Vol.4.Issue.3.2016 (July-Sept.,)

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

http://www.bomsr.com

Email:editorbomsr@gmail.com

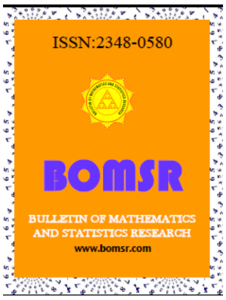
RESEARCH ARTICLE

ISSN:2348-0580

PERSISTENCE OF A STAGE STRUCTURED PREY-PREDATOR MODEL WITH REFUGE

AZHAR ABBAS MAJEED[1], ZAHRAA JAWAD KADHIM[2]

[1.2]Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq.



ABSTRACT

In this paper, sufficient conditions are derived for the uniformly persistence of a stage structured prey predator model with refuge. By constructing appropriate Lyapunov functions, a set of easily verifiable sufficient conditions are obtained for the global asymptotic stability of nonnegative equilibria of the model. Numerical analysis are presented to illustrate the validity of our main results.

Keywords: Equilibrium points, stage structured, refuge, average Lyapunov function, uniformly persistence.

©KY PUBLICATIONS

1. INTRODUCTION

Prey-predator interactions is an important subject in ecology and mathematical ecology for which many problems still remain open [1]. Lotka-Volterra model was the first in this context to describe the interaction of species. After that many complex models are developed to study prey-predator systems. The predator-prey system is an important population model, which has received extensive attention (for example see [2,3,4,5,6]). But all of these works ignore the stage structure of species. However, in the real world, almost animals have the stage structure of immature and mature. Therefore, in recent years, several predator-prey models based on age-structure are developed and studied by many authors (for example see [7,8,9,10,11,12,13,14]). Dynamic nature of refuge has been studied in different models. Actually in prey-predator interaction prey population are at the verge of extinction due to over predation, environmental pollution, mismanagement of natural resources so as to save these species, suitable measures such as restriction on harvesting, creating reserve zones/refuges should be implemented. Thus study of persistence is important from the biological point of view. Biologically, persistence means the long term survival of all populations. In mathematical language, persistence of a system means that strictly positive solutions do not have omega limit points on the boundary of the non-negative cone[15].

In this paper, we present the occurrence of persistence in the mathematical model proposed by Zahraa, J.K., et al [16].

2. Mathematical model [16]

Consider the food web model consisting of two predators-stage structure prey in which the prey species growth logistically in the absence of predation, while the predators decay exponentially in the absence of prey species. It is assumed that the prey population divides into two compartments: immature prey population N_1(t) that represents the population size at time t and mature prey population N_2(t) which denotes to population size at time t. Furthermore the population size of the first predator at time t is denoted by N_3(t), while N_4(t) represents the population size of second predator at time t , see [16].

Now, the dynamics of the model can be represented by the following differential equations [16].

$$\begin{aligned}
\frac{dN_1}{dT} &= \alpha\, N_2 \left(1 - \frac{N_2}{k} \right) - \beta\, N_1 \\
\frac{dN_2}{dT} &= \beta\, N_1 - d_1\, N_2 - c_1\,(1-m)\, N_2\, N_3 - c_2\,(1-m)\, N_2\, N_4 \\
\frac{dN_3}{dT} &= -d_2\, N_3 + e_1\, c_1\,(1-m)\, N_2\, N_3 - c_3\, N_3\, N_4 \\
\frac{dN_4}{dT} &= -d_3\, N_4 + e_2\, c_2\,(1-m)\, N_2\, N_4 - c_4\, N_3\, N_4
\end{aligned} \qquad (2.1)$$

Now, for further simplification of system (2.1) the following dimensionless variables are used in [16].

$$t = \alpha\, T\,,\ u_1 = \frac{\beta}{\alpha}\,,\ u_2 = \frac{d_1}{\alpha}\,,\ u_3 = \frac{d_2}{\alpha}\,,\ u_4 = \frac{e_1 c_1 k}{\alpha}, u_5 = \frac{c_3}{c_2}\,,\ u_6 = \frac{d_3}{\alpha}\,,\ u_7 = \frac{e_2 c_2 k}{\alpha}, u_8 = \frac{c_4}{c_1}\ ,$$
$$x = \frac{N_1}{k}\ , y = \frac{N_2}{k}\ , z = \frac{c_1\, N_3}{\alpha}\ , w = \frac{c_2\, N_4}{\alpha}\ .$$

