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RESEARCH ARTICLE

INCIDENCE MATRIX, DEGREE MATRIX, CIRCUIT MATRIX AND LINE GRAPH OF A GRAPH AND SOME PROPERTIES

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ABSTRACT

Mathematics has become part of our life. Graph Theory is a branch of mathematics that plays a major role in every field of human being. In this paper we define different matrices associated with the Graphs along with some properties of a graph. And Rank of rank of an incidence matrix.

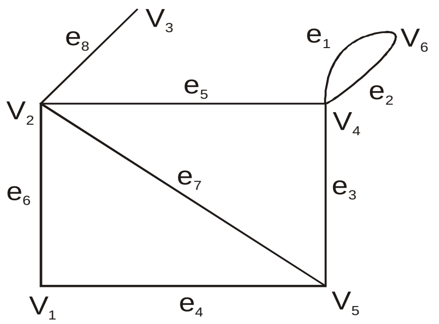
Key Words: Adjacency matrix, Incidence matrix, Degree matrix, Circuit matrix and Line graph.

INTRODUCTION

Incidence Matrix is one of methods of representing the graph. Let G = (V,E) be graph whose vertices and edges are labeled as $v_1, v_2, \dots\dots.. v_n$ and $e_1, e_2, \dots\dots.. e_m$. An incidence matrix B associated with graph G of order $n \times m$ is defined by

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident with } e_j \\ 0 & \text{Otherwise} \end{cases} \quad [4]$$

The graph and its incidence matrix are shown as follows:



$$B_{n\times m} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The incidence matrix contains only two elements 0 and 1. Such a matrix is called a binary matrix or (0 1) – matrix. For a given any geometric representation of a graph without self-loops we can easily write its corresponding incidence matrix. On the other hand if we were given an incidence matrix B we can construct a graph without any difficulty. The graph and its incidence matrix are simply two alternative ways of representing the same graph. **[1]**

PROPERTIES:

Some observations of incidence matrix and it's graph.

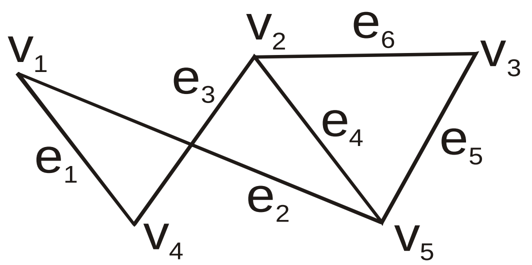
a) Since every edge is incident on exactly two vertices, each column of $B_{n\times m}$ has exactly two 1's.
b) The number of 1's in each row equals the degree of the corresponding vertex.
c) A row with all 0's therefore represents an isolated vertex.
d) Parallel edges in a graph produce identical columns in an incidence matrix as in column 01 and 02 of $\mathrm{B}_{\mathrm{n}\times\mathrm{m}}$
e) If a graph G is disconnected and consists of two components G_1 and G_2, the incidence matrix B of a graph G can be written in a block-diagonal form as follows:

$$\mathrm{B} = \begin{bmatrix} \mathrm{B}(\mathrm{G}_1) & 0 \\ 0 & \mathrm{B}(\mathrm{G}_2) \end{bmatrix}$$

Where $B(G_1)$ and $B(G_2)$ are incidence matrices corresponding to components G_1 and G_2. This result is due to reason that no edge in G_1 is incident on to the vertices of G_2. This is true for any disconnected graph with any number of components.

INCIDENCE MATRIX OF A GRAPH AND ITS RANK:

Consider the following graph and the corresponding incidence matrix. We would like discuss the rank of incidence matrix



$$E(G) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Considering each row of an incidence matrix $E(G)$ as vector defined over mod 2 in the vector space of a graph. If $X_1, X_2, \dots\dots X_n$ are denoting the vectors whose elements are the elements of the first row, second row etc of the matrix $E(G)$. It is clear that there are exactly two 1's in each column of $E(G)$ and the sum of all these vectors is zero with respect to mod 2. Thus $X_1, X_2, \dots\dots X_n$ are not linearly independent.

Thus the rank of $E(G)$ is less than n

This implies that rank of $\mathrm{E(G)} \leq \mathrm{n}-1$ **[I]**

Consider the sum of k of these n vectors where $\mathrm{k} \leq \mathrm{n}-1$. If the graph were connected it is clear that $E(G)$ cannot be portioned such that $E(G_1)$ is of k rows and $E(G_2)$ is of $\mathrm{n}-\mathrm{k}$ rows.

Now, No $k \times k$ sub matrix can be obtained for $\mathrm{k} \leq \mathrm{n}-1$ so that, modulo 2 sum of these k rows is equal to zero. As we observed there are exactly two constants 0 and 1 in the field the sum of all vectors taken k at a time for $\mathrm{k} = 1,2 \dots\dots. \mathrm{n}-1$, exhausts all possible linear combinations of $n-1$ row vectors.

