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RESEARCH ARTICLE

ON THE SYMMETRY GROUP OF DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper we find Lie algebra of infinitesimal generators of symmetry group of heat equation and it is found general traveling wave solutions in explicitly form.

Key words: heat equation, symmetry group, Lie algebra, traveling wave solutions

1. INTRODUCTION

Suppose we are given a differential equation of order m

$$\Delta\left(x,u^{(m)}\right)=0 \qquad (1)$$

for n the independent $x=\left(x^1,x^2,...,x^n\right)$ and q dependent variables $u=\left(u^1,u^2,...,u^q\right)$, containing derivatives of the function u up to order m.

Definition. Group G of transformations acting on an open subset of M independent and dependent variables of the differential equation is called the group of symmetries of the equation (1) if for each solution $u=f(x)$ of equation (1) and for $g\in G$ such that $g\circ f$ it is determined that the function $\tilde{u}=g\circ f$ is also a solution of equation .

For the heat equation $u_t=u_{xx}$ group of translations

$$(x,t,u)\to(x+as,t+bs,u), s\in R$$

is the group of symmetries, as if the function $u=f(x)$is a solution, then the function $u=f(x-as,t-bs)$ is also a solution of the heat equation.

One advantage of the knowledge the group of symmetries of differential equations is that if we know the solution $u=f(x)$ then, in accordance with the definition $\tilde{u}=g\circ f$ also is solution for every element g of G so that we have the opportunity to build a whole family of solutions, exposing known solution to the action of various elements of the group.

To do this, we "continue" the main space, which represents the independent and dependent variables to the space, representing also all the various partial derivatives appearing in the equation. Suppose we are given a smooth real function $f(x)=f(x^1,x^2,...,x^n)$ of the independent variables. This function has a different $n_k=C^k_{n+k-1}$ partial derivatives of k-th order. We use a multi-index notation

$$\partial_J f(x)=\frac{\partial^k f}{\partial x^{j_1}\partial x^{j_2}...\partial x^{j_k}}$$

for these derivatives. *In this notation* $J=(j_1,j_2,...,j_k)$*- unordered set of* k *integers such that* $1\le j_k\le n$*, indicating* which *derivatives are being taken. For function* $u=f(x)=\left(f^1(x),f^2(x),...f^q(x)\right)$ *requires* qn_k *numbers* $u^\alpha_j=\partial_j f^\alpha$ *to represent all the various partial derivatives of order* k of *all components of* f *at* x.

Let $U_k=R^{qn_k}$ - Euclidean space with coordinates $u^\alpha_j=\partial_j f^\alpha$, corresponding to $\alpha=1,2,\dots,q$ and to multiindices $J=(j_1,j_2,...,j_k)$ of order k to represent these derivatives.

Consider the space $U^m=U\times U_1\times U_2...\times U_m$ *in which the coordinates are* derivatives of $u=f(x)$ *of all orders from* 0 *up* m. *The space* U^m *is a Euclidean space of dimension*

$q+qn_1+qn_2+\cdots+qn_m=qC^m_{n+m}$

Let $n^{(m)}=C^m_{n+m}$ The point of the space U^m will be denoted by $u^{(m)}$, its coordinates are $u^\alpha_j=\partial_j f^\alpha$ and the number of coordinates equal $qn^{(m)}$.

Given a smooth function $u=f(x)$ there is an induced function $u^{(m)}=pr^{(m)}f(x)$ called the m*-th prolongation of* $f(x)$, which is defined by the equations $u^\alpha_j=\partial_j f^\alpha(x)$.Thus $u^{(m)}=pr^{(m)}f(x)$ is a function from X to the space $U^{(m)}$,and for each $x\in X$ the function $pr^{(m)}f(x)$*)* is a vector whose qp^n entries represent the values of f and all its derivatives up to order m at the point x.

Now we can replace the differential equation $\Delta\left(x,u^{(m)}\right)=0$ by an algebraic equation, which is determined by the vanishing of the function, which is the right-hand side of the equation $\Delta\left(x,u^{(m)}\right)=0$ defined on $X\times U^m$.

A smooth solution of the differential equation $\Delta\left(x,u^{(m)}\right)=0$ - a smooth function $u=f(x)$ such that $\Delta\left(x,pr^{(m)}u\right)=0$. This means that the function $u=f(x)$and its derivatives $u^\alpha_j=\partial_j f^\alpha$ must satisfy the algebraic equation $F\left(x,t,pr^{(m)}u(x)\right)=0$ in space $X\times U^m$.

2. MAIN PART

Consider the heat equation with the source

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[k(u)\frac{\partial u}{\partial x}\right]+Q(u) \tag{2}$$

where the function $Q(u)$ describes the heat dissipation process, if $Q(u)>0$ the process of heat absorption when $Q(u)<0$.