Thus, system (2.1) can be written in the following dimensionless form:

$$\begin{aligned}
\frac{dx}{dt} &= x \left[\frac{y\,(1-y)}{x} - u_1 \right] = x\ f_1(x\,,y\,,z\,,w) \\
\frac{dy}{dt} &= y \left[\frac{u_1\, x}{y} - u_2 - (1-m)\, z - (1-m)\, w \right] = y\ f_2(x\,,y\,,z\,,w) \\
\frac{dz}{dt} &= z\ \left[-u_3 + u_4\,(1-m)\, y - u_5\, w \right] = z\ f_3(x\,,y\,,z\,,w) \\
\frac{dw}{dt} &= w\ \left[-u_6 + u_7\,(1-m)\, y - u_8\, z \right] = w\ f_4(x\,,y\,,z\,,w)
\end{aligned} \qquad (2.2)$$

with $x(0) \geq 0\,, y(0) \geq 0\,, z(0) \geq 0\ \ and\ \ w(0) \geq 0$. It is observed that the number of parameters have been reduced from thirteen in the system (2.1) to nine in the system (2.2) .

Obviously the interaction functions of the system (2.2) are continuous and have continuous partial derivatives on the following positive four dimensional space.

$R_+^4 = \{ (x\,,y\,,z\,,w) \in R^4 :\ x(0) \geq 0\,, y(0) \geq 0\,, z(0) \geq 0\,, w(0) \geq 0 \}$.

Therefore these functions are Lipschitzian on R_+^4 , and hence the solution of the system (2.2) exists and is unique. Further, all the solutions of system (2.2) with non-negative initial conditions are uniformly bounded as shown in the following theorem [16].

Theorem 1: All the solutions of system (2.2) which initiate in R_+^4 are uniformly bounded.

3. The stability analysis of system (2. 2) [16]

The mathematical model given by system (2.2) has at most five equilibrium points, which are mentioned with their existence conditions in [16] as the following:

1. The vanishing equilibrium point $E_0 = (0,0,0,0)$, always exists and it is locally asymptotically stable in the R_+^4 provided that the following condition holds:

$$u_2 > 1 \tag{3a}$$

However, it is a saddle point otherwise. More details see [16].

2. The first equilibrium point $E_1 = (\bar{x}, \bar{y}, 0, 0)$ exists uniquely in Int. R_+^2 (Interior of R_+^2) of $xy -$ plane under the following condition:

$$u_2 < 1 \tag{3b}$$

And it is locally asymptotically stable provided that:

$$\frac{1-u_2}{2} < \bar{y} < \min\left\{\frac{u_3}{u_4(1-m)}, \frac{u_6}{u_7(1-m)}\right\} \tag{3c}$$

holds. However, it is a saddle point otherwise. More details see [16].

3. The second three species equilibrium point $E_2 = (\check{x}, \check{y}, \check{z}, 0)$ exists uniquely in the $Int.R_+^3$ of $xyz - space$ provided that the following condition holds:

$$u_3 < \min\{u_4(1-m), u_4(1-m)(1-u_2)\} \tag{3d}$$

where

$$\check{x} = \frac{u_3}{u_1 u_4(1-m)}\left[\frac{u_4(1-m)-u_3}{u_4(1-m)}\right], \check{y} = \frac{u_3}{u_4(1-m)} \ and \ \check{z} = \frac{u_4(1-m)(1-u_2)-u_3}{u_4(1-m)^2}.$$

However, according to the Jacobian matrix J_2 given in[16], the characteristic equation of J_2 can be written as:

$$[\lambda^3 + \check{A}_1\lambda^2 + \check{A}_2\lambda + \check{A}_3](-u_6 + u_7(1-m)\check{y} - u_8\check{z} - \lambda) = 0 \ , where$$

$$\check{A}_1 = 1 + u_1 - \frac{u_3}{u_4(1-m)},$$

$$\check{A}_2 = \frac{u_3[u_1 + u_4(1-m)(1-u_2) - u_3]}{u_4(1-m)},$$

$$\check{A}_3 = \frac{u_1 u_3[u_4(1-m)(1-u_2) - u_3]}{u_4(1-m)}.$$

Now, by using Routh-Hawirtiz criterion we obtain that E_2 is locally asymptotically stable if and only if the following conditions

$$u_7(1-m)\check{y} < u_6 + u_8\check{z} \tag{3e}$$

and condition (3d) are hold. For otherwise, it is a saddle point.

