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RESEARCH ARTICLE

# ON THE GEOMETRY OF FOLIATED MANIFOLDS

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**ABSTRACT**

In this paper we study geometry of geodesic lines and topology of the group of isometries of foliated manifold. In first part we prove that the limit of geodesic lines of foliated manifold is geodesic line of foliated manifold. In second part we study topology of the set of isometries of foliated manifold.

**Keywords***:* Riemannian manifold, foliation, isometric mapping of foliated manifold, geodesic line of foliated manifold, foliated compact - open topology.



## 1. INTRODUCTION

The foliation theory is a branch of the geometry which has arisen in the second half of the XX th century on a joint of ordinary differential equations and the differential topology. Now the foliation theory is intensively developed, has wide applications in various areas of mathematics - such, as the optimal control theory, the theory of dynamic polysystems. There are numerous researches on the foliation theory. The review of the last scientific works on the foliation theory and very big bibliography is presented in work of Ph. Tondeur (Tondeur Ph., 1988).

The purpose of first part of this paper is to study the geometry of the geodesics of foliated manifold, namely the question on limit of geodesics of foliated manifold.

The purpose of second part of this paper is to study the group $G_F^r(M)$ of isometries of foliated manifold $(M,F)$ with a certain topology, which was introduced in the paper (Narmanov A. & Sharipov A., 2009), depending on the foliation $F$, such that it coincides with the compact - open topology if $F$ is an $n$- dimensional foliation. If the codimension of the foliation $F$ is equal to $n$, convergence in the introduced topology coincides with the pointwise convergence. Papers (Skorobogatov D., 2000), (Narmanov A. & Skorobogatov D., 2004) are devoted to isometric mappings of foliated manifold. In these papers it is investigated the question under what conditions any isometry of the foliation will be an isometry of the manifold. In addition it is proved that the existence of a diffeomorfism of a foliated manifold onto itself which is an isometry of the foliation, but it is not an isometry of the manifold. It is constructed an example of a diffeomorfism of three - dimensional sphere which is an isometry of the Hopf fibration but is not an isometry of the three - dimensional sphere.

## 2. MAIN PART

### I. GEOMETRY OF THE GEODESICS OF FOLIATED MANIFOLD

In modern Riemannian geometry, one of the main objectives is the geometry of the geodesic lines of the Riemannian manifold, i.e. in particular the question on the limit of the geodesic lines. It is known that the limit of the geodesic lines of Riemannian manifold is a geodesic line. In the work (Helgason S., 2001) it is proved a remarkable theorems on the geodesic lines. In the case of a foliated manifold, this question is complicated by the fact that a geodesic line of foliated manifold is not necessary geodesic line of the manifold.

We have the following theorem which is a analog of classic theorem of Riemannian geometry.

**Theorem-1.** Let $M$ be a smooth complete Riemannian manifold of dimension $n$ with a smooth foliation $F$ of dimension $k$, where $0<k<n$. Let $\gamma_m : R^1 \to L_m$ be a sequence of geodesics (determined by the induced Riemannian metrics) on leaves $L_m$. If $\gamma_m(s_o) \to p$ for $m \to \infty$ for some $s_o \in R^1$, then there exists a subsequence $\gamma_{m_l}$ of the sequence $\gamma_m$ which pointwise converges to some geodesic $\gamma : R^1 \to L_{(p)}$ of the leaf L(p), passing through the point p at $s = s_0$.

**Proof of Theorem-1.** Let $M$ be a $n$- dimensional smooth connected Riemannian manifold with a Riemannian metric $g$, $F$ a smooth $k$ dimensional foliation on $M$.

Let denote by $L(p)$ the leaf of the foliation $F$, passing through the point $p$, by $T_pF$ tangent space to the leaf $L(p)$ at $p$, and by $H_pF$ it's orthogonal complement of $T_pF$ in $T_pM$, $p \in M$. We get two subbundles (smooth distributions) $TF = \{T_pF : p \in M\}$, $HF = \{H_pF : p \in M\}$ of the tangent bundle $TM$ of the manifold $M$ and, as a result, the tangent bundle $TM$ of the manifold $M$ decomposing into the sum of two orthogonal bundles, i.e. $TM = TF \oplus HF$ (Sharipov A., 2015).