$\Rightarrow$ No linear combination of k row vectors of $E(G)$ for $\mathrm{k} \leq \mathrm{n}-1$ can be equal to zero.

$\Rightarrow$ This implies rank of $\mathrm{E(G)} \geq \mathrm{n}-1$ **[ii]**

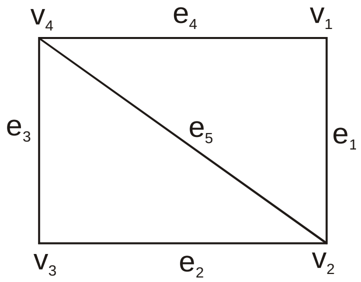
Hence from the results (I) and (II), rank of $\mathrm{E(G)} = \mathrm{n} - 1$.

CIRCUIT MATRIX:

Let G be a graph with number of different circuit in G and edges in G. Let q be the number of circuit and e be the number of edges then the circuit matrix $\mathrm{C} = \mathrm{C_{ij}}$ of G is a $\mathrm{q} \times \mathrm{e}$ matrix defined as follows:

$$\mathrm{C_{ij}} = \begin{cases} 1 & \text{if } \mathrm{i^{th}} \text{ circuit includes } \mathrm{j^{th}} \text{ edge} \\ 0 & \text{Othrwise} \end{cases}$$ **[3] & [4]**

Consider the following graph and its corresponding circuit matrix. Here G is having three different circuits as $(e_1 e_4 e_5), (e_2 e_3 e_5)$, and $(e_1 e_2 e_3 e_4)$.



$$C(G) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

The following observations are made regarding the circuit matrix of a graph G.

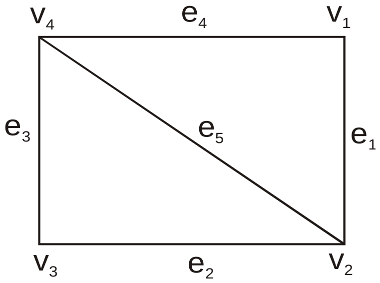
a) Each row of C(G) is a circuit vector.
b) With the help of circuit matrix, similar to that of incidence matrix. If a graph is disconnected and consists of two blocks $G_1$and G_2then the circuit matrix of G can be expressed in the block-diagonal form as

$$\mathrm{C(G)} = \begin{bmatrix} \mathrm{C(G_1)} & 0 \\ 0 & \mathrm{C(G_2)} \end{bmatrix}$$

Where $\mathrm{C(G_1)}$ and $\mathrm{C(G_2)}$ corresponds to G_1 and G_2 of G respectively. **[6]**

Here we define another important matrix called fundamental circuit matrix which is a sub matrix of circuit matrix in which all rows corresponds to a set of fundamental circuits of graph and is denoted by $\mathrm{C_f}$. If n is the number of vertices and e be the number of edges in a connected graph. Then C_f is always $(e - n + 1) \times e$ matrix.This is because there are $(e - n + 1)$ independent circuits. **[5]**

The graph and its fundamental circuit matrix are shown as follows:



$$C_f = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Consider the arrangement of the rows such that, the first row corresponds to the fundamental circuit represented by the edge e_1 and second row corresponds to the fundamental circuit represented by the edge e_2 and so on. Here C_f can be represented by $[\mathrm{I_2} \quad \mathrm{C_l}]$ where I_2 is an identity matrix of order 2 equal to $(e - n + 1)$, and C_l is a sub matrix of order $(\mathrm{e} - \mathrm{n} + 1) \times (\mathrm{n} - 1)$. This implies rank of $\mathrm{C_f} = \mathrm{e} - \mathrm{n} + 1 = 2$.

Since c_f is a sub matrix of the circuit matrix C, then rank of

$$\mathrm{C(G)} \geq \mathrm{e} - \mathrm{n} + 1 = 2 \qquad \textbf{(III)}$$

Since the rank of any matrix is equal to the number of linearly independent rows (columns) in the matrix. That is Rank of C = Number of linearly independent rows (columns) in C.

Let $X_1 = (10011),\ X_2 = (01101)$ and $X_3 = (11110)$

Be the set of vectors obtained by the circuit matrix corresponding to the graph in the above fig 4.