4. The second three species equilibrium point $E_3 = (\tilde{x}, \tilde{y}, 0, \widetilde{w})$ exists uniquely in the $Int.R_+^3$ under the following condition:

$$u_6 < \min\{u_7(1-m), u_7(1-u_2)(1-m)\} \tag{3f}$$

where

$$\tilde{x} = \frac{u_6}{u_1 u_7(1-m)}\left[\frac{u_7(1-m)-u_6}{u_7(1-m)}\right], \tilde{y} = \frac{u_6}{u_7(1-m)}, \widetilde{w} = \frac{u_7(1-u_2)(1-m)-u_6}{u_7(1-m)^2}.$$

Moreover, according to the Jacobian matrix J_3 given in[16], the characteristic equation of J_3can be written as:

$$[\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3](-u_3 + u_4(1-m)\tilde{y} - u_5\widetilde{w} - \lambda) = 0 \ , where$$

$$B_1 = 1 + u_1 - \frac{u_6}{u_7(1-m)},$$

$$B_2 = \frac{u_6[u_1 + u_7(1-m)(1-u_2) - u_6]}{u_7(1-m)},$$

$$B_3 = \frac{u_1 u_6[u_7(1-u_2)(1-m) - u_6]}{u_7(1-m)},$$

Note that, by using Routh-Hawirtiz criterion we obtain that E_3 is locally asymptotically stable if and only if the following conditions

$$u_4\,(1-m)\,\tilde{y} < u_3 + u_5\,\widetilde{w} \tag{3g}$$

and condition (3f) are hold. For otherwise, it is a saddle point.

5. The positive equilibrium point $E_4 = (x^*, y^*, z^*, w^*)$ exists uniquely in the $Int.R_+^4$ under the following condition:

$$\max\left\{\frac{u_3}{u_4(1-m)}, \frac{u_6}{u_7(1-m)}\right\} < y^* < 1 \tag{3h}$$

And it is locally asymptotically stable if and only if the following conditions are hold:

$$y^* < \min\left\{\frac{u_3 + u_5\,w^*}{u_4\,(1-m)}, \frac{u_6 + u_8\,z^*}{u_7\,(1-m)}\right\} \tag{3i}$$

$$\beta_4 > -(\,d_{22}\ell_2 + \ell_4\,) \quad with \quad y^* < \frac{1}{2} \tag{3j}$$

where $\ell_2 = d_{11}\Gamma_2 + \Gamma_6$, $\ell_3 = d_{23}d_{32} - d_{24}d_{42}$ and $\ell_4 = d_{11}(\,\Gamma_6 - \ell_3\,) + d_{24}\Gamma_8$

$$\beta_5 > -\beta_6 \tag{3k}$$

$$-[\,\beta_4(2\,\beta_3 + \beta_1\beta_2) + \beta_1^2\beta_6\,] > \beta_1^2\beta_5 + \beta_3(\beta_1\beta_2 + \beta_3) \tag{3l}$$

However,

$$x^* = \frac{y^*}{u_1}(\,1 - y^*\,),\, y^* = \frac{1 - u_2 + \frac{u_6}{u_8}\,(1-m) + \frac{u_3}{u_5}\,(1-m)}{1 + \frac{u_7}{u_8}\,(1-m)^2 + \frac{u_4}{u_5}\,(1-m)^2},\, z^* = \frac{u_7}{u_8}(1-m)\,y^* - \frac{u_6}{u_8}\ and$$

$$w^* = \frac{u_4}{u_5}(1-m)\,y^* - \frac{u_3}{u_5}\,.$$

4. The persistence of system (2. 2)

In this section, we will establish conditions for the persistence of the global dynamics in the boundary plans xy and in the $IntR_+^3$ of xyz and xyw respectively by using the method of average Lyapunov function[17] as shown in the following theorems.

Theorem3: Suppose that the equilibrium point $E_1 = (\,\bar{x}, \bar{y}, 0, 0\,)$ is locally asymptotically stable in the $IntR_+^2$, then it is a globally asymptotically stable in the $IntR_+^2$ of $xy - plane$.

Proof: Consider the following subsystem

$$\frac{dx}{dt} = x\left[\frac{y\,(1-y)}{x} - u_1\right] = f(x,y) \tag{2.3}$$

$$\frac{dy}{dt} = y\left[\frac{u_1\,x}{y} - u_2\right] = g(x,y)$$

where E_1 represent the positive equilibrium point of subsystem (2.3) in the $IntR_+^2$ of $xy - plane$.

