 1$

$$d(f^n(x), f^{n+1}(x)) \leq h\, d(f^{n-1}(x), f^n(x))$$
$$\leq h^2\, d(f^{n-2}(x), f^{n-1}(x))$$
$$\leq \ldots\ldots\ldots \leq h^n\, d(x, f(x))$$

As $h < 1 \Rightarrow d(f^n(x), f^{n+1}(x)) \to 0$, as $n \to \infty$, $\forall x \in X$

It follows by Lemma (2.8) $F_\varepsilon(f) \neq \emptyset$

Theorem 4.2 - Let (X,d) be a dislocated quasi-metric space and $f : X \to X$ such that f is asymptotically regular if

$$d(f(x), f(y)) \leq \frac{\alpha d(y,f(y))\,.[1+ d(x,f(x))}{1 + d(x,y)} + \beta\, d(x,y)$$

Then $\forall \varepsilon > 0$, $F_\varepsilon(f) \neq \emptyset$ where $\alpha + \beta < 1$

Proof- Let $\varepsilon > 0$ and $x \in X$. So

$$d(f^n(x), f^{n+1}(x)) = d(f(f^{n-1}(x)), f(f^n(x)))$$
$$\leq \frac{\propto d(f^n(x), f(f^n(x))).\ [1+d(f^{n-1}(x), f\, f^{n-1}(x))]}{1+ d\big(f^{n-1}(x), f^n(x)\big)}$$
$$+ \beta\, d(f^{n-1}(x), f^n(x))$$
$$= \frac{\propto d\big(f^n(x), (f^{n+1}(x)\big).\ [1+d\big(f^{n-1}(x), f^n(x)\big)}{1+ d\big(f^{n-1}(x), f^n(x)\big)}$$
$$+ \beta\, d(f^{n-1}(x), f^n(x))$$

or $(1-\propto)\, d(f^n(x), f^{n+1}(x)) \leq \beta\, d(f^{n-1}(x), f^n(x))$

or $d(f^n(x), f^{n+1}(x)) \leq \frac{\beta}{1-\propto} d(f^{n-1}(x), f^n(x))$

Put $\frac{\beta}{1-\propto} = h < 1$

$$d(f^n(x), f^{n+1}(x)) \leq h\, d(f^{n-1}(x), f^n(x))$$
$$\leq h^2 d(f^{n-2}(x), f^{n-1}(x))$$
$$\leq \ldots\ldots\ldots \leq h^n\, d(x, f(x))$$

as $h < 1 \Rightarrow d(f^n(x), f^{n+1}(x)) \to 0$, as $n \to \infty$, $\forall x \in X$

It follows by Lemma (2.8) $F_\varepsilon(f) \neq \emptyset$

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