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RESEARCH ARTICLE

A CLASS OF LINEAR BOUNDARY VALUE PROBLEM FOR k-REGULAR VECTOR-VALUED FUNCTION IN CLIFFORD ANALYSIS

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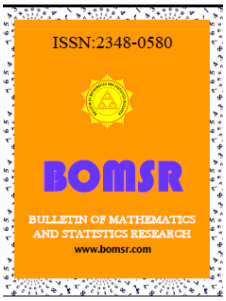

ABSTRACT

In this paper, we introduce the linear boundary value problem for $\boldsymbol{k}$ -regular vector-valued function,and give solution to this problem.

Key words: k-regular vector – valued function, Clifford analysis.

1.INTRODUCTION

The boundary value problem is one of the important aspects in Clifford Analysis. This problem on bounded domain has seen great achievement [1-14]. But the one on unbounded domain has not, which is widely used in practical applications. So, it is necessary to discuss the properties and boundary value problems of functions on unbounded domains. [3-5] have discussed Riemann-Hilbert boundary value problems of regular function on bounded domains, [6] characterized boundary value problems of k – regular functions. In this paper, we introduce a class of linear boundary value problem of k-regular vector – valued function, and give an unique solution to this problem.

Let n be a positive integer, and $\{e_0, e_1, \ldots\ldots, e_n\}$ be basis for the Euclidean space $\mathcal{R}^{n+1}$. We denote by $\mathcal{A}$ the 2^n dimensional real Clifford algebra, which is generated by $\mathcal{R}^{n+1}$; denote the basis of A by $e_A = e_{\alpha_1,\alpha_2,\ldots\ldots,\alpha_h}$, A = $\alpha_1,\alpha_2, \ldots \ldots .., \alpha_h \subseteq$ { 1, 2, …, n}, $1 \le \alpha_1,\alpha_2,.,\alpha_h \le n$.

In particular, if A = $\emptyset$, $e_\emptyset = e_0$. So, for an arbitrary u ϵ $\mathcal{A}$, we have u = $\sum_A u_A e_A$ with $u_A \in \mathbb{R}$.

In $\mathcal{A}$, we have

$$e_i^2 = -1,\ e_1 e_j = -\,e_j\, e_i,\ \text{for } i \neq j,\ i, j = 1,2,\ldots,n,$$

that is so-called combinative and incommutable multiplication rule of Clifford algebra. For u ϵ $\mathcal{A}$, we have $u^\star = \sum_A (-1)^{\frac{|A|(|A|-1)}{2}} u_A e_A$, $u' = \sum_A (-1)^{\frac{|A|}{2}} u_A e_A$ and |u| for its module; where |A|is the cardinality of index set. Define | A $|^2$; $\bar{u}$ =($u^\star$)$'$, where $u^* = \sum_A (-1)^{\frac{|A|(|A|-1)}{2}} u_A e_A$, and u' = $\sum_A (-1)^{\frac{|A|(|A|-1)}{2}} u_A e_A$. Herein. For u, v ϵ $\mathcal{A}$ we have

$$| u + v | \leq |u| + |v|, |uv| \leq 2^n |u||v|,$$

Let D be a region in $\mathbb{R}^{n+1}$. For a differentiable function f : D → A with f(x) = $\sum_A f_A(x)e_A$, we say f is a regular function if

$$\bar{\partial}f = \sum_{i=0}^{n} e_i \frac{\partial f}{\partial x_i} = \sum_{i=0}^{n} \sum_A e_i e_A \frac{\partial f_A}{\partial x_i} = 0,$$

and a k – regular function if $\bar{\partial}^k$f = 0. Let $\Omega \subset \mathcal{R}^{n+1}$ *be a* unbounded domain with smooth oriented Liapunove boundary $\partial\,\Omega$ and Ω^c, the complementary set of Ω containing a non – empty openset. We denote the bounded holder continuous function on $\partial\,\Omega$ in order of $\beta(0 < \beta < 1)$ by H $(\partial\,\Omega, \beta)$. For $f \,\epsilon$ H $(\partial\,\Omega, \beta)$, we define its norm by

$$\| f \|_\beta = \sup_{t\,\epsilon \partial\,\Omega} |f(t)| + \sup_{t_1 \neq t_2} \frac{|f(t) - f(t))}{t_1 - t_2}.$$

Then H$(\partial\,\Omega,\ \| f \|_\beta)$ is a Banach space. And for f, g ϵ H$(\partial\,\Omega,\ \| f \|_\beta)$, we have

$$\| f + g \|_\beta \leq \| f \|_\beta \| g \|_\beta, \| f g \|_\beta \leq 2^n \| f \|_\beta \| g \|_\beta.$$

2. Main result

In what follows, we denote by Ω a non-empty connected open set in $\mathcal{R}^{n+1}$ with smooth oriented Liapunove boundary $\partial\,\Omega$. We first give the linear boundary value problem for k-regular vector-valued function.