The restriction of the Riemannian metric g on $T_pF$ for all $p$ induces a Riemannian metric on the leaves. The induced Riemannian metric a defines distance function on every leaf. Further, throughout in this paper, under the distance on a leaf is understood this distance. This distance on a leaf is different from distance induced by the distance on $M$.

Let $\pi : TM \to TF$ be orthogonal projection, $V(M)$, $V(F)$, $V(H)$ be the set of smooth sections of bundles $TM$, $TF$, $HF$ respectively.

We put $\tilde{\nabla}_X Y = \pi(\nabla_X Y)$ for vector fields $X \in V(M), Y \in V(F)$, where $\nabla$ is a Levi-Civita connection, determined by the Riemannian metric $g$ on $M$. It is known, that $\tilde{\nabla}_X Y$ is a connection on $TF$, and it's restriction to each leaf $L_\alpha$ coincides with connection on $L_\alpha$, determined by the induced Riemannian metric on $L_\alpha$ from $M$ (Tondeur Ph., 1988). Therefore, the smooth parametric curve $\mu : (a,b) \to M$, lying on a Leaf $L_\alpha$ of the foliation F, is geodesic on $L_\alpha$ (determined by the induced Riemannian metric) if and only if

$$\tilde{\nabla}_{\dot\mu}\dot\mu = 0. \tag{1}$$

If $\mu$ lies in the foliated neighborhood $U$, it's equations have the form:

$$\begin{cases} x^i = x^i(s) \\ y^\alpha = const \end{cases}$$

where $1 \le i \le k, k+1 \le \alpha \le n$.

So, for $\nabla$ we have

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \tilde{A}^l_{i,j} \frac{\partial}{\partial x^l} + \tilde{A}^a_{i,j} \frac{\partial}{\partial y^a} \tag{2}$$

hence,

$$\tilde{\nabla}_{\frac{\partial}{\partial x^i}} = \tilde{A}^l_{i,j} \frac{\partial}{\partial x^l} \tag{3}$$

where $1 < i, j, l \le k,\ k+1 \le \alpha \le n$, $\tilde{A}^{\beta}_{i,j}$ are Christoffel symbols.

From here using properties of the operator $\tilde{\nabla}$ it follows, that equation (1) is equivalent to the following system of the differential equations of the 2-nd order:

$$\frac{d^2 x^i}{ds^2} + \tilde{A}^i_{l,j} \frac{dx^l}{ds} \frac{dx^j}{ds} = 0. \tag{4}$$

If we put $u^i = \dfrac{dx^i}{ds}$ then it is possible to write this system as:

$$\begin{cases} \dfrac{dx^i}{ds} = u^i \\ \dfrac{du^i}{ds} = -\tilde{A}^i_{l,j} u^l u^j \end{cases} \tag{5}$$

With no loss of generality, we can assume those geodesics $\gamma_m : R^1 \to L_m$ are parameterized by length of the arc and $s_0 = 0$.

Now we show that there exists a subsequence of the sequence of tangential vectors $\dot{\gamma}_m(0)$, which converges. Since $|\dot{\gamma}_m(0)| = 1$ for each m, we follow equality

$$\sum_{i,j=1}^{k} g_{ij}(p_m) u^i_m u^j_m = 1 \tag{6}$$

for enough large integer number $m$, where $p_m = \gamma_m(0)$, $g_{ij}$ are coefficients of the Riemannian metric g, $\{u^i_m\}$ are the first $k$ coordinates of vector $\dot{\gamma}_m(0)$ on the foliated neighborhood $U$ of the point $p$.