Assume that $B(x,y) = \frac{1}{x\,y}$. Clearly, $B(\,x, y\,)$ is a C^1 positive definite function. Further

$$\Delta(\,x\,,y\,) = \frac{\partial}{\partial x}(\,Bf\,) + \frac{\partial}{\partial y}(\,Bg\,) = -\left[\frac{u_2}{x^2} + \frac{u_1}{y^2}\right]$$

Note that, $\Delta(\,x\,,y\,)$does not change sign and is not identically zero in the $IntR_+^2$ of $xy - plane$. Then according to Bendixson-Dulac criterion subsystem (2.3) has no periodic dynamic in the interior of positive quadrant of $xy - plane$. Further, since E_1 is the only positive equilibrium point of subsystem (2.3) in the interior of positive quadrant of $xy - plane$. Hence according to Poincare-Bendixson theorem E_1 is a globally asymptotically stable in the $Int\ R_+^2$ of $xy - plane$ and the proof is complete.

Theorem 4: Assume that the equilibrium point $E_2 = (\,\check{x}\,,\check{y}\,,\check{z}\,,0\,)$ of the system (2.2) is locally asymptotically stable in the $Int\ R_+^3$ and the following conditions

$$\frac{1}{x}+\frac{u_1}{y}-\frac{(y+\breve{y})}{\breve{x}} \leq 2\sqrt{\frac{u_1}{x\,y}} \qquad (4a)$$

$$\frac{y+\breve{y}}{\breve{x}} < \frac{1}{x}+\frac{u_1}{y} \qquad (4b)$$

$$\frac{y^2\,(x-\breve{x})^2}{x\,\breve{x}} < \left[\sqrt{\frac{\breve{y}}{x\,\breve{x}}}\,(x-\breve{x}) - \sqrt{\frac{u_1\,\breve{x}}{y\,\breve{y}}}\,(y-\breve{y})\right]^2 \qquad (4c)$$

are hold. Then the equilibrium point E_2 of the system (2.2) is globally asymptotically stable in the $Int\ R_+^3$ of $xyz-space$.

Proof: Consider the following subsystem

$$\frac{dx}{dt} = x\left[\frac{y\,(1-y)}{x} - u_1\right] = x\,f_1 \qquad (2.4)$$

$$\frac{dy}{dt} = y\left[\frac{u_1\,x}{y} - u_2 - (1-m)z\right] = y\,f_2$$

$$\frac{dz}{dt} = z\,[\,-u_3 + u_4(1-m)y\,] = z\,f_3$$

Now, consider the following function

$$V_1(\,x\,,y\,,z\,\,) = c_1\left(\,x-\breve{x}-\breve{x}\,\ln\frac{x}{\breve{x}}\right) + c_2\left(\,y-\breve{y}-\breve{y}\,\ln\frac{y}{\breve{y}}\right) + c_3\left(\,z-\breve{z}-\breve{z}\,\ln\frac{z}{\breve{z}}\right).$$

Clearly $V_1\colon R_+^3 \to R$ is a C^1 positive definite function. Now by differentiating V_1 with respect to time t and doing some algebraic manipulation by choosing $c_1 = c_2 = 1\ and\ c_3 = \frac{1}{u_4}$ gives that:

$$\frac{dV_1}{dt} < -\left[\sqrt{\frac{\breve{y}}{x\,\breve{x}}}\,(x-\breve{x}) - \sqrt{\frac{u_1\,\breve{x}}{y\,\breve{y}}}\,(y-\breve{y})\right]^2 + \frac{y^2}{x\,\breve{x}}\,(x-\breve{x})^2\,.$$

However, the conditions $(4a)\ and\ (4b)$ guarantee the completeness of the quadratic term between $x\ and\ y$. So, if condition $(4c)$ holds. Then, $\frac{dV_1}{dt}$ is negative and hence V_1 is strictly Lyapunov function. Thus E_2 is globally asymptotically stable in the $Int\ R_+^3\ of\ xyz-space$ and the proof is complete.