Researchs shows the thermal conductivity $k(u)$ in a fairly wide range of parameters can be described by a power function of the temperature, t. e. it has the form. $k(u)=u^{\sigma}$.

Self-similar solutions of the equation (2) studied in papers (Aripov M.M. 1988, Bratus A.S., Novojilov A.C.,Platonov A.P.2011, Volosevich P.P., Lavanov E. I. 1997, Samarskiy A.A., Galaktionov V.A. , Kurdyumov S.P., Mikhailov A.P. 1987) when $\sigma>0$.

We will explore solutions that are invariant under the group of symmetries of the equation.

In the paper (Olver P.J. 1993) developed a computational method, clearly defining the full symmetry group of an arbitrary differential equation.

We find symmetry group with infinitesimal technique developed in (Olver P.J. 1993).

Let us consider the case of $k(u)=1, Q(u=u)$. In this case the equation (2) has the following form

$$u_t = u_{xx} + u \tag{3}$$

Infinitesimal generator of the symmetry group of (3) we will seek in the form of the following vector field

$$X=\xi\frac{\partial}{\partial x}+\tau\frac{\partial}{\partial t}+\varphi\frac{\partial}{\partial u} \tag{4}$$

Prolongation of the vector field (3) is as follows (Olver P.J. 1993):

$$pr^{(2)}X = X+\varphi^{x}\frac{\partial}{\partial u_x}+\varphi^{t}\frac{\partial}{\partial u_t}+\varphi^{xx}\frac{\partial}{\partial u_{xx}}+\varphi^{xt}\frac{\partial}{\partial u_{xt}} \tag{5}$$

The differential equation (3) we replace by the algebraic equation in space $X\times U^2$

$$F(x,t,u,u_x,u_t,u_{xx})=0 \tag{6}$$

Where $F(x,t,u,u_x,u_t,u_{xx})=u_t-u_{xx}-u$.

From the equality $Y(F)=0$, where $Y=pr^{(2)}X$, we get the following equation for the unknown functions

$$-\varphi+\varphi^{t}-\varphi^{xx}=0 \tag{7}$$

For components of the vector field Y we use expressions found in (Olver P.J. 993):

$$\varphi^{t}=\varphi_t-\xi_t u_x+(\varphi_u-\tau_t)u_t-\xi_u u_x u_t-\tau_u u_t^2$$

$$\varphi^{xx}=\varphi_{xx}+(2\varphi_{xu}-\xi_{xx})u_x-\tau_{xx}u_t+(\varphi_{uu}-2\xi_{xu})u_x^2-2\tau_{xu}u_x u_t-\xi_{uu}u_x^3-\tau_{uu}u_x^2u_t+$$
$$+(\varphi_u-2\xi_x)u_{xx}-2\tau_x u_{xt}-3\xi_u u_x u_{xx}-\tau_u u_t u_{xx}-2\tau_u u_x u_{xt}$$

Substituting these expressions in (7) and taking into account equality

$$u_t=u_{xx}+u,\ u_{xt}=u_{tx}=u_{xxx}+u_x \tag{8}$$

we obtain a polynomials in the right-hand side of equation (7) with respect to the variables u, u_x, u_{xx}, u_{xxx}.

Equating the coefficients of variuos monomials zero, we get the following equation for determining the group of symmetries of the heat equation.

№	monomial	defining equations	№	monomial	defining equations
1	1	$\varphi+\varphi_{xx}-\varphi_t=0$	9	u_{xx}^2	$-\tau_u+\tau_u=0$
2	u	$-\tau_{xx}-\varphi_u+\tau_t=0$	10	uu_{xx}	$-2\tau_u=0$
3	u^2	$\tau_u=0$	11	$u_x u_{xxx}$	$-2\tau_u=0$
4	u_x	$2\varphi_{xu}-\xi_{xx}-2\tau_x+\xi_t=0$	12	u_x^3	$-\xi_{uu}=0$
5	u_x^2	$\varphi_{uu}-2\xi_{xu}-2\tau_u=0$	13	uu_x^2	$-\tau_{uu}=0$
6	u_{xx}	$-\tau_{xx}+\varphi_u-2\xi_x-\varphi_u+\tau_t$	14	$u_{xx}u_x^2$	$-\tau_{uu}=0$
7	uu_x	$-2\tau_{xu}+\xi_u=0$	15	u_{xxx}	$-2\tau_x=0$
8	$u_x u_{xx}$	$-2\tau_{xu}-3\xi_u+\xi_u=0$			

From the defining equation (1) of the table we find that the function φ is a solution of equation (3).From equations (3) and (15) (and the equations (10) and (11)) we find that, $\tau_u=0,\ \tau_x=0$ that function τ depends on t, $\tau=\tau(t)$. From the equations (7) and (12) we get that. $\xi_{uu}=0,\ \xi_u=0$. From equation (5) we get that $\varphi_{uu}=0$.From the equations (6) and (2) get that $\tau_t=2\xi_x,\ \tau_t=\varphi_u$ .Consequently, $2\xi_{xx}=(\tau_t)_x=0$ i.e. ξ is a linear function of x.