Equality (6) is meaningful for $\gamma_m(0) \in U$. Since $p_m \to p$ at $m \to \infty$, for each $\varepsilon > 0$ there exists $m_0$ such that, $|g_{ij}(p_m) - g_{ij}(p)| < \varepsilon$, at $m \ge m_0$. From here we have

$$\sum_{i,j=1}^{k} (g_{ij}(p) - \varepsilon) |u^i_m| |u^j_m| \le \sum_{i,j=1}^{k} g_{ij}(p_m) |u^i_m| |u^j_m|.$$

By virtue of that a matrix $\{g_{ij}(p_m)\}$ is positively determined, each equation

$$\sum_{i,j=1}^{k} g_{ij}(p_m) u^i u^j = 1$$

determines ellipsoid with the center of symmetry at the origin of coordinates.

Therefore, we have

$$\sum_{i,j=1}^{k} g_{ij}(p_m) |u^i_m| |u^j_m| = 1$$

and hence,

$$\sum_{i,j=1}^{k} (g_{ij}(p) - \varepsilon) |u^i_m| |u^j_m| \le 1$$

For a sufficiently small number $\varepsilon > 0$ , the matrix $\{(g_{ij}(p)-\varepsilon)\}$ is positively determined too. From here we have that points $u_m = \{u_m^1, u_m^2, ..., u_m^k\}$ belong to a compact set. Therefore there exists a converging subsequence $u_{m_l}^i$ of the sequence $u_m^i$.

We denote by $u_0^i$ limit of the $u_{m_l}^i$. Then for vector $v = (u_0^1, u_0^2, ..., u_0^k)$ has places

$$\sum_{i,j=1}^{k} g_{ij}(p)|u_0^i||u_0^j| = 1$$

Let us consider a geodesic $\gamma$ on the leaf $L(p)$ passing through the point $p$ at s = 0 in a direction of vector $v$. This curve satisfies the equation (5).

Let $K_0 \subset R^1$ be compact containing $s_0 = 0$ such, that $\gamma(K_0) \subset U$. Then coordinate functions of the curve $\gamma : K_0 \to L(p)$ satisfy to the system of the differential equations (5) with the initial conditions: $x^i(0) = p^i$, $u^i(0) = v^i$; where $i = 1, 2, ..., k$, $p = (p^1, p^2, ..., p^n)$, $v = (u_0^1, u_0^2, ..., u_0^k)$. As $\gamma_{m_l}(0) \to p$, $\dot{\gamma}_{m_l}(0) \to v$ for $m \to \infty$ under the theorem of continuous dependence of the solution of the differential equations from the initial dates the sequence $\gamma_m$ is convergent to $\gamma$ uniformly on compact $K_0 \subset R^1$. Further for every compact $K \subset R^1$ containing $K_0$ covering $\gamma(K)$ with the foliated neighborhoods, we shall receive, that $\gamma_{m_l}$ is convergent to $\gamma$ uniformly on compact K. The Theorem-1 has been proved.

## II. ON THE TOPOLOGY OF THE GROUP OF ISOMETRIES OF FOLIATED MANIFOLD

Let $(M, F_1)$ and $(N, F_2)$ be $n$ - dimensional smooth foliated manifolds with $k$ dimensional foliations, where $0 < k < n$. If for some $C^r$ - diffeomorphism $f : M \to N$ the image $f(L_\alpha)$ of any leaf $L_\alpha$ of the foliation $F_1$ is a leaf of the foliation $F_2$, we say that pairs $(M, F_1)$ and $(N, F_2)$ $C^r$ - diffeomorphic foliated manifolds. In this case the mapping $f$ is called $C^r$ diffeomorphism, preserving foliation and is written as
$f : (M, F_1) \to (N, F_2)$.

In the case where $M = N$, $F_1 = F_2$, f is called a diffeomorphism of the foliated manifold $(M, F)$.

Diffeomorphisms, preserving foliation, are investigated in (Tondeur Ph., 1988), (Aranson S., 1992).

**Definition 1.** Diffeomorfism $\varphi : M \to M$ of the class $C^r (r \geq 0)$; preserving foliation, is called a foliation isometry $F$ (an isometry of the foliated manifold $(M, F)$) if it is an isometry on each leaf of the foliation $F$, i.e. for each leaf $L_\alpha$ of the foliation $F$, $\varphi : L_a \to \varphi(L_a)$ is an isometry.