Theorem5: Assume that the equilibrium point $E_3 = (\,\tilde{x}\,,\tilde{y}\,,0\,,\widetilde{w}\,)$ of the system (2.2) is locally asymptotically stable in the $Int\ R_+^3$ and the following conditions

$$\frac{1}{x}-\frac{(y+\tilde{y})}{\tilde{x}}+\frac{u_1}{y} \leq 2\sqrt{\frac{u_1}{x\,y}} \qquad (4d)$$

$$\frac{y+\tilde{y}}{\tilde{x}} < \frac{1}{x}+\frac{u_1}{y} \qquad (4e)$$

$$\frac{y^2}{x\,\tilde{x}}\,(x-\tilde{x})^2 < \left[\sqrt{\frac{\tilde{y}}{x\,\tilde{x}}}\,(x-\tilde{x}) - \sqrt{\frac{u_1\,\tilde{x}}{y\,\tilde{y}}}\,(y-\tilde{y})\right]^2 \qquad (4f)$$

are hold. Then the equilibrium point E_3 of the system (2.2) is globally asymptotically stable in the $Int\ R_+^3$ of $xyw-space$.

Proof: Consider the following subsystem

$$\frac{dx}{dt} = x\left[\frac{y\,(1-y)}{x} - u_1\right] = x\,f_1 \qquad (2.5)$$

$$\frac{dy}{dt} = y\left[\frac{u_1\,x}{y} - u_2 - (1-m)w\right] = y\,f_2$$

$$\frac{dw}{dt} = w\,[\,-u_6 + u_7(1-m)y\,] = w\,f_3$$

Now, consider the following function

$$V_2(\,x\,,y\,,w\,\,) = c_1\left(\,x - \tilde{x} - \tilde{x}\ln\frac{x}{\tilde{x}}\right) + c_2\left(\,y - \tilde{y} - \tilde{y}\,\ln\frac{y}{\tilde{y}}\right) + c_3\left(\,w - \widetilde{w} - \widetilde{w}\,\ln\frac{w}{\widetilde{w}}\right).$$

Clearly V_2: $R_+^3 \to R$ is a C^1 positive definite function. Now by differentiating V_2 with respect to time t and doing some algebraic manipulation by letting $c_1 = c_2 = 1\ and\ c_3 = \frac{1}{u_7}$ gives that:

$$\frac{dV_2}{dt} < -\left[\sqrt{\frac{\tilde{y}}{x\,\tilde{x}}}\,(x-\tilde{x}) - \sqrt{\frac{u_1\,\tilde{x}}{y\,\tilde{y}}}\,(y-\tilde{y})\right]^2 + \frac{y^2}{x\,\tilde{x}}\,(x-\tilde{x})^2\,.$$

However, the conditions $(4d)\ and\ (4e)$ guarantee the completeness of the quadratic term between $x\ and\ y$. So, if condition $(4f)$ holds. Then, $\frac{dV_2}{dt}$ is negative and hence V_2 is strictly Lyapunov function. Thus E_3 is globally asymptotically stable in the $Int\ R_+^3\ of\ xyw - space$ and the proof is complete.

In the next theorem we show that system (2.2) is uniformly persistence. By the permanence or persistence of a system, we mean that all the species are present and non of them will go to extinction. The persistence of a system have been studied by several researchers for example see[18,19,20,21,22].

Theorem6: Assume that there are no periodic dynamics of system (2.2) in the boundary of the solution. Further, if the following conditions

$$u_4u_6(1-m)^2 + u_8[u_4(1-m)(1-u_2) - u_3] < u_3u_7(1-m)^2 \qquad (4g)$$

$$u_3u_7(1-m)^2 + u_5[u_7(1-m)(1-u_2) - u_6] < u_4u_6(1-m)^2 \qquad (4h)$$

are hold. Then system (2.2) is uniformly persistent.

Proof: Consider the following average Lyapunov function

$$\delta(\,x\,,y\,,z\,,w\,) = x^{P_1}\,y^{P_2}\,z^{P_3}\,w^{P_4}\,,$$

where each $P_i\,, i = 1,2,3,4$ is a positive constant. Obviously, $\delta(\,x\,,y\,,z\,,w\,)$ is a nonnegative C^1 defined in R_+^4 . Then we have

$$\begin{aligned}\Psi(\,x,y,z,w\,) &= \frac{\acute{\delta}(\,x,y,z,w\,)}{\delta(\,x,y,z,w\,)} \\ &= P_1\left[\frac{y\,(1-y)}{x} - u_1\right] + P_2\left[\frac{u_1\,x}{y} - u_2 - (1-m)\,z - (1-m)\,w\right] + \\ &\quad P_3[-u_3 + u_4\,(1-m)\,y - u_5\,w] + P_4[-u_6 + u_7\,(1-m)\,y - u_8\,z]\ .\end{aligned}$$

Now, violate condition $(3a)$ imply that E_0 is unstable and then we obtain that these equilibrium point does not belong to the omega limit set of system (2.2), then the only possible omega limit set of system (2.2) are the equilibrium points $E_i\,, i = 1,2,3$.

