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RESEARCH ARTICLE

CELLULAR FOLDING OF SOME NEW *CW*-COMPLEXES

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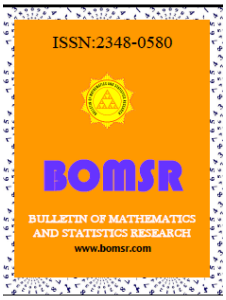


ABSTRACT

In this paper we obtained the conditions satisfied by a cellular folding of a given *CW*-complex to be able to cellular fold some new *CW*-complexes generated by some known operations like quotient, suspension of a regular *CW*-complex, Cartesian product, join product, and wedge sum of two *CW*-complexes.

Keywords: Cellular folding, quotient, suspension, join product, and wedge sum

1. INTRODUCTION

A cellular folding is a folding defined on regular *CW*-complexes first defined by, E. El-Kholy and H. Al-Khurasani, [1] and various properties of this type of folding are also studied by them.

Let *K* and *L* be complexes, a continuous map $f : K \to L$ is called cellular if $f(K^n) \subset L^n$ for $n = 0, 1, 2, \ldots, n$, where K^n and L^n denote the *n*-skeletons of *K* and *L* respectively.

Now, let *K* and *L* be regular *CW*-complexes of the same dimension $\boldsymbol{n}$, a cellular map $f : K \to L$ is a cellular folding if and only if f satisfies the following:

(i) For each *i*-cells $e^i \in K$, $f(e^i)$ is an *i*-cell in *L*, i.e. , f maps *i*-cells to *i*-cells ;

(ii) If $\bar{e}$ contains *n* vertices, then $\overline{f(e)}$ must contains *n* distinct vertices, [1].The set of regular *CW*-complexes together with cellular folding form a category denoted by $C(K, L)$. If $f \in C(K, L)$, then $x \in K$ is said to be a singularity of f iff f is not a local homeomorphism at *x*. The set of all singularities of f is denoted by $\sum f$. This set corresponds to the "folds" of the map. It is noticed that for a cellular folding *f*, the set $\sum f$ of singularities of f is a proper subset of the union of cells of dimension $\leq n-1$. Thus, when we consider any $f \in C(K, L)$, where *K* and *L* are connected regular *CW*-complexes of dimension 2, the set $\sum f$ will consist of 0-cells, and 1-cells, each of 0-cells (vertices) has even valences [2]. Of course $\sum f$ need not be connected. Thus in this case $\sum f$ has the structure of a locally finite graph Γ_f embedded in *K*, for which every vertex has an even valency.

From now on by a complex we mean regular *CW*-complexes.

2. Cellular folding of the Cartesian product of complexes:

If *X* and *Y* are cell complexes, then $X \times Y$ has the structure of a cell complex with cell products $e_\alpha^m \times e_\beta^n$ where e_α^m ranges over the cells of *X* and e_β^n ranges over the cells of *Y* [3].

2.1. Theorem

Let *K* and *X* be complexes of the same dimension *n*, *L* and *Y* be complexes of the same dimension m. Let $f: K \to X$ and $g: L \to Y$ be cellular maps. Then $f \times g \in C(K \times L, X \times Y)$ if and only if *f* and *g* are cellular foldings.

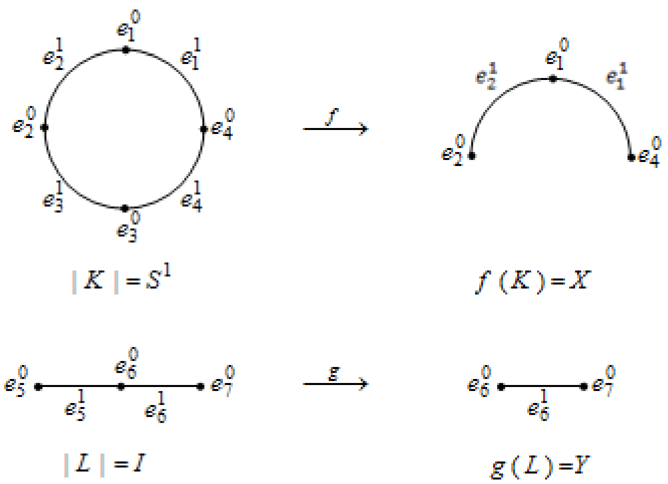
Proof:

If f and g are cellular foldings, then each will maps cells to cells of the same dimension hence do $f \times g$. Also $\bar{e}$ and $\overline{f \times g(e)}$ contains the same number of vertices because each of *f* and *g* are cellular foldings.