Let us denote by $G_F^r(M)$ the set of all $C^r$ - isometries of a foliated manifold $(M, F)$, where $r \geq 0$. The following remarks show that the notion of an isometry of a foliated manifold is correctly defined.

**Remark 1.** If $r \geq 1$, for each element $\varphi \in G_F^r(M)$ the differential $d\varphi$ preserves the length of each tangent vector $v \in T_p F$, i.e. $|d\varphi_p(v)| = |v|$ at any $p \in M$

**Remark 2.** If $r = 0$, each element $\varphi$ from $G_F^r(M)$ is homeomorphism of manifold $M$. A Riemannian metric of the manifold $M$ induces a Riemannian metric on each leaf $L_a$ which defines a distance on it. In this case, $\varphi$ is an isometry between the metric spaces $L_a$ and $\varphi(L_\alpha)$. Then, according to the known theorem, $\varphi$ is a diffeomorphism of $L_a$ onto $\varphi(L_\alpha)$ for each leaf $L_a$ and it's differential preserves the length of each tangent vector $v \in T_p F, i.e. |d\varphi_p(v)| = |v|$ at any $p \in M$ (Helgason S., 2001). But as shown by a simple example, from differentiability of a mapping on each leaf can not imply it's differentiability on the entire manifold $M$.

**Example**. Let $M = R^2(x, y)$ be the Euclidean plane with Cartesian coordinates $(x, y)$. Leaves $L_\alpha$ of foliation $F$ are given by the equations $y = \alpha = const$. Then the plan homeomorphism $\varphi : R^2 \to R^2$ determined by formula

$$\varphi(x, y) = (x + y, y^{\frac{1}{3}})$$

is an isometry of the foliation $F$, but is not a diffeomorphism of the plane.

The set $Diff^r(M)$ of all diffeomorphisms of a manifold $M$ onto itself is group with the operations of composition and taking the inverse. The set $G_F^r(M)$ is subgroup of the group $Diff^r(M)$.

The group of diffeomorfism of manifolds is studied by many mathematics, in particular in monography (Rokhlin V. & Fuks D., 1977).

It is known that the following theorem holds (Rokhlin V. & Fuks D., 1977),(Narmanov A. & Sharipov A., 2009).

**Theorem.** Let $M$ be a smooth, connected and finite-dimensional manifold. Then the group of homeomorphisms $Homeo(M)$ is a topological group with compact - open topology.

In particular, the subgroups $Diff^r(M),\ G_F^r(M)$ are topological groups in this topology.

Now we will intoduce new topology which is connected with foliation.

Let $\{K_\lambda\}$ be a family of all compact sets where each $K_\lambda$ ¸ is a subset of some leaf of foliation $F$ and let $\{U_\beta\}$ be a family of all open sets on M.

We consider, for each pair $K_\lambda$ ¸ and $U_\beta$ the set of all mappings $f \in G_F^r(M)$ such that $f(K_\lambda) \subset U_\beta$. This set of mappings is denoted by $\left[K_\lambda, U_\beta\right] = \left\{ f : M \to M \,|\, f(K_\lambda) \subset U_\beta \right\}$.

It isn't difficult to show that all finite intersections of sets of the form $\left[K_\lambda, U_\beta\right]$ forms a base for some topology. This topology we will call the foliated compact open topology or, briefly $F$-compact open topology. This topology depends on the foliation $F$ and coincides with the compact open topology, when $F$ is an $n$-dimensional foliation. If codimension of a foliation is equal to $n$, then the convergence in this topology coincides with the pointwise convergence.

**Proposition 1.** The set $G_F^r(M)$ with the $F$-compact open topology is a Hausdorff space.

**Proof**. Get an arbitrary $f_1 \in G_F(M)$ and $f_2 \in G_F(M)$ with $f_1 \neq f_2$. This means that there is a point $x \in M$ such that images of this point under $f_1$ and $f_2$ are distinct, i.e. if we get $x_1 = f_1(x)$ and $x_2 = f_2(x)$, then $x_1 \neq x_2$. Since $M$ is a Hausdorff space, there are disjoint neighborhoods $U_1$ and $U_2$ of points $x_1 \in U_1$ and $x_2 \in U_2$ respectively. Put $C = \{x\}$. Then by definition of the $F$-compact open topology $W(C, U_1)$ and $W(C, U_2)$ are open and are disjoint, i.e. $W(C, U_1) \cap W(C, U_2) = \varnothing$. Indeed, if we assume $W(C, U_1) \cap W(C, U_2) \neq \varnothing$, then there is an element $f \in G_F(M)$ from the intersection $f \in W(C, U_1) \cap W(C, U_2)$.

Consequently, $f(C) \subset U_1,\ f(C) \subset U_2$. Thence $f(C) \in U_1 \cap U_2 = \varnothing$. This contradiction proves that the space $G_F(M)$ is Hausdorff.

The following theorem shows some property of group $G_F^r(M)$ with $F$ - compact open topology.

**Proposition 2.** Assume that a sequence $\{f_m\} \in G_F^r(M)$ converges pointwise on a set $A \subset L_\alpha$ where $L_\alpha$ some leaf of the foliation $F$. Then $\{f_m\}$ also converges pointwise on $A$ (where $\bar{A}$ - the closure of the set $A$ in $L_\alpha$).

**Proof**. Let $p \in \bar{A}, \varepsilon > 0$ At first we select a point $p_1 \in A$, that $d_a(p, p_1) < \frac{\varepsilon}{3}$ and an integer $N$ such that $d(f_1(p_1), f_m(p_1)) < \frac{\varepsilon}{3}$ for $l, m \geq N$ (where $d_a(p, p_1)$ is the distance between points $p$ and $p_1$ on the leaf $L_\alpha$). Then

$d(f_l(p), f_m(p)) \leq d(f_l(p), f_l(p_1)) + d(f_l(p_1), f_m(p_1)) + d(f_m(p_1), f_m(p)) < \varepsilon,$

for all $m, n$ at $m, n \geq N$. Hence, $\{f_m(p)\}$ is fundamental and from completeness of $M$ follows that $\{f_m(p)\}$ is convergent.

**Proposition 3.** Let $A$ be a set of points on a leaf $L_\alpha$ such that for each point $p \in A$ there exists a converging subsequence $f_{m_l}(p)$ of the sequence $f_m(p)$ . If the set $A$ is nonempty set, then $A = L_\alpha$ .

**Proof**. Let $p \in L_\alpha$ , $p^* \in A$, $r = d_\alpha(p, p^*)$, where $d_\alpha(p, p^*)$ is the distance between points $p$ and $p^*$ on the leaf $L_\alpha$ . Assume that $\{f_{m_l}\}$ is a subsequence that $\{f_{m_l}(p^*)\}$ is convergent. Since $f_{m_l}$ an isometries of foliation, the distance $d_\alpha(p, p^*)$ between points $p$ and $p^*$ on the leaf $L_\alpha$ is preserved and holds equality $d_{f_l(\alpha)}(f_{m_l}(p), f_{m_l}(p^*)) = d_\alpha(p, p^*)$ where $d_{f_l(\alpha)}$ is the distance on the leaf $f_{m_l}(L_\alpha)$.

Let $q^* = \lim_{l \to \infty}(f_{m_l}(p^*))$

Then

$d(q^*, f_{m_l}(p)) \leq d(q^*, f_{m_l}(p^*) + d(f_{m_l}(p^*), f_{m_l}(p)) \leq \varepsilon + r$

Hence, from completeness of $M$ follows that the set $\{f_{m_l}(p)\}$ has compact closure. It follows that $p \in A$.

**Theorem-2.** Let $M$ be a complete smooth $n$ - dimensional manifold with the smooth $k$ - dimensional foliation $F, f_m \in G_F^r(M), r \geq 0, \quad m = 1,2,3,....$ Suppose, that for each leaf $L_\alpha$ there exists a point $o_\alpha \in L_\alpha$ such that the sequence $f_m(o_\alpha)$ is convergent. Then there exists a subsequence $f_{m_l}$ of the sequence $f_m$ which converges in a $F$ - compact open topology.