Definition 2.1: Let G be a (n + 1) x (n + 1) number matrix, and $H_{i,}$ 0 ≤ l ≤ k – 1be a vector – valued function with each $H_1 \,\epsilon$ H $(\partial\,\Omega, \beta)$. Write $\Omega^+ = \Omega$, $\Omega^- = \mathcal{R}^{n+1} \setminus \bar{\Omega}$ with $\bar{\Omega} = \Omega \cup \partial\,\Omega$. If there exists some vector – valued function ϕ such that

1) ϕ is a k-regular vector – valued function on $\Omega^{\pm}$;

2)
$$\begin{cases} \phi^+(x) = \phi^-(x)G + \mathbf{H}_0(x) \\ \bar{\partial}\phi^+(x) = \bar{\partial}\phi^-(x)G + \mathbf{H}_1(x) \\ \quad\vdots \\ \bar{\partial}^{k-1}\phi^+(x) = \bar{\partial}^{k-1}\phi^-(x)G + \mathbf{H}_{k-1}(x) \end{cases} \tag{2.1}$$

for x ϵ $\partial\,\Omega$. Then we say ϕ is a solution to the linear boundary problem. And this problem is also called boundary problem for k – regular vector- valued function.

The following lemmas are borrowed from [6]:

Lemma 2.2 write

$$\phi(x) = \frac{1}{w_n} \int_{\partial\Omega} \frac{\overline{\tau - x}}{|\tau - x|^{n+1}} \sum_{j=0}^{k-1} \frac{(-1)^{j+k-1}}{(k-j-1)!} (\tau_1 - x_1)^{k-j-1} m(\tau) \varphi_{k-j-1} ds(\tau) \quad x \notin \partial\Omega, \tag{2.2}$$

Where m(u) is the unit vector in $\partial\,\Omega$'s normal direction, and $\varphi_j \,\epsilon$ H $(\partial\,\Omega, \alpha)$, j = 0,1,…..k-1. Then φ is a k-regular function on $\mathcal{R}^{n+1} \setminus \partial\,\Omega$.

By Lemma 2.1, we also have:

Lemma 2.2. Let $\varphi \,\epsilon$ H $(\partial\,\Omega, \beta\,)$

$$\phi(\mathbf{x}) = \frac{\mathbf{1}}{\mathbf{w_n}} \sum_{\mathbf{j}=0}^{\mathbf{n}} \int_{\partial\Omega} \frac{\overline{\tau - \mathbf{x}}}{|\tau - \mathbf{x}|^{\mathbf{n}}} \mathbf{m}(\tau) \frac{(-\mathbf{1})^{\mathbf{j}+\mathbf{k}-\mathbf{1}} (\tau_1 - \mathbf{x}_1)^{\mathbf{k}-\mathbf{j}-\mathbf{1}}}{(\mathbf{k} - \mathbf{j} - \mathbf{1})!} \varphi_{\mathbf{k}-\mathbf{j}-\mathbf{1}} \mathbf{d}_{\mathbf{s}_\tau}, \tag{2.3}$$

Where m(u) is the unit vecto in $\partial\,\Omega$'s normal direction. Then we have

1)ϕ is a k-regulal vector-valued function;

2)

$$\begin{cases} \bar{\partial}^m \phi^+ + \bar{\partial}^m \phi^- = \frac{2}{w_n} \int_{\partial\Omega} \frac{\overline{\tau - x}}{|\tau - x|^n} \sum_{j=0}^{k-m-1} \frac{(-1)^{j+m+k-1}}{(k-j-m-1)!} (\tau_1 - x_1)^{k-j-m-1} \varphi_{k-j-1} d_{s_\tau} \\ \bar{\partial}^m \phi^+ - \bar{\partial}^m \phi^- = \varphi_m, \qquad t \in \partial\Omega, m = 0, 1, \cdots, k-1 \end{cases} \tag{2.4}$$

Proof: By Definition 2.1, we have

$$\phi_i(x) = \frac{1}{w_n} \sum_{j=0}^{n} \int_{\partial\Omega} \frac{\overline{\tau - x}}{|\tau - x|^n} m(\tau) \frac{(-1)^{j+k-1} (\tau_1 - x_1)^{k-j-1}}{(k-j-1)!} \varphi^i_{k-j-1} d_{s_\tau} \quad (x \in R^{n+1} \backslash \partial\Omega, i = 1, \cdots, n), \tag{2.5}$$

and $\overline{\partial^k}\phi = 0$.

The following lemmas are borrowed from [2]:

Lemma2.3: Let ϕ be a k-regular function on $\mathcal{R}^{n+1}$, We have

1) If ϕ(∞) is bounded, then ϕ = YX + CX with C being a hypercomplex constant on$\mathcal{R}^{n+1}$;
2) If $\phi(\infty) = 0$, then $\phi = 0$ on $\mathcal{R}^{n+1}$.