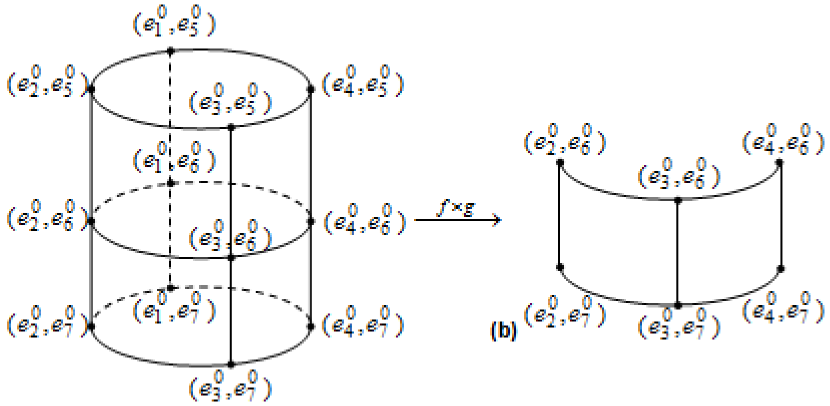
Suppose now $f \times g$ is a cellular folding, then $f \times g$ maps p-cells to p-cells, i.e., if (e, e') is a p-cell in $K \times L$, then $(f \times g)(e, e') = (f(e), g(e'))$ is a p-cell in $X \times Y$. Let e be an *i*-cell in *K* and e' be a $(p-i)$-cell in *L*. The cellular map must maps *i*-cells to *j*-cells such that $j \leq i$. If $j = i$ nothing to prove, so let $i > j$. In this case g will maps $(p-i)$-cells to $(p-j)$-cells and hence is not a cellular map. This is a contradiction and hence $i = j$ is the only possibility. The second condition of cellular folding certainly satisfied in this case.

2.2. Example:

Suppose that *K* and *L* are complexes such that $|K| = S^1, |L| = I$ with cell decomposition shown in Fig. (1-a). Let $f: K \to K$ be a cellular folding defined by $f(e_3^0) = e_1^0$, $f(e_3^1, e_4^1) = (e_2^1, e_1^1)$, where the omitted cells through the paper are mapped into themselves. The image of *f* is a complex consisting of three vertices and two 1-cells. Let $g: L \to L$ be a cellular folding defined by $(e_5^0) = e_7^0$, $g(e_5^1) = e_6^1$. Then $f \times g: K \times L \to K \times L$ is a cellular folding. The cell decomposition of $K \times L$ and $(f \times g)(K \times L)$ are shown in Fig.(1-b).



(a)



(b)

Fig. (1)

3. Cellular folding of the quotient of a complex:

If (X, A) is a pair consisting of a cell complex *X* and a subcomplex *A*, then the quotient space X/A inherits a natural cell complex structure from *X*. The cells of X/A are the cells of $X - A$ plus one new 0-cell, the image of *A* in X/A, [3].

3.1. Example:

If $X = D^2 = \{(x, y) \in E^2 : x^2 + y^2 \le 1\}$ is a disc with the cell structure consisting of two 0-cells, two 1-cells and one 2-cell, and let $A = S^1 = \partial D^2$. Then D^2/A is a sphere S^2 with one 2-cell and one 0-cell, see Fig. (2).

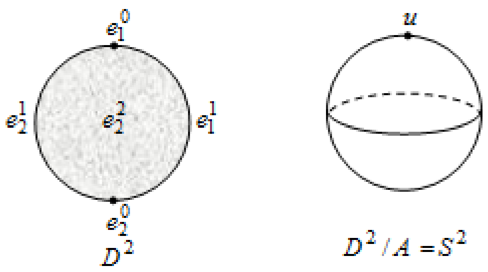


Fig. (2)

Generally, if we give S^{n-1} any cell structure and build D^n from S^{n-1} by attaching an n-cell, then the quotient D^n/S^{n-1} is S^n with its usual cell structure [3].

3.2. Theorem

Let *X* be a complex, $A \subset X$ a subcomplex, $f : X \to X$ a cellular map. Let $g : X/A \to X/A$ be defined by, for each *i*-cell e in $X - A$, $g(e) = f(e)$, $g(e^0) = e^0$, where e^0 is the new 0-cell of X/A. Then g is a cellular folding if and only if both *f* and $f|A$ is a cellular folding. In this case $g(X/A) = f(X)/f(A)$.

Proof:

Let $f : X \to X$ be a cellular folding, *e* an *i*-cell in X/A such that $\bar{e}$ has *n* distinct vertices $g(e) = f(e)$ is an *i*-cell such that $\overline{g(e)} = \overline{f(e)}$ has *n* distinct vertices, $g(e^0) = e^0$. Thus $g : X/A \to X/A$ is a cellular folding.

Now suppose $g : X/A \to X/A$ is a cellular folding and $f : X \to X$ is a cellular map, if *e* is an *i*-cell in $X - A$, then $f(e) = g(e)$ is an *i*-cell in *X*, but if *e* is an *i*-cell in *A*, then $f(e)$ might be a *j*-cell in *X*, $j < i$ while if $f \mid A : A \to A$ is a cellular folding, then for any *i*-cell in *X*, $f(e)$ is an *i*-cell in *X* and consequently *f* is a cellular folding.

3.3. Example

Let $X = D^2$ be a disc with cellular subdivision consisting of two 0-cells, three 1-cells and two 2-cells, and l et $A = S^1 = \partial D^2$, $f : D^2 \to D^2$ be a cellular folding defined as follows : $f(e_1^0, e_2^0) = (e_1^0, e_2^0)$, $f(e_2^1) = (e_1^1)$, $f(e_2^2) = f(e_1^2)$.

The map $f|A$ is the cellular folding shown in Fig.(3). Now $g : X/A \to X/A$ is a cellular folding defined by, $g(e^0) = e^0$, $g(e_3^1) = e_3^1$, $g(e_2^2) = e_1^2$, see Fig. (3).

