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RESEARCH ARTICLE

FURTHER RESULTS ON TOTAL DOMINATING COLOR TRANSVERSAL NUMBER OF GRAPHS

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ABSTRACT

In this contribution, we investigate some relations between Total Dominating Color Transversal number, Chromatic number and Total Domination number of a graph. To be precise, we determine the conditions under which Υ_{tstd} of a graph becomes equal or unequal to Υ_t or χ of the graph (Where Υ_{tstd}, Υ_t and χ are, respectively, the Total Domination Color Transversal number, Total Dominating number and Chromatic number of the graph). We show that when $\Upsilon_t < \chi$ or $\chi < \Upsilon_t$ does not imply $\Upsilon_{tstd} = \chi$ or Υ_t. We also give different examples to justify our theorems and statements. Also we determine the upper bound of Υ_{tstd} in terms of χ and Υ_t and prove related results. Additionally, we determine an upper bound of Υ_{tstd} for perfect graphs.

Keywords: Total Dominating Color Transversal number, χ – Partition of a graph and Transversal.

AMS Subject classification code (2010): 05C15, 05C69

1. INTRODUCTION

In [1], Manoharan introduced the concept of Dominating Color Transversal Set of a graph. We know that proper coloring of vertices of graph G partitions the vertex set V of G into equivalence classes (also called the color classes of G). Using minimum number of colors to properly color all the vertices of G yields χ equivalence classes. Transversal of a χ - Partition of G is a collection of vertices of G that meets all the color classes of the χ – Partition. That is, if T is a subset of V (the vertex set of G and $\{V_1, V_2, ..., V_\chi\}$ is a χ - Partition of G then T is called a Transversal of this χ - Partition if $T \cap V_i \neq \emptyset$, $\forall\ i \in \{1, 2, ..., \chi\}$. Dominating Color Transversal Set of graph G is a Dominating Set with the extra property that it is also Transversal of some such χ - Partition of G. Dominating Color Transversal Set of G with minimum cardinality is called Minimum Dominating Color Transversal Set of G and its cardinality, denoted by γ_{st}(G) or just by γ_{st}, is called the Dominating Color Transversal number of G. In [1], Manoharan proved several results regarding this number. Motivated by this concept, in [2], we analogously defined the concept of Totalal Dominating Color Transversal Set and Total Dominating Color Transversal number of a graph and proved several results regarding this number.

From the definition it is clear that Total Dominating Color Transversal number must have some relations with Total Domination number and Chromatic number of the graph. We investigate the same in this contribution.

Throughtout this paper we assume that graphs are simple, finite, connected and undirected without isolated vertices.

First let us go through some definitions.

2. Definitions

Definition 2.1[4]: (Total Dominating Set) Let G = (V, E) be a graph. Then a subset S of V (the vertex set of G) is said to be a Total Dominating Set of G if for each v ∈ V, v is adjacent to some vertex in S.

Definition 2.2[4]: (Minimum Total Dominating Set/Total Domination number) Let G = (V, E) be a graph. Then a Total Dominating Set S is said to be a minimum Total Dominating Set of G if $|S|$ = minimum {$|D|$: D is a Total Dominating Set of G}. Here S is called γ_t —set and its cardinality, denoted by γ_t(G) or just by γ_t, is called the Total Domination number of G.

Definition 2.3[2]: (χ -partition of a graph) Proper coloring of vertices of a graph G, by using minimum number of colors, yields minimum number of independent subsets of vertex set of G called equivalence classes (also called color classes of G). Such a partition of a vertex set of G is called a χ - Partition of the graph G.

Definition 2.4[2]: (Transversal of a χ - Partition of a graph) Let G = (V, E) be a graph with χ – Partition {V_1, V_2, ..., V_χ}. Then a set S ⊂ V is called a Transversal of this χ – Partition if $S \cap V_i \neq \emptyset$, ∀ i ∈ {1, 2, 3, ..., χ}.

Definition 2.5[2]: (Total Dominating Color Transversal Set) Let G = (V, E) be a graph. Then a Total Dominating Set S ⊂ V is called a Total Dominating Color Transversal Set of G if it is Transversal of at least one χ - Partition of G.

Definition 2.6[2]: (Minimum Total Dominating Color Transversal Set/Total Dominating Color Transversal number) Let G = (V, E) be a graph. Then S ⊂ V is called a Minimum Total Dominating Color Transversal Set of G if $|S|$ = minimum {$|D|$: D is a Total Dominating Color Transversal Set of G}. Here S is called γ_{tstd} —Set and its cardinality, denoted by γ_{tstd}(G) or just by γ_{tstd}, is called the Total Dominating Color Transversal number of G.

Definition 2.7: (Clique) Let G = (V, E) be a graph. A clique is a subset of vertices, S ⊂ V, such that every two distinct vertices are adjacent. This is equivalent to a condition that the sub graph induced by S, denoted by <S>, is complete.

Definition 2.8: (Maximum Clique/ Clique number) Let G = (V, E) be a graph. A clique S is called Maximum clique of G if its cardinality is maximum among all cliques that the graph G contains. Here cardinality of S is called Clique number of G and it is denoted by ω (G) or just by ω.

Definition 2.9: (Perfect Graph) Let G = (V, E) be a graph. Then G is said to be perfect if for every subset S of V, $\chi(<S>) = \omega(<S>)$.

Definition 2.10: (Pendant Vertex) Let G = (V, E) be a graph. Then a vertex v of G is called pendant if its degree is one.

Definition 2.11: (Support Vertex) Let G = (V, E) be a graph. Then a vertex v of G is called support if it is adjacent to a pendant vertex.