***Proof of*** **theorem -2.** Let $p \in L_\alpha$ for a some leaf $L_\alpha$. Under conditions of the theorem there exists a point $o_\alpha \in L_\alpha$ such that $f_m(0_\alpha)$ is convergent. From Proposition 3 for each point $p \in L_a$ the sequence $\{f_m(p)\}$ containing converging subsequence. Hence the sequence $\{f_m(p)\}$ containing subsequence which converges for all points $p \in M$.

Since Riemannian manifold $M$ is a separable metric space it contains everywhere dense countable subset $A = \{p_i\}$ For each point $p_i$ there is a converging subsequence $\{f_{m_l}^i(p_i)\}$ of the sequence $\{f_m(p)\}$. Using diagonal process, we can find a subsequence $\{f_m(p)\}$ which converges at all points. Therefore for each leaf $L_\alpha$ there exists a point $o_\alpha \in L_\alpha$, such that sequence $\{f_{m_l}(o_\alpha)\}$ is convergent. By Proposition 3 this sequence is convergent at all points of $L_\alpha$. Now by putting $\varphi(p) = \lim_{l \to \infty} f_{m_l}(p).$ we have the map $\varphi : M \to M$ .

Let $p \in L_\alpha$ for some leaf $L_\alpha$, $\varphi : [0, l] \to L_\alpha$ be a geodesic which realizes the distance $d_0 = d_\alpha(o_\alpha, p)$ on the leaf $L_\alpha$, and is parameterized by length of the arc, $\gamma(0) = o_\alpha$, $\gamma(l) = p$. If we consider $\gamma_l = f_{m_l}(\gamma)$, then they are geodesic lines on the leaf $f_{m_l}(L_a)$. From condition of theorem we have $\gamma_l(0) \to p_0$ at $l \to \infty$, where $p_0$ is a some point on $M$. From Theorem 1 follows that there exists subsequence of sequence $\{\gamma_l(s)\}$, which pointwise converges to some geodesic $\gamma(0): R^1 \to L(p_0)$ of the leaf $L(p_0)$ passing through the point $p_0$ at s = 0.

With no loss of generality, we can suppose that sequence $\{\gamma_l(s)\}$ is convergent to geodesic $\gamma_0(s)$ for each $s \in [0;l]$. Therefore follows that $\varphi(\gamma) = \gamma_0$ i.e. map $\varphi$ is an isometry of $L_\alpha$ to $L(p_0)$.

Now we show that $f_{m_l} \to \varphi$ uniformly on each compact, lying on a leaf of foliation $F$. Let $K$ be a compact set on a leaf $L$ and $\varepsilon > 0$. As $K$ is a compact set there exist finite points $p_1, p_2, ..., p_m$ on $L$ such that each point from $p \in K$ some $p_i$ has distance less than $\varepsilon$. For each point $p_i$ there is number $N_i$ such that $d(f_{m_l}(p_i), \varphi(p_i) < \frac{\varepsilon}{3}$ for any $m_l \geq N_i$. Besides for each point $p \in K$ there exists $p_i$ such that $d_L(p, p_i) < \frac{\varepsilon}{3}$ where $d_L(p, p_i)$ distance between points $p$ and $p_i$ determined by induced Riemannian metric on $L$. Therefore follows that

$$d(f_{m_l}(p), \varphi(p)) \leq d(f_{m_l}(p), f_{m_l}(p_1)) + d(f_{m_l}(p_i), \varphi(p_i)) + d(\varphi(p_i), \varphi(p) \leq$$

$$\leq d_{m_l}(f_{m_l}(p), f_{m_l}(p_i)) + d(f_{m_l}(p_i), \varphi(p_i) + d_i(\varphi(p_i), \varphi(p) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

at $m_l > N = \max_{1 \leq i \leq m} \{N_i\}$, where $1 \leq i \leq m$.

From here follows that $f_{m_l} \to \varphi$ in $F$ - compact open topology. The Theorem 2 has been proved.

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