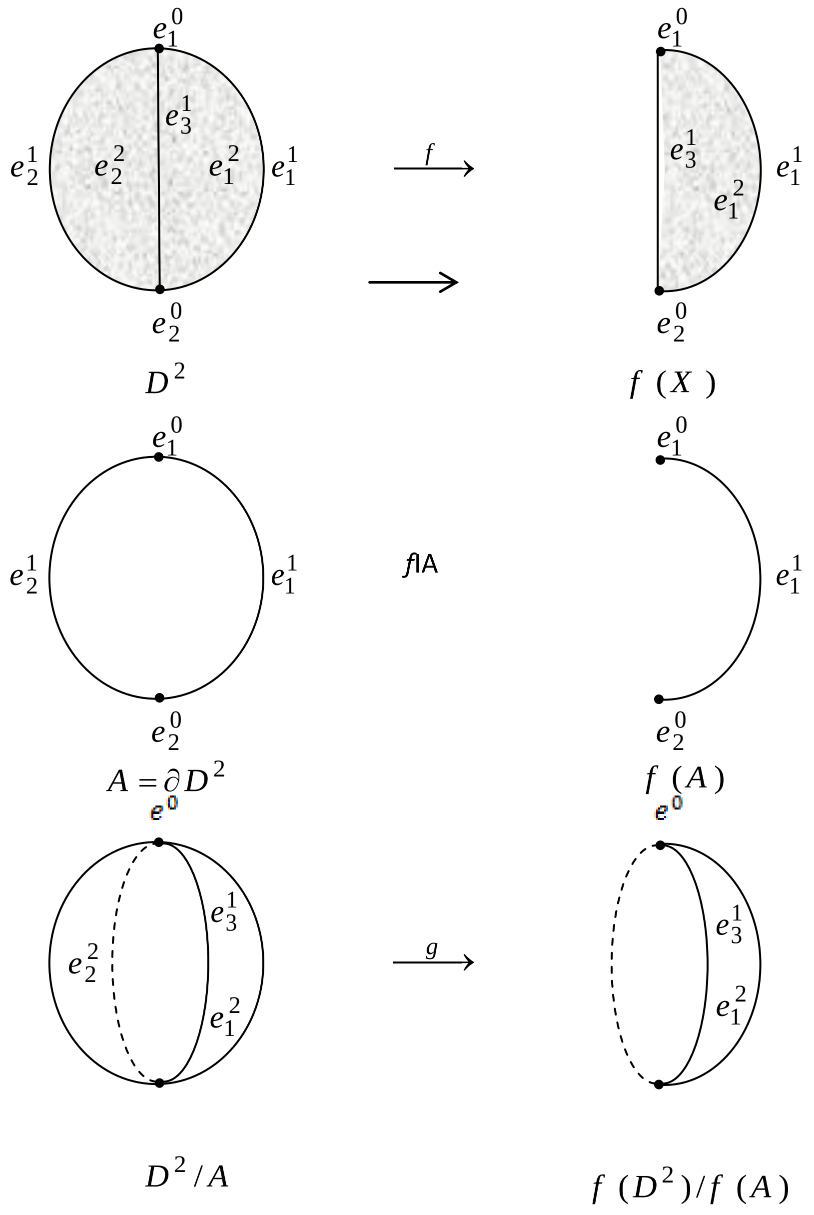


Fig. (3)

3.4. Example

Let X be a complex, such that $|X| = T$ is a torus, with cellular subdivision consisting of two 0-cells, four 1-cells, and two 2-cells, $A \subset X$ be the subcomplex shown in Fig. 4. Let $f : X \to X$ be a cellular folding defined as follows: $f(e_i^0) = e_i^0$, $i = 1, 2$; $f(e_1^1) = (e_2^1)$, $f(e_1^2) = (e_2^2)$. The map $f|A$ is the cellular folding shown in Fig. (3).

Now $g : X/A \to X/A$ is a cellular folding defined by, $g(e^0) = e^0$, $g(e_3^1) = e_3^1$, $g(e_1^2) = e_2^2$, see Fig.(4).