1) For $E_1 = (\,\bar{x}\,,\bar{y}\,,0\,,0\,)$ we have

$$\Psi(E_1) = P_3[-u_3 + u_4(1-m)(1-u_2)] + P_4[-u_6 + u_7(1-m)(1-u_2)]$$

Violate condition $(3c)$ imply that $\Psi(\,E_1) > 0$ for any $P_3 > 0$ and $P_4 > 0$.

2) For $E_2 = (\,\check{x}\,,\check{y}\,,\check{z}\,,0\,)$ we have

$$\Psi(E_2) = P_4\left[\frac{u_3u_7(1-m)^2 - [\,u_4u_6(1-m)^2 + u_8[u_4(1-m)(1-u_2) - u_3]\,]}{u_4\,(1-m)^2}\right]$$

So, $\Psi(E_2) > 0$ for any $P_4 > 0$ provided that condition $(4g)\ and\ (3d)$ are hold.

3) For $E_3 = (\,\tilde{x}\,,\tilde{y}\,,0\,,\widetilde{w}\,)$ we have

$$\Psi(E_3) = P_3\left[\frac{u_4u_6(1-m)^2 - [\,u_3u_7(1-m)^2 + u_5[u_7(1-m)(1-u_2) - u_6]\,]}{u_7\,(1-m)^2}\right]$$

So, $\Psi(E_3) > 0$ for any $P_3 > 0$ provided that condition $(4h)$ and $(3f)$ are hold.
Hence, system (2.2) is uniformly persistent and that completes the proof.

5. Numerical analysis

In this section the global dynamics of system (2.2) is studied numerically. The system (2.2) is solved numerically, for different sets of parameters and different sets of initial conditions, using predictor-corrector method with six order Runge-Kutta method [23], and then the time series for the trajectories of system (2.2) are drown.

Now, for the following set of hypothetical parameters that satisfies stability conditions of the positive equilibrium point, system (2.2) has a globally asymptotically stable positive equilibrium point as shown in Fig.1.

$$u_1 = 0.1, u_2 = 0.1, u_3 = 0.1, u_4 = 0.8, u_5 = 0.2, u_6 = 0.1, u_7 = 0.8\ , u_8 = 0.2, m = 0.8 \qquad (5.1)$$

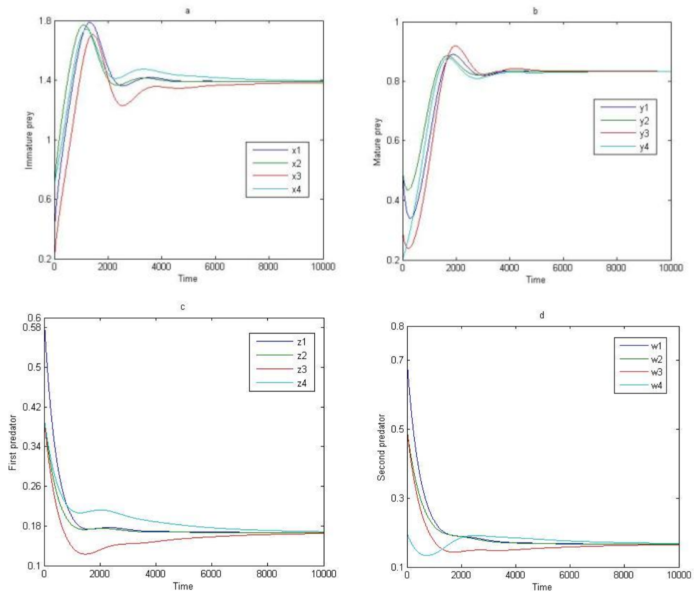


Fig.1: Time series of the solution of system (2.2) that started from four different initial points $(0.4\ ,0.5\ ,0.6\ ,0.7)$, $(0.7\ ,0.8\ ,0.4\ ,0.5)$, $(0.2\ ,0.3\ ,0.4, 0.5)$ and $(0.7\ ,0.2\ ,0.4\ ,0.2)$ for the data given by (5.1).(a) trajectories of x as a function of time,(b) trajectories of y as a function of time,(c) trajectories of z as a function of time,(d) trajectories of w as a function of time.

