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RESEARCH ARTICLE

OPTIMAL CONVEX COMBINATION BOUNDS OF ARITHMETIC AND CENTROIDAL MEANS FOR NEUMAN-SÁNDOR MEAN

HE HAIBIN, LIU CHUNRONG

College of Mathematics and Information Science, Hebei University,

Baoding, P. R. China

Corresponding Author: He Haibin, mchehb@126.com

He Haibin

ABSTRACT

In this paper, we present the least value α and the greatest value β such that the double inequality

$$\alpha A(a,b) + (1-\alpha)\overline{C}(a,b) < M(a,b) < \beta A(a,b) + (1-\beta)\overline{C}(a,b)$$

holds for all $a,b > 0$ with $a \neq b$. Here $A(a,b)$,$\overline{C}(a,b)$ and $M(a,b)$ denote the arithmetic, centroidal and Neuman-Sándor means of two positive numbers a and b, respectively.

Keywords: Inequality, Neuman-Sándor mean, Harmonic mean, Arithmetic mean, Centroidal mean

1. INTRODUCTION

For $a,b > 0$ with $a \neq b$, the Neuman-Sándor mean $M(a,b)$ [1] was defined by

$$M(a,b) = \frac{a-b}{2\sinh^{-1}[(a-b)/(a+b)]}, \tag{1.1}$$

where $\sinh^{-1} x = \log(x + \sqrt{1+x^2})$ is the inverse hyperbolic sine function.

Recently, the Neuman-Sándor mean has been the subject of intensive research. In particular, many remarkable inequalities for $M(a,b)$ can be found in the literature [1, 2]. Let $H(a,b) = (2ab)/(a+b)$, $G(a,b) = \sqrt{ab}$,

$L(a,b) = (a-b)/(\log a - \log b)$, $N(a,b) = \left[(\sqrt{a}+\sqrt{b})/2\right]^2$, $P(a,b) = (a-b)/(4\tan^{-1}\sqrt{a/b} - \pi)$,

$A(a,b) = (a+b)/2$,

$T(a,b) = (a-b)/[2\tan^{-1}(a-b)/(a+b)]$, $\overline{C}(a,b) = 2/3\cdot(a^2+ab+b^2)/(a+b)$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$ and $C(a,b) = (a^2+b^2)/(a+b)$ be the harmonic, geometric, logarithmic, square-root, first Seiffert, arithmetic, second Seiffert, centroidal, quadratic and contra-harmonic harmonic, geometric, logarithmic,, first Seiffert, arithmetic, second Seiffert, quadratic and contra-harmonic mean of a and b, respectively. Then

$$\min\{a,b\} < H(a,b) < G(a,b) < L(a,b) < N(a,b) < P(a,b) < A(a,b)$$
$$< M(a,b) < T(a,b) < \overline{C}(a,b) < Q(a,b) < C(a,b) < \max(a,b) \quad (1.2)$$

hold for all $a,b>0$ with $a \neq b$.

In [3], Neuman proved that the double inequalities

$$\alpha Q(a,b) + (1-\alpha)A(a,b) < M(a,b) < \beta Q(a,b) + (1-\beta)A(a,b) \quad (1.3)$$

and

$$\lambda C(a,b) + (1-\lambda)A(a,b) < M(a,b) < \mu C(a,b) + (1-\mu)A(a,b) \quad (1.4)$$

hold for all $a,b>0$ with $a \neq b$ if and only if $\alpha \le [1-\log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})] = 0.3249\cdots$, $\beta \ge 1/3$,

$\lambda \le [1-\log(1+\sqrt{2})]/\log(1+\sqrt{2})$ and $\mu \ge 1/6$.

In [4], Li etc showed that the double inequality

$$L_{p_0}(a,b) < M(a,b) < L_2(a,b) \quad (1.5)$$

holds for all $a,b>0$ with $a \neq b$, where $L_p(a,b) = [(a^{p+1}-b^{p+1})/(p+1)(a-b)]^{1/p}$ $(p \neq -1,0)$, $L_0(a,b) = 1/e[(a^a)/b^b]^{1/(a-b)}$ and $L_{-1}(a,b) = (a-b)/(\log a - \log b)$ is the p-th generalized logarithmic mean of a and b, and $p_0 = 1.843\cdots$ is the unique solution of the equation $(p+1)^{1/p} = \log(1+\sqrt{2})$.