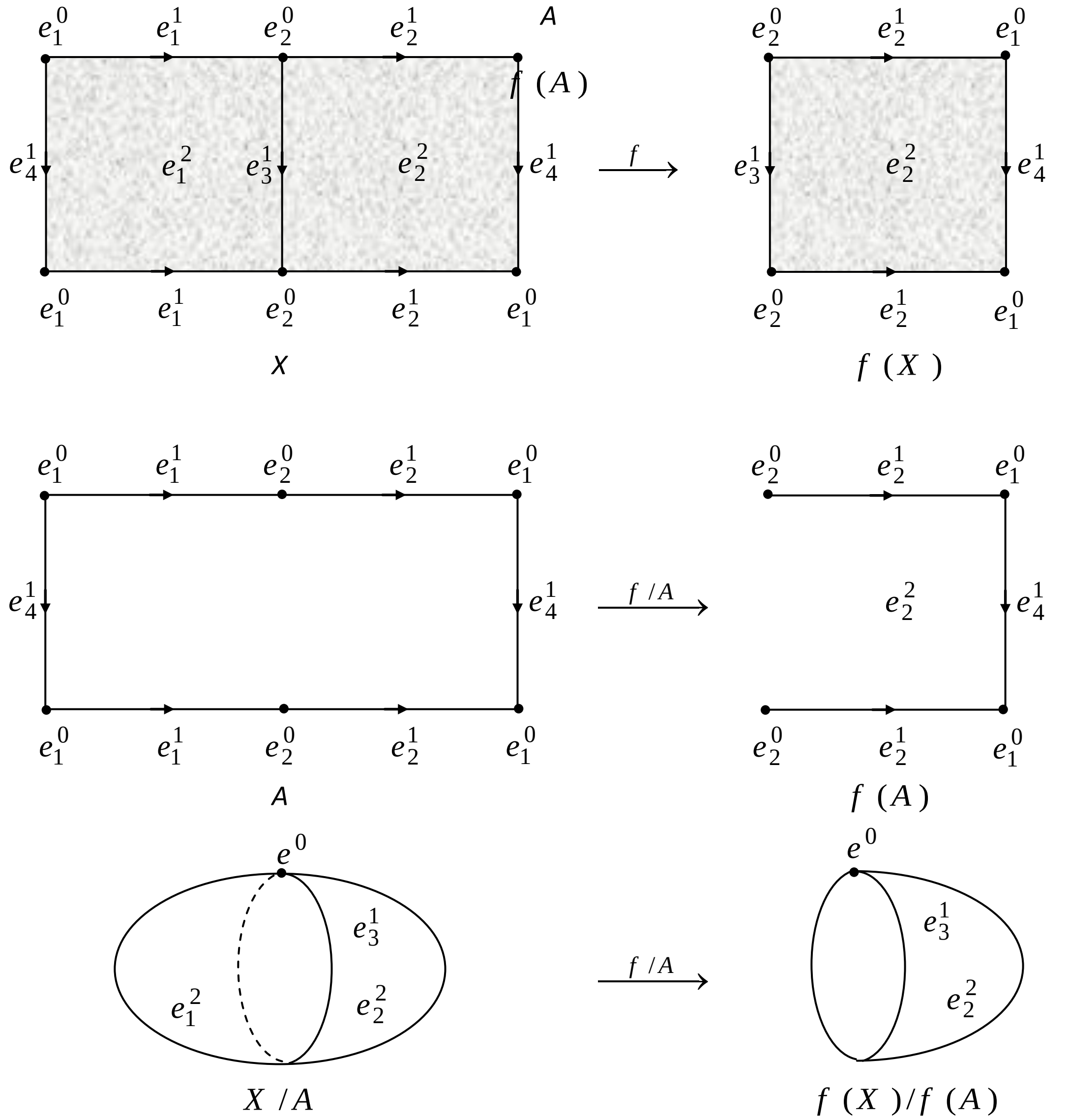


Fig.(4)

4. Cellular folding of the suspension

For a space X, the suspension SX is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point. If X is a complex, so is SX as quotient of $X \times I$ with its product cell structure, I being given the standard cell structure of two 0-cells joined by a 1-cell [3]. Thus we can define the suspension SX as the union of all line segments joining points of *X* to two external vertices called "suspension points".

Useful property of suspension is that not only spaces but also maps can be suspended, a map $f : X \to Y$ suspends to $Sf : SX \to SY$, the quotient map of $f \times I : X \times I \to Y \times I$, [3].

4.1. Example

If $X = S^1$, circle, then $S(S^1) = S^2$, see Fig.(5)

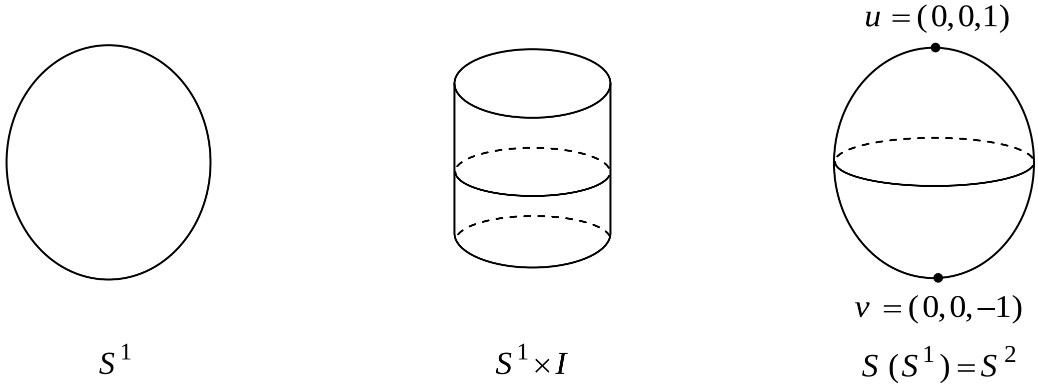


Fig. (5)

Generally $S(S^n) = S^{n+1}$.

4.2. Theorem

Let *X* and *Y* be complexes of the same dimension *n*, let $f : X \to Y$ be a cellular map. Then $g = Sf : SX \to SY$ mapping suspension points (vertices) *u*, *v* into itself, and for each *i*-cell $(e, e') \in SX$, $g(e, e') = (f(e), e')$, where e' is a zero or a one-cell of *I*, is a cellular folding if and only if *f* is a cellular folding.