Clearly, Fig.1 shows that system (2.2) has a globally asymptotically stable as the solution of system (2.2) approaches asymptotically to the positive equilibrium point $E_4 = (\ 1.39\ , 0.83\ , 0.17\ , 0.17\ )$ starting from four different initial points and this is confirming our obtained analytical results.

Now, in order to discuss the effect of the parameters values of system (2.2) on the dynamical behavior of the system, the system is solved numerically for the data given in (5.1) with varying one parameter at each time. It is observed that varying the parameters values u_i , $i = 1, 5,6,7,8$ the solution still approaches to a positive equilibrium point $E_4 = (x^*, y^*, z^*, w^*)$, we obtain that system (2.2) persists as shown in Fig.2 for typical value $u_1 = 0.1$.

By varying the parameter u_2 , m and keeping the rest of parameters values as in (5.1), it is observed that for $0.1 < u_2 < 0.5\ and\ 0.1 \leq m < 0.9$, the solution of system (2.2) approaches asymptotically to a positive equilibrium point E_4 . while for $0.5 \leq u_2 < 1\ and\ 0.9 \leq m < 1$, system (2.2) losses the persistence and the solution of system (2.2) approaches asymptotically to $E_1 = (\ \bar{x}, \bar{y}, 0, 0\ )$ as shown in Fig.3 for typical value $u_2 = 0.7$.

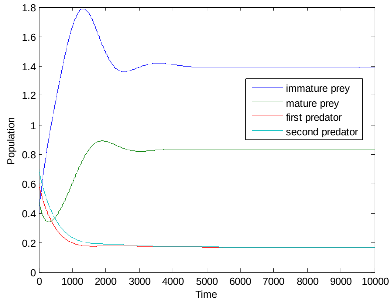


Fig.2: Time series of the solution of system (2.2) for the data given by (5.1) with $u_1 = 0.1$ which approaches to (1.39 ,0.83 ,0.17 ,0.17) in the interior of R_{+}^{4} .

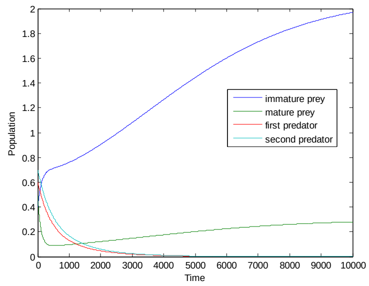


Fig.3: Times series of the solution of system (2.2) for the data given by (5.1) with $u_2 = 0.7$ which approaches to (1.97 , 0.28 , 0 , 0) in the interior of the positive quadrant of $\mathrm{xy-plane}$.

6. Conclusions

In this paper, the conditions of occurrence of persistence of a mathematical model consists of a stage structured prey-predator model incorporating a prey refuge are established. Now, we shall discuss the effects of changing the parameters on the dynamical behaviour of system (2.2) according to the numerical results in section 5:

1. For the set of hypothetical parameters values given in (5.1), the system (2.2) approaches asymptotically to global stable positive equilibrium point .
2. It is observed that system (2.2) has no effect on the dynamical behavior for the data given in (5.1) with varying the parameter value u_1 in the range $0.1 < u_1 < 1$ and the system still approaches to the positive equilibrium point and the system persists.
3. As the natural death rate u_2 of the mature prey increasing in the range $0.1 < u_2 < 0.5$ and keeping other parameters fixed as in (5.1), then again the solution of system (2.2) approaches asymptotically to the positive equilibrium point. However, increasing u_2 in the range $0.5 \le u_2 < 1$ will cause extinction in the predators and the solution of system (2.2) approaches asymptotically to $E_1 = (\bar{x}, \bar{y}, 0, 0)$ and the system is not persists.
4. As the natural death rate u_3 of the first predator increasing in the range $0.1 < u_3 < 1$ and keeping the rest of parameters as in (5.1), then again the solution of system (2.2) approaches asymptotically to E_1. Consequently, for $u_3 > 0.1$, the system is not persists.
5. As the predation rate u_4 increasing in the range $0.1 \le u_4 \le 0.4$ causes extinction in the predators and the solution of system (2.2) approaches asymptotically to $E_1 = (\bar{x}, \bar{y}, 0, 0)$. while, increasing u_4 in the range $0.4 < u_4 \le 1$ then the system is persist and the solution of system (2.2) approaches asymptotically to the positive equilibrium point.
6. As the competition rate u_5 and the predation rate u_7 increasing in the range $0.1 \le u_5 \le 1$, $0.1 \le u_7 \le 1$ and keeping the rest of parameters as in (5.1), then again the solution of

system (2.2) approaches asymptotically to the positive equilibrium point. It is observed that the natural death rate u_6 of the second predator and the competition rate u_8 have the same effect as u_1 and u_5 .