Let $C_1, C_2,$ and C_3 be the scalars, then by the expression $C_1X_1 + C_2X_2 + C_3X_3 = 0$ we get $C_1 + C_3 = 0,\ \ C_2 + C_3 = 0,$ and $C_1 + C_2 = 0$

That implies $C_1 + C_2 = 0 \ \Rightarrow C_1 = -C_2 = C_3 = \propto$ (say)

$\Rightarrow X_1, X_2\ and\ X_3$ are not linearly independent

$\Rightarrow$ Rank of $C \leq 2 = e - n + 1$ **(IV)**

By the results **(III)** and **(IV)** Rank of $C(G) = e - n + 1 = 2$. **[5]**

ADJACENCY MARIX, DEGREE MATRIX AND LINE GRAPH:

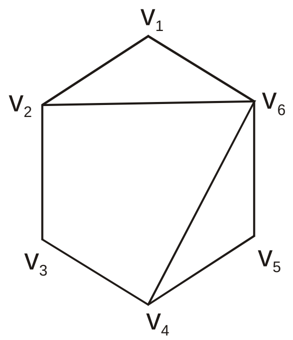
Here we consider the definitions of adjacency matrix, degree matrix and cycle matrix.

Adjacency matrix of a graph:

If G be graph with vertices $v_1, v_2, \ldots\ldots\ldots\ldots..v_n$. then adjacency matrix of a graph is defined as

$$A = a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ adjacent to } v_j \\ 0 & \text{Otherwise} \end{cases}$$

Graph and it's Adjacency matrix



$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

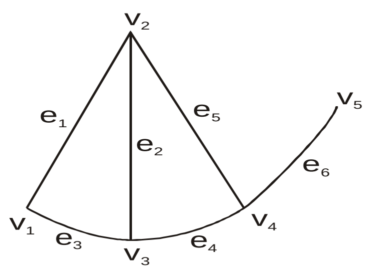
Degree matrix of a graph;

Let $G = (V, E)$ be a graph whose vertices and edges are labeled as $v_1v_2 \ldots\ldots.v_n$ and $e_1e_2 \ldots\ldots\ldots e_n$. Then the degree matrix $D = d_{ij}$ of graph G is a sequence matrix of order n where d_{ij} are given by

$$d_{ij} = \begin{cases} deg v_i & if\ i = j \\ 0 & Otherwise \end{cases} \quad \textbf{[4]}$$

The line graph $L(G)$ of a graph G is derived from G by taking the edges of G as the vertices and joining two vertices in $L(G)$ if the corresponding edges in G have a common vertex. **[2]**

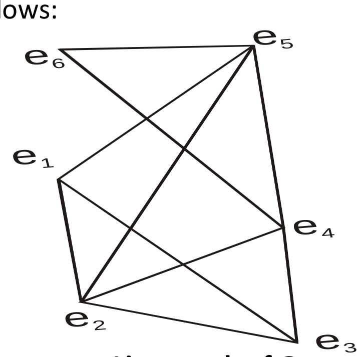
The degree matrix and the line graph of a graph G are shown as follows:



Graph G

$$D(G)=\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Degree matrix of G



Line graph of G

The following theorem relates the adjacency matrix, adjacency matrix of the line graph, incidence matrix and its transpose and also degree matrix.

THEOREM

Let B be the incidence matrix, B^t be the transpose of matrix$B$, A be the adjacency matrix of graph G and A_L be the adjacency matrix of graph $L(G)$ where $L(G)$ is the line graph of a graph of G. Then,

a) $B^tB = A_L + 2I_n$

b) $BB^t = A + B$ **[3]**

c) If G is a regular graph of degree r then $BB^t = A + rI_n$

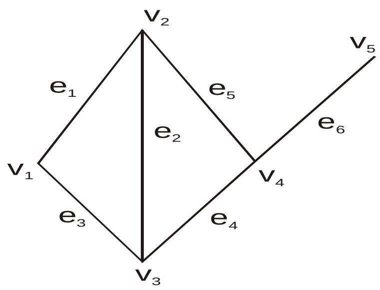
Proof:

(a) The proof follows from the fact that $(ij)^{th}$ entry of B^tB is the inner product of i^{th} column and j^{th} column and B which is zero. If the edges e_i and e_j do not share a common vertex and 1 if e_i and e_j are incident with a common vertex and inner product of i^{th} column of B with itself will always leads 2, since each edge e_i is incident with two vertices. Thus the $(ij)^{th}$ entry of B^tB is nothing but $(ij)^{th}$ entry of the adjacency matrix $A_L$of $L(G)$ for $i + j$ and for $i = j$ it is 2.

(b) In a similar fashion, one can prove the matrix equation in (b) by considering the inner product of i^{th} row and j^{th} row of B and $(ii)^{th}$ entry is the degree of the vertex v_i of G and hence by the definition of degree matrix (b) follows:

(c) This follows from (b) since $(ii)^{th}$ entry of D is r , as the degree of every vertex in G is r .

Considering the adjacency matrix, incidence matrix of a graph and also adjacency matrix of a line graph $\mathrm{L}(\mathrm{G})$. We have



$$A(G)=\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad B=\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad A_L(G)=\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

(a) LHS $$B^tB=\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

RHS $A_L(G)+2I_n = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} +2 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix}$

Hence LHS = RHS

Similarly other results can also be proved.

Conclusion: The properties of Graphs can be studied well with help of matrix algebra.

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