Proof:

If *f* is a cellular folding, then it will maps cells to cells of the same dimension, and hence does *g*. Also $\overline{(e, e')}$ and $\overline{g(e, e')} = \overline{(f(e), e')}$ contains the same number of vertices because *f* is a cellular folding.

Suppose now g is a cellular folding, then g maps *i*-cell to *i*-cell, i.e., if (e, e') is an *i*-cell in SX, then $g(e, e') = (f(e), e')$ is an *i*-cell in SY. Let e be a *j*-cell in *X*, and e' be an $(i-j)$-cell in *I*. The cellular map must maps *j*-cells to *k*-cells such that $k \le j$. If $k = j$ nothing to prove, so let $k < j$. In this case g will maps $(i-j)$-cells to $(i-k)$-cells and hence is not a cellular map. This is a contradiction, and hence $k = j$ is only possibility. The second condition of cellular folding certainly satisfied in this case.

4.3. Example:

Let $X = S^1$ be a complex with cellular subdivision shown in Fig.(6-a), and $f : X \to X$ be a cellular folding defined by: $f(e_3^0) = (e_1^0)$, $f(e_2^1, e_3^1) = (e_1^1, e_4^1)$.

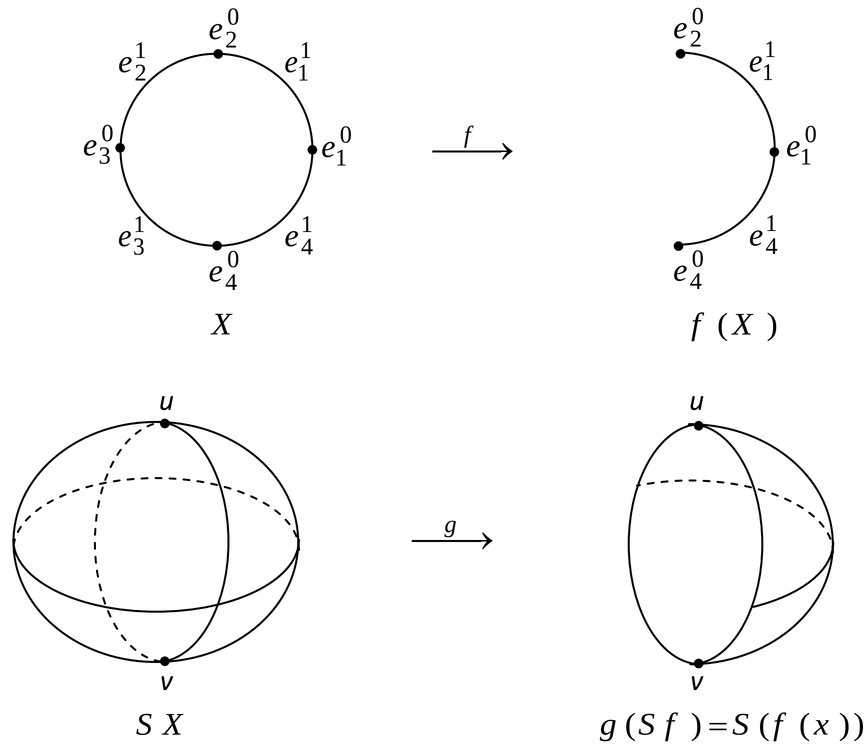


Fig. (6)

$g(u,v)=(u,v),\ g(e,e')=(f(e),e'),\ e\in X,\ e'\in I$

Then g=Sf: $SX \to SY$ is a cellular folding defined by $g(u,v)$=(u,v), $(e,\acute{e})=(f(e),\acute{e}), e\in X, \acute{e}\in I$.

5. Cellular folding of the join of complexes

The join $X * Y$ of the two spaces *X* and *Y* is the quotient space $X \times Y \times I$ under the identification $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 0) \sim (x_2, y, 0)$. Thus we are collapsing the subcomplex $X \times Y \times \{0\}$ to *X* and $X \times Y \times \{1\}$ to *Y*, [3]. One can define this space as the space of all line segments joining points in *X* to points in *Y*.

Note that if *X* and *Y* are complexes, then there is a natural *CW* structure on $X * Y$ having the subspaces *X* and *Y* as a subcomplexes, with the remaining cells being the produce cells of $X \times Y \times (0,1)$.

5.1. Example

If *X* and *Y* are both closed intervals, then we are collapsing two opposite faces of a cube onto line segments so that the cube becomes a tetrahedron, see Fig.(7).

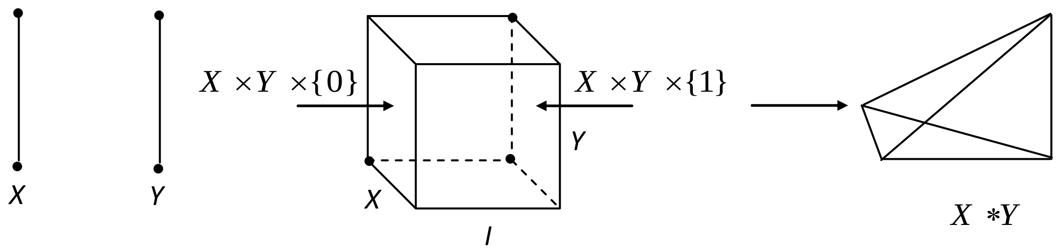


Fig. (7)

5.2. Theorem

Let *X* and *Y* be complexes of the same dimension *n*, let $f: X \to X$, $g: Y \to Y$ be cellular maps. Then $h = f * g : X * Y \to X * Y$ defined as the quotient map of $f \times g \times I : X \times Y \times I \to X \times Y \times I$ under the identifications $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 0) \sim (x_2, y, 0)$ is a cellular folding if and only if *f* and g are both cellular foldings.