In [5], Chu etc proved that the double inequalities

$$\alpha_1 L(a,b) + (1-\alpha_1)Q(a,b) < M(a,b) < \beta_1 L(a,b) + (1-\beta_1)Q(a,b) \quad (1.6)$$

and

$$\alpha_2 L(a,b) + (1-\alpha_2)C(a,b) < M(a,b) < \beta_2 L(a,b) + (1-\beta_2)C(a,b) \quad (1.7)$$

hold for all $a,b>0$ with $a \neq b$ if and only if $\alpha_1 \ge 2/5$, $\beta_1 \le 1-1/[\sqrt{2}\log(1+\sqrt{2})] = 0.1977\cdots$, $\alpha_2 \ge 5/8$ and $\beta_2 \le 1-1/[2\log(1+\sqrt{2})] = 0.4327\cdots$.

The main purpose of this paper is to found the least value α and the greatest value β such that the double inequality

$$\alpha A(a,b) + (1-\alpha)\overline{C}(a,b) < M(a,b) < \beta A(a,b) + (1-\beta)\overline{C}(a,b)$$

holds for all $a,b>0$ with $a \neq b$.

2. MAIN RESULTS

THEOREM 2.1. The double inequality

$$\alpha A(a,b) + (1-\alpha)\overline{C}(a,b) < M(a,b) < \beta A(a,b) + (1-\beta)\overline{C}(a,b)$$

holds for all $a,b>0$ with $a \neq b$ if and only if $\alpha \ge 4-3/\log(1+\sqrt{2}) = 0.5962\cdots$ and $\beta \le 1/2$.

Proof. Without loss of generality, we assume that $a>b>0$.

Let $x = (a-b)/(a+b) \in (0,1)$, $\lambda = 4-3/\log(1+\sqrt{2}) = 0.5962\cdots$ and $\theta \in \{1/2, \lambda\}$. Then

$$\frac{M(a,b)}{A(a,b)} = \frac{x}{\sinh^{-1}x}, \frac{\overline{C}(a,b)}{A(a,b)} = 1+\frac{x^2}{3}. \quad (2.1)$$

Firstly, we prove that

$$M(a,b) < \frac{1}{2}[A(a,b) + \overline{C}(a,b)], \quad (2.2)$$

and

$$\lambda A(a,b) + (1-\lambda)\overline{C}(a,b) < M(a,b), \quad (2.3)$$

From (2.1) we have

$$\frac{qA(a,b)+(1-q)\overline{C}(a,b)-M(a,b)}{A(a,b)}=\frac{3+(1-q)x^2}{3\log(x+\sqrt{1+x^2})}E_q(x), \tag{2.4}$$

where

$$E_q(x)=\log(x+\sqrt{1+x^2})-\frac{3x}{3+(1-q)x^2}. \tag{2.5}$$

(2.5) lead to

$$\lim_{x\to 0^+}E_q(x)=0, \tag{2.6}$$

$$\lim_{x\to 1^-}E_q(x)=\log(1+\sqrt{2})-\frac{3}{4-q}. \tag{2.7}$$

and

$$E_q'(x)=\frac{1}{\left[3+(1-q)x^2\right]^2}J_q(x), \tag{2.8}$$

where

$$J_q(x)=\frac{(1-q)^2x^4+6(1-q)x^2+9}{\sqrt{1+x^2}}+3(1-q)x^2-9. \tag{2.9}$$

Let $x=\sqrt{t}$, $t\in(0,1)$, then

$$J_q(x)=\frac{(1-q)^2t^2+6(1-q)t+9}{\sqrt{1+t}}+3(1-q)t-9=K_q(t). \tag{2.10}$$

Now we distinguish between two cases:

Case 1. $q=1/2$. (2.10) leads to

$$K_{1/2}(t)=\frac{(t+6)^2}{4\sqrt{1+t}}-\left(9-\frac{3}{2}t\right)=\frac{\left[\frac{(t+6)^2}{4\sqrt{1+t}}\right]^2-\left(9-\frac{3}{2}t\right)^2}{\frac{(t+6)^2}{4\sqrt{1+t}}+\left(9-\frac{3}{2}t\right)}=\frac{t^2[(t-6)^2+576]}{16(1+t)\left[\frac{(t+6)^2}{4\sqrt{1+t}}+\left(9-\frac{3}{2}t\right)\right]}>0, \tag{2.11}$$

for $t\in(0,1)$. From (2.8), (2.10) and (2.11) we clearly see that $E_{1/2}(x)$ is strictly increasing in $(0,1)$. Therefore the inequality (2.2) follows from (2.4) and (2.6) together with the monotonicity of $E_{1/2}(x)$.