7. As the number of preys inside the refuge m increasing in the range $0.1 \leq m < 0.9$ and keeping other parameters fixed as in (5.1), then the system persists and again the solution of system (2.2) approaches asymptotically to the positive equilibrium point. However, increasing m in the range $0.9 \leq m < 1$ will cause extinction in the predators and the solution of system (2.2) approaches asymptotically to E_1 and the system losses persistence.

References

[1]. A.A. Berryman, The origins and evolutions of predator-prey theory, Ecology,73:1530-1543,1992.

[2]. Chen, L., Mathematical ecology modeling and research methods, Beijing: Science Press, 1988.

[3]. Goh, B.S., Global stability in two species interaction, J Math Biol.,3: 313–8,1976.

[4]. Kuang, Y., Delay differential equations with applications in population dynamics, New York: Academic Press, 1993.

[5]. Murray, JD., Mathematical biology, Springer-Verlag, p. 9, 1989.

[6]. Takeuchi, Y., Global dynamical properties of Lotka–Volterra systems, Singapore: World Scientific, 1996.

[7]. Gourley, S. and Yang, K., A stage structured predator–prey model and its dependence on maturation delay and death rate, J Math Biol., 49: 188–200, 2004.

[8]. Liu S, Chen L and Agarwal R., Recent progress on stage-structured population dynamics, Math. Comput. Model, 36: 1319-60,2002.

[9]. Pang, G., Wang, F. and Chen, L., Extinction and permanence in delayed stage-structure predator–prey system with impulsive effects, Chaos Solution Fract, 39(5): 2216-24, 2009.

[10]. Song, X. and Chen, L., Optimal harvesting stability for a predator–prey system with stage structure, Acta Math. Appl. Sin Eng Ser, 18(3): 423–30, 2002.

[11]. Tang, S. and Chen, L., Quasi periodic solutions and chaos in a periodically predator–prey system model with age structure for predator, Int. J Bifurcat Chaos, 13(4): 973–80, 2003.

[12]. Wang, F. and Pang, G., The global stability of a delayed predator–prey system with two stage-structure, Chaos Solitons Fract., 40 (2): 778–85, 2009.

[13]. Wang, W. and Chen L., A predator–prey system with stage-structure for predator, Comput. Math. Appl., 33: 83–91, 1997.

[14]. Zhang, X., Chen, L. and Neumann, A., The stage-structured predator–prey model and optimal harvesting policy, Math. Biosci., 168:201–10, 2000.

[15]. Debasis Mukherjee, Persistence in a Generalized Prey-Predator Model with Prey Reserve, International Journal of Nonlinear Science, Vol.14, No.2,pp.160-165, 2012.

[16]. Zahraa, J.K., Azhar, A.M. and Raid, K.N., Stability analysis of two predator-one stage-structured prey model incorporating a prey refuge, IOSR Journal of Mathematics, V.11, pp.42-52, 2015.

[17]. Grad T.C. and Hallam T.C., Persistence in food webs-I. Lotka Volterra food chains, Bull. Math.Biol.,41,877-891, 1979.

[18]. Freedman H.I. and Waltman P., “Persistence in models of three interacting predator prey populations”, Math. Biosci., 68, 213–231, 1984.

[19]. Mitra D., Mukherjee D. and Roy A.B., “Permanent coexistence in a resource-based competition system”, Ecol. Model., 60, 77–85, 1992.

[20]. Butler G., Freedman H.I. and Waltman P., “Uniformly persistent system”, In: Proc. Am. Math. Soc., 96, 420–430, 1986.

[21]. Gard T.C. and Hallam T.G., “Persistence in food webs-I. Lotka-Volterra food chains”, Bull. Math. Biol., 41, 285–299, 1979.

[22]. Hutson V. and Vickers G.T., “A criterion for permanent coexistence of species with an application of two prey one predator system”, Math. Biosci., 63, 253–269, 1983.

[23]. R.K. Naji, “Order and chaos in multi species ecological systems”, Ph. D. Thesis, Indian institute of Technology Roorkee, 2003.