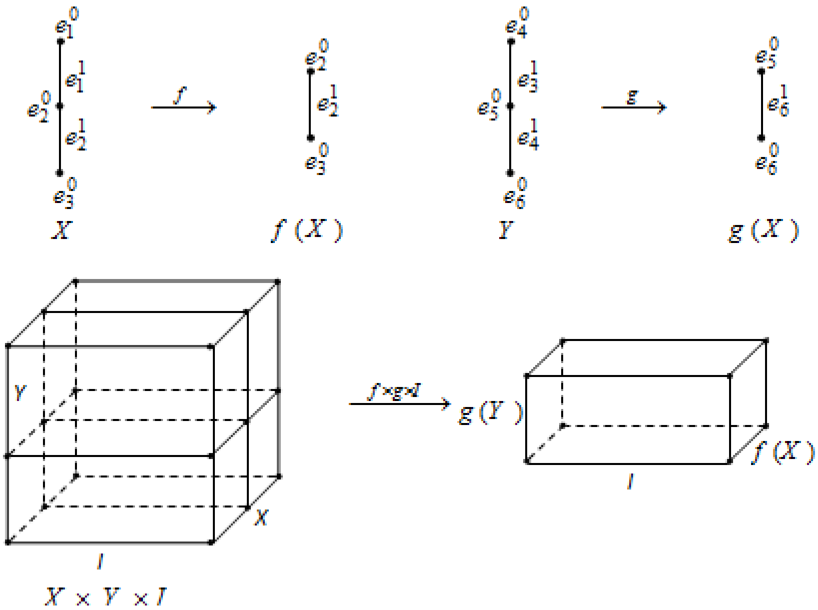
Proof

Suppose that f and g are cellular foldings. Let e be an *i*-cell in *X* and σ be a *j*-cell in *Y*. Then (e, σ) is an $(i + j + 1)$-cell in $X * Y$. Now $(f * g)(e, \sigma) = (f(e), g(\sigma))$, but since each of *f* and *g* are cellular foldings, then $f(e)$ is an *i*-cell in $f(X)$ and $g(\sigma)$ is a *j*-cell in $g(Y)$. Thus $(f * g)(e, \sigma)$ is an $(i + j + 1)$-cell in $f(X) * g(Y)$, i.e., $f * g$ sends cells to cells of the same dimension. Also $\overline{(e, \sigma)}$ and $\overline{f * g(e, \sigma)}$ contains the same number of vertices because each of *f* and *g* is a cellular folding.

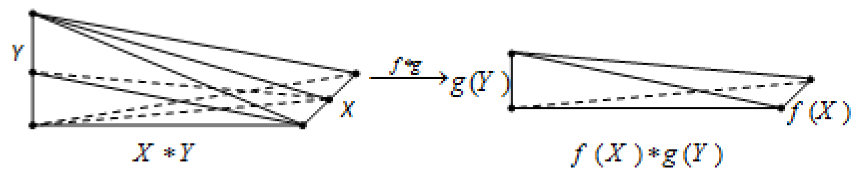
To prove the converse, suppose $f * g$ is a cellular folding, then $f * g$ maps cells of $X * Y$ to cells of the same dimension, so if (e, σ) is a *p*-cell in $X * Y$, then $(f * g)(e, \sigma) = (f(e), g(\sigma))$ is a *p*-cell in $f(X) * g(Y)$. Now let e be an *i*-cell in *X*, then σ is a $(p - i - 1)$-cell in *Y*. But any cellular map maps *i*-cells to *j*-cells where $j \le i$. If $i = j$, then nothing to prove, so let $i > j$. In this case g will maps a $(p - i - 1)$-cell to $(p - i - 1)$-cell and hence it is not a cellular folding, which is a contradiction and hence $i = j$ is the only possibility. The second condition of cellular folding is certainly satisfied in this case ,then *f*, *g* are cellular foldings.

5.3. Example

Let *X* and *Y* be complexes such that $|X| = |Y| = I$ with cellular divisions shown in Fig. (8), and $f: X \to X$, $g: Y \to Y$ be cellular foldings defined as follows: $f(e_1^0) = e_3^0$, $f(e_1^1) = e_2^1$ and $g(e_4^0) = e_6^0$, $g(e_3^1) = e_4^1$



(a)



(b)

Fig. (8)

Then the map $f * g : X * Y \to X * Y$ is a cellular folding, see Fig.(8-b).

6. Cellular folding of the wedge sum of two complexes

Given two complexes *X* and *Y* with chosen zero cells $u \in X$ and $v \in Y$, then the wedge sum $X \vee Y$ is the quotient of the disjoint union $X \bigcup Y$ obtained by identifying *u* and *v* to a single 0-cell, [3].We will call this 0-cell, the identifying 0-cell.