Case 2. $q=\lambda$. (2.10) leads to

$$K_\lambda(t)=\frac{(1-\lambda)^2t^2+6(1-\lambda)t+9}{\sqrt{1+t}}+3(1-\lambda)t-9, \tag{2.12}$$

Simple computations yield

$$\lim_{t\to 0^+}K_\lambda(t)=0. \tag{2.13}$$

$$\lim_{t\to\frac{1}{4}}K_\lambda(t)=\frac{\sqrt{5}}{40}(13-\lambda)^2-\frac{3}{4}(11+\lambda)<\frac{\sqrt{5}}{40}\left(13-\frac{14}{25}\right)^2-\frac{3}{4}\left(11+\frac{14}{25}\right)=\frac{96721\sqrt{5}-216750}{25000}<0, \tag{2.14}$$

$$\lim_{t\to 1^-}K_\lambda(t)=\frac{\sqrt{2}}{2}(4-\lambda)^2-3(2+\lambda)>\frac{\sqrt{2}}{2}\left(4-\frac{3}{5}\right)^2-3\left(2+\frac{3}{5}\right)=\frac{289\sqrt{2}-390}{50}>0,. \tag{2.15}$$

$$K_\lambda'(t)=\frac{3(1-\lambda)t^2+2(2\lambda^2-7\lambda+5)t+3(1-4\lambda)}{2(1+t)^{3/2}}+3(1-\lambda), \tag{2.16}$$

and

$$K_\lambda''(t)=\frac{3(1-\lambda)t^2+2(4\lambda^2-5\lambda+1)t+8\lambda^2+8\lambda+11}{2(1+t)^{5/2}}>0, \tag{2.17}$$

for $t \in (0,1)$. From (2.17) we clearly see that $K_\lambda(t)$ is a strictly convex function in $(0,1)$. It follows from (2.13)- (2.15) and convexity of $K_\lambda(t)$ that there exists $t_0 \in (0,1)$ such that $K_\lambda(t) < 0$ for $t \in (0,t_0)$ and $K_\lambda(t) > 0$ for $t \in (t_0,1)$, this fact together with (2.8) and (2.10) result in the conclusion that $E_\lambda'(x) < 0$ for $x \in (0,x_0)$ and $E_\lambda'(x) > 0$ for $x \in (x_0,1)$, where $x_0 = \sqrt{t_0}$, hence $E_\lambda(x)$ is strictly decreasing in $(0,x_0)$ and strictly increasing in $(x_0,1)$.

Notice that (2.7) becomes

$$\lim_{x\to 1^-} E_\lambda(x) = 0. \tag{2.18}$$

Therefore the inequality (2.3) follows from (2.4), (2.6) and (2.18) together with the monotonicity of $E_\lambda(x)$.

Finally we prove that $\lambda A(a,b) + (1-\lambda)\overline{C}(a,b)$ is the best possible lower convex combination bound and $\frac{1}{2}[A(a,b) + \overline{C}(a,b)]$ is the best possible upper convex combination bound of the arithmetic and centroidal means for the Nueman-Sándor mean.

(2.1) leads to

$$\frac{\overline{C}(a,b) - M(a,b)}{\overline{C}(a,b) - A(a,b)} = \frac{(1+\frac{x^2}{3}) - \frac{x}{\sinh^{-1} x}}{(1+\frac{x^2}{3}) - 1} = B(x), \tag{2.19}$$

From (2.19) one has

$$\lim_{x\to 1^-} B(x) = \lambda, \tag{2.20}$$

and

$$\lim_{x\to 0^+} B(x) = \frac{1}{2}. \tag{2.21}$$

If $\alpha < \lambda$, then (2.19) and (2.20) lead to the conclusion that there exists $0 < \delta_1 < 1$ such that $\alpha A(a,b) + (1-\alpha)\overline{C}(a,b) > M(a,b)$ for all $a,b > 0$ with $(a-b)/(a+b) \in (1-\delta_1,1)$.

If $\beta > 1/2$, then (2.19) and (2.21) lead to the conclusion that there exists $0 < \delta_2 < 1$ such that $M(a,b) > \beta A(a,b) + (1-\beta)\overline{C}(a,b)$ for all $a,b > 0$ with $(a-b)/(a+b) \in (0,\delta_2)$.

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