Lemma2.4: Let ϕ in (2.1) Then we have

$$\begin{cases} \bar{\partial}^m \phi^+(t) + \bar{\partial}^m \phi^-(t) = \frac{2}{w_n} \int_{\partial\Omega} \frac{\overline{\tau - x}}{|\tau - x|^{n+1}} \sum_{j=0}^{k-m-1} \frac{(-1)^{j+m+k-1}}{(k-j-m-1)!} (\tau_1 - t_1)^{k-j-m-1} m(\tau) \varphi_{k-j-1} ds(\tau), \\ \bar{\partial}^m \phi^+(t) - \bar{\partial}^m \phi^-(t) = \varphi_m(t), \end{cases} \tag{2.6}$$

for m=0,1,2,…..,k-1 and t ϵ $\partial\ \Omega$.

Theorem 2.1 . let $\Omega \subset \mathcal{R}^{n+1}$. For the linear boundary value problem in Definition 2.2, we have the solution to(2.1): 1) if ϕ (∞)is bounded, then ϕ = YX + CX with C being a hypercomplex constant on $\mathcal{R}^{n+1}$, where

$$Y(x) = \frac{1}{w_{n-1}} \sum_{j=0}^{k-1} \int_{\Sigma} \frac{\overline{\tau - x}}{|\tau - x|^n} m(\tau) \frac{(-1)^{j+k-1} (\tau_1 - x_1)^{k-j-1} g_{k-j-1}(\tau)}{(k-j-1)! X^+(\tau)} d_{s_\tau} \tag{2.7}$$

for x ϵ $\mathcal{R}^{n+1} \backslash \partial\ \Omega$.

2)If ϕ (∞) = 0 the ϕ (x) = Y(x)X(x).

Proof:We first deal with the case of H_1 = 0 (0 ≤ ι ≤ k -1) in (2.1). Let

$$X(x) = \begin{cases} G, & x \in \Sigma^+ \\ 1, & x \in \Sigma^- \end{cases}$$

Then

$$G = \frac{X^+(x)}{X^-(x)} \tag{2.8}$$

Substituting (2.8) into (2.1), we have

$$\begin{cases} \frac{\phi^+}{X^+} = \frac{\phi^-}{X^-} \\ \frac{\bar{\partial}\phi^+}{X^+} = \frac{\bar{\partial}\phi^-}{X^-} \\ \vdots \\ \frac{\bar{\partial}^{k-1}\phi^+}{X^+} = \frac{\bar{\partial}^{k-1}\phi^-}{X^-} \end{cases} .$$

Let

$$Y_1^+(x) = \frac{\phi^+(x)}{X^+(x)}, Y_1^-(x) = \frac{\phi^-(x)}{X^-(x)}.$$

Then we have

$$Y_1^+(\cdot) = Y_1^-(\cdot), \bar{\partial} Y_1^+(\cdot) = \bar{\partial} Y_1^-(\cdot), \cdots, \bar{\partial}^{k-1} Y_1^+(\cdot) = \bar{\partial}^{k-1} Y_1^-(\cdot)$$

On $\partial\ \Omega$. It follows that

$$Y(x) = \frac{\phi(x)}{X(x)},$$

and
$$\frac{\phi}{X} = C$$

When $\phi(\infty)$ is bounded, which implies that ϕ = CX.

Next we turn to the homogeneous case. By(2.8), (2.1) can be rewritten as

$$\begin{cases} \frac{\phi^+(x)}{X^+(x)} - \frac{\phi^-(x)}{X^-(x)} = \frac{H_0(x)}{X^+(x)} \\ \frac{\bar{\partial}\phi^+(x)}{X^+(x)} - \frac{\bar{\partial}\phi^-(x)}{X^-(x)} = \frac{H_1(x)}{X^+(x)} \\ \vdots \\ \frac{\bar{\partial}^{k-1}\phi^+(x)}{X^+(x)} - \frac{\bar{\partial}^{k-1}\phi^-(x)}{X^-(x)} = \frac{H_{k-1}(x)}{X^+(x)} \quad x \in \Sigma \end{cases} \tag{2.9}$$

Let Y = $\frac{\phi}{X}$. Then the above equation turns to be

$$\begin{cases} Y^+ - Y^- = \frac{H_0(x)}{X^+(x)} \\ \bar{\partial}Y^+ - \bar{\partial}Y^- = \frac{H_1(x)}{X^+(x)} \\ \vdots \\ \bar{\partial}^{k-1}Y^+ - \bar{\partial}^{k-1}Y^- = \frac{H_{k-1}(x)}{X^+(x)}. \end{cases} \tag{2.10}$$

By Lemma 2.4, we have

$$Y(x) = \frac{1}{w_{n-1}} \sum_{j=0}^{k-1} \int_\Sigma \frac{\overline{\tau - x}}{|\tau - x|^n} m(\tau) \frac{(-1)^{j+k-1}(\tau_1 - x_1)^{k-j-1} H_{k-j-1}(\tau)}{(k-j-1)! X^+(\tau)} d_{s_\tau}. \tag{2.11}$$

It follows that

1) If $\phi(\infty)$ is bounded, then
$$\phi = YX + CX;$$
2) If $\phi(\infty)$ = 0 then
$$\phi = YX \tag{2.12}$$

This gives the proof.

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