Note that for any cell complex *X*, the quotient X^n / X^{n-1} is a wedge sum of *n*-spheres $V_\alpha S_\alpha^n$, with one sphere for each n-cell of *X* .

6.1. Example

Let *X*, *Y* be two complexes such that $|X| = |Y| = S^1$. Then $X \vee Y = S^1 \vee S^1$ is the figure eight (8), see Fig. (9).

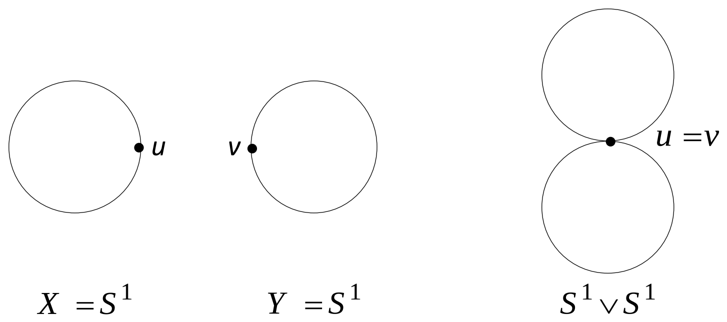


Fig. (9)

More generally one could form the wedge sum $V_\alpha X_\alpha$ of an arbitrary collection of spaces X_α by starting with the disjoint $\bigcup_\alpha X_\alpha$ and identifying points $x_\alpha \in X_\alpha$ to a single point. In case the spaces X_α are cell complexes and the points x_α are 0-cells, then $V_\alpha X_\alpha$ is a cell complex since it is obtained from the cell complex $\bigcup_\alpha X_\alpha$ by collapsing a subcomplex to a point.

6.2. Theorem

Let *X* and *Y* be complexes of the same dimension *n*, let $f : X \to X$ and $g : Y \to Y$ be cellular maps. Let $h = f \vee g : X \vee Y \to X \vee Y$ be defined as follows: for each *i*-cell *e*,

$$h(e) = \begin{cases} f(e), & e \in X \\ g(e), & e \in Y \end{cases},$$

$f(e^0) = g(e^0) = e^0$, e^0 is the identifying 0-cell. Then *h* is a cellular folding if and only if *f* and *g* are cellular foldings.

Proof

Suppose *f* and *g* are cellular foldings. Let *e* be an *i*-cell of $X \vee Y$ such that $\bar{e}$ has *r* distinct vertices, then we have:

(i) If $e \in X$, then $h(e) = f(e)$ is an *i*-cell in *Y*, $\overline{f(e)}$ has *r* distinct vertices, since *f* is a cellular folding.

(ii) If $e \in X$, then $h(e) = g(e)$ is an *i*-cell in *X*, $\overline{g(e)}$ has *r* distinct vertices, since *g* is a cellular folding. Thus $h = f \vee g$ is a cellular folding.

Conversely, let $h = f \vee g$ be a cellular folding, then $f \vee g$ maps *p*-cells to *p*-cells. Let *e* be an *i*-cell in *X* and *f* a cellular map, then it will maps *i*-cells to *j*-cells such that, $j \le i$. If $j = i$ nothing to prove, so let $j < i$. In this case $h = f \vee g$ will maps *i*-cells to *j*-cells and hence it is not a cellular folding. Which is a contradiction and hence $j = i$ is the only possibility. The second condition of cellular foldings is certainly satisfied in this case, then *f*, *g* are cellular foldings.

6.3. Examples

(1) Let *X* and *Y* be two complexes such that $|X| = |Y| = S^1$, and $f : X \to X$, $g : Y \to Y$ be cellular foldings defined as follows:

$f(e_i^0) = (e_i^0)$, $i = 1, 2$; $f(e_2^1) = (e_1^1)$, $g(e_5^0, e_6^0) = (e_3^0, e_4^0)$, $g(e_i^1) = e_3^1, i = 3,4,5$. See Fig. (10)

Then the map $f \vee g : X \vee Y \to X \vee Y$ is defined by:

$(f \vee g)(e_5^0, e_6^0) = (e_3^0, e_4^0)$, $(f \vee g)(e_2^1, e_4^1, e_5^1, e_6^1) = (e_1^1, e_3^1, e_3^1, e_3^1)$ is a cellular folding, see Fig. (10)

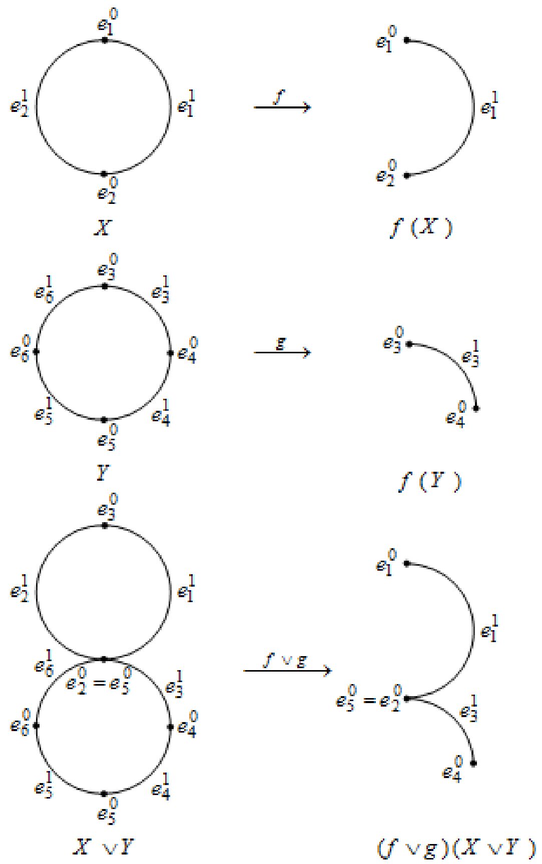
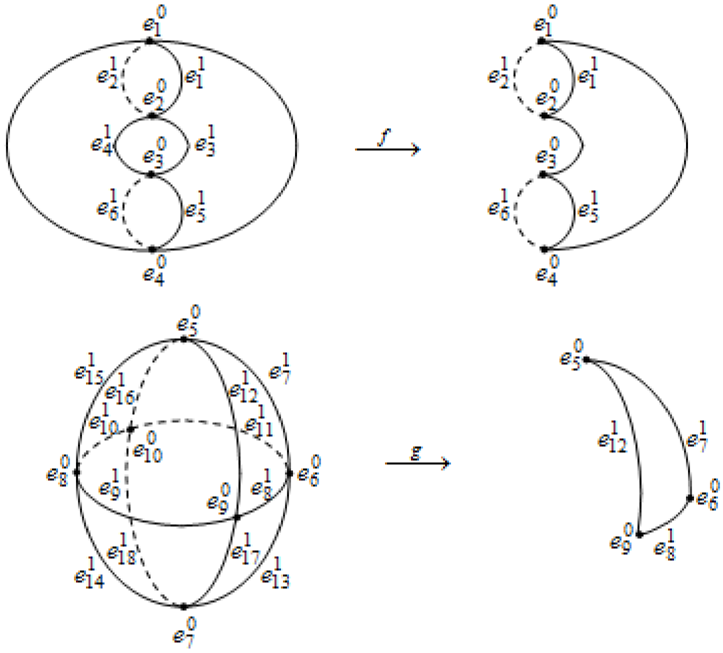


Fig. (10)

(2) Let X and Y be two complexes such that $|X| = T^2$, $|Y| = S^2$ with the cellular subdivision shown in Fig. (11-a). Let $f : X \to X$, $g : Y \to Y$ be cellular foldings defined as follows: f$(e_1^0, e_2^0, e_3^0, e_4^0)$=$(e_1^0, e_2^0, e_3^0, e_4^0)$,f$(e_1^1, e_2^1, e_3^1, e_4^1, e_5^1, e_6^1)$=$(e_1^1, e_2^1, e_3^1, e_3^1, e_5^1, e_6^1)$ g$(e_5^0, e_6^0, e_7^0, e_8^0, e_9^0, e_{10}^0)$=g$(e_5^0, e_6^0, e_5^0, e_9^0, e_9^0, e_6^0)$, g$(e_7^1, e_8^1, e_9^1 .... e_{18}^1)$=$(e_7^1, e_8^1, e_{12}^1)$



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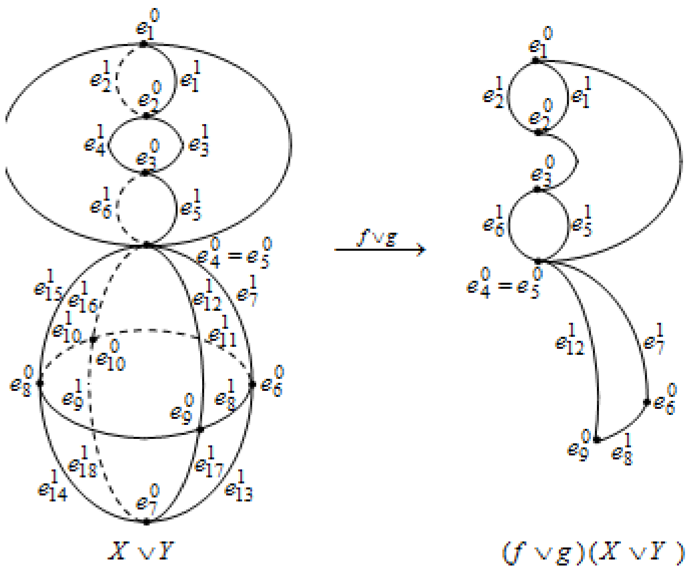

(b)

Fig. (11)

Then the map $f \vee g : : X \vee Y \to X \vee Y$ defined by $f \vee g\ (e_1^0, e_2^0, e_3^0)$=$(e_1^0, e_2^0, e_3^0)$, $e_4^0 = e_5^0$ $f \vee g(e_6^0,\ e_7^0, e_8^0, e_9^0, e_{10}^0)$ = $(e_6^0, e_9^0, e_6^0, e_9^0, e_6^0)$, $f \vee g(e_1^1, e_2^1, e_3^1, \dots e_{18}^1)$=$(e_1^1, e_2^1, e_3^1, e_3^1, e_5^1, e_6^1, e_7^1, e_8^1, e_{12}^1)$ $f \vee g\ (e_i^2) = e_1^2, i = 1,..8$ is a cellular folding, see Fig.(11-b).

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