From equation (4) we get that $2\varphi_{xu}=-\xi_t$. Given that $\tau_t=\varphi_u$ we have the equality $2(\tau_t)_x=-\xi_t=0$, ξ is independent of t, i.e. $\xi=\xi(x)$.

Hence, taking into account equality $\tau_t=2\xi_x$, we find $2\xi_{xt}=(\tau_t)_t=0$ that τ is a linear function of t i.e: $\tau=c_1t+b$. As $\tau_t=2\xi_x$, we get that $\xi=\frac{c_1}{2}x+a$.Finally, taking into account $\tau_t=\varphi_u$ we see that $\varphi=c_1u+\alpha(x,t)$. From the defining equation (1) of the table we find that $c_1=0$.

And so we found all the components of the vector field X : $\xi=a=const$, $\tau=b$, $\varphi=\alpha(x,t)$, where a,b- are constants, and $\alpha(x,t)$- an arbitrary solution of the equation (3).

Thus we have

$$X=a\frac{\partial}{\partial x}+b\frac{\partial}{\partial t}+\alpha(x,t)\frac{\partial}{\partial u}.$$

One of Lie algebra's infinitesimal generators of the symmetry group is a two-dimensional Lie algebra generated by the vector fields

$$X_1=\frac{\partial}{\partial x},\ X_2=\frac{\partial}{\partial t}$$

Second Algebra - infinite dimensional Lie algebra generated by the vector fields of the form

$$X_3=\alpha(x,t)\frac{\partial}{\partial u}.$$

The second infinite Lie algebra gives not much information about the fact that if we have solution $u=f(x)$ of the equation (2), $u(x,t)+\alpha(x,t)s$ is also a solution of the equation (2).

Therefore, basic information provides the first algebra, since the Lie brackets of vector fields X_1 and X_2 vanishes. And so we have proved the following theorem.

Theorem. The Lie algebra infinitesimal generators of the group of symmetries of the equation (2) is a two-dimensional algebra, which gives rise to a Lie group, consisting of parallel translations in the space of the independent variables.

3 CONCLUSION

In conclusion, using proved above theorem, we find the solution of equation (3) traveling wave type. Recall that the solution partial differential equations are invariant under the group of translations in the space of independent variables, called traveling wave solutions of the form.

Consider an arbitrary element of the Lie algebra of infinitesimal generators of the group of symmetries of the equation (3)

$$Z=a\frac{\partial}{\partial x}+b\frac{\partial}{\partial t}.$$

The flow generated by this vector field consists from parallel translations

$$(x,t)\to(x+ab,t+bs,u).$$

Invariants of this group are the functions

$$y=bx-at,\quad v=u$$

Therefore, the solution of equation (3) has the form

$$u=v(y),\quad y=bx-at$$

Then $u_t=-av_y,\quad u_x=bv_y,\quad u_{xx}=b^2v_{yy}$, and we get a linear ordinary differential equation of the second order

$$b^2v_{yy}+av_y+v=0 \tag{9}$$

Characteristic equation of (9) has the form

$$b^2\lambda^2+a\lambda+1=0.$$

Depending on the sign of the discriminant of the characteristic equation has either two real or two complex or one double real root of multiplicity.

Let λ_1,λ_2 the roots of the characteristic equation are real and distinct. Then the general solution of equation (9) has the form

$$v(y)=C_1\exp(\lambda_1 y)+C_2\exp(\lambda_2 y).$$

In this case, the general solution of equation (3) has the form

$$u(x,t)=C_1\exp(\lambda_1(bx-at))+C_2\exp(\lambda_2(bx-at)).$$

If $\lambda_1=\lambda_2=\lambda$, then the general solution of equation (9) has the form

$$v(y)=C_1\exp(\lambda y)+C_2 y\exp(\lambda y),$$

Respectively

$$u(x,t)=C_1\exp(\lambda(bx-at))+C_2(bx-at)\exp(\lambda(bx-at)).$$

If $\lambda_{1,2} = \mu \pm \varphi i i$, then the general solution of equation (9) has the form

$$v(y) = \exp(\mu y)[C_1 \cos\varphi y + C_2 \sin\varphi y],$$

respectively, the general solution of equation (3) has the form

$$u(x,t) = \exp(\varphi(bx - at))[C_1 \cos\varphi(bx - at) + C_2 \sin\varphi(bx - at))$$

Consider the case of a multiple root $a = 2, b = 1, \lambda = -1$. In this case the general solution of equation (9) has the form

$$v(y) = \exp(-y)[C_1 + C_2 y],$$

and the solution of the equation (3) has the form

$$u(x,t) = \exp(2t - x)[C_1 + C_2(x - 2t)].$$

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