Definition 2.12: (Isolated Vertex) Let G = (V, E) be a graph. Then a vertex v of G is isolated vertex if its degree is 0.

Definition 2.13: (Isolate vertex with respect to a set) Let G = (V, E) be a graph and S be a subset of V. Then a vertex v in S is called isolate with respect to S if v is not adjacent to any vertex in S.

3. Main results

Remark 3.1: For any graph G, χ (G) $\leq \gamma_{tstd}$(G) and γ_t(G) $\leq \gamma_{tstd}$(G).

Theorem 3.2 [2]: If χ (G) = 2 then γ_{tstd}(G) = γ_t(G).

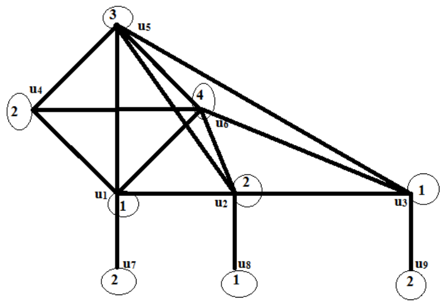
Theorem 3.3 [2]: If γ_t(G) = 2 then γ_{tstd}(G) = χ (G).

First let us note down one basic theorem which will prove very important elementary result later on.

Result 3.4: If a graph G has k distinct support vertices then Υ_{tstd} (G) $\geq$ k.

Proof: As any Total Dominating Set must contain all the support vertices of the graph G, $\Upsilon_t \geq$ k and hence Υ_{tstd} (G) $\geq$ k.

Example 3.5: For the below given graph G, Υ_t (G) < χ (G) but Υ_{tstd} (G) $\neq \chi$ (G).

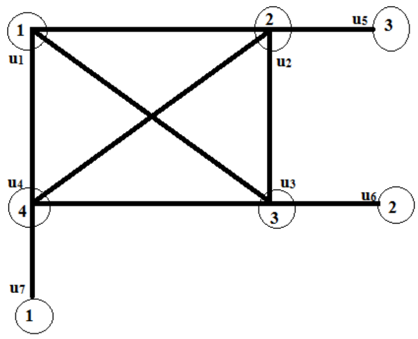


G

Fig.1

Υ_t (G) = 3 and χ (G) = 4 but Υ_{tstd} (G) = 5 $\neq$ 4 = χ (G).

Example 3.6: For the below given graph G, Υ_t (G) < χ (G) and Υ_{tstd} (G) = χ (G).



G

Fig. 2

Υ_t (G) = 3, χ (G) = 4 and Υ_{tstd} (G) = 4 = χ (G).

Theorem 3.7: Let G be a graph. Then Υ_{tstd} (G) = χ (G) > Υ_t (G) if and only if following two conditions hold:

(1) There exists a Total Dominating Set D of G and a χ – Partition {V_1, V_2,...., V_χ} of G such that $|D \cap V_i|$ = 1, $\forall$ i = 1, 2, ... , χ.

(2) No Υ_t - Set of G is a transversal of any χ – Partition of G.

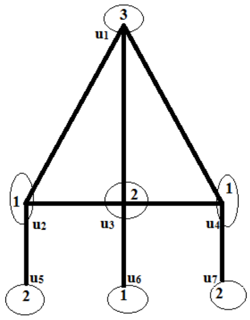
Proof: We know that Υ_{tstd} (G) $\geq \chi$ (G).

Suppose (1) is not true. Then Υ_{tstd} (G) $> \chi$ (G) , which is contradiction to Υ_{tstd} (G) = χ (G). Hence (1) is true.

Suppose (2) is not true. Then there exists a Υ_t - Set of G which is a transversal of some χ – Partition of G. Therefore Υ_{tstd} (G) = Υ_t (G) which is contradiction to Υ_{tstd} (G) > Υ_t (G) . Hence (2) is true.

Conversely assume (1) and (2). (1) implies that Υ_{tstd} (G) $\leq \chi$ (G) which implies that Υ_{tstd} (G) = χ (G) and (2) implies that Υ_{tstd} (G) > Υ_t (G). So χ (G) = Υ_{tstd} (G) > Υ_t (G).

Example 3.8: For the below given graph G, Υ_t (G) = χ (G) = k but Υ_{tstd} (G) $\neq$ k.

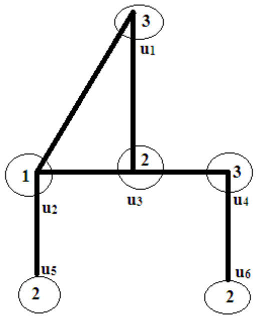


G

Fig.3

Υ_t(G) = 3, χ (G) = 3 but Υ_{tstd} (G) = 4 $\neq$ 3 = χ (G) = Υ_t(G).

Example 3.9: For the below given graph G, Υ_{tstd} (G) = χ (G) = Υ_t(G).



G

Fig.4

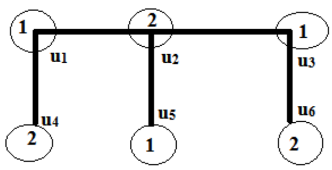
Υ_{tstd} (G) = χ (G) = Υ_t(G) = 3.

Theorem 3.10: Let G be a graph. Then Υ_{tstd} (G) = χ (G) = Υ_t (G) if and only if there exists a Minimum Total Dominating Set D of G and a χ – Partition {V_1, V_2,...., V_χ} of G such that $|D \cap V_i|$ = 1, $\forall$ i = 1, 2, ... , χ.

Proof: Obvious.

Remark 3.11: Let G be a graph. If there exists a Υ_t - Set of G such that it is a transversal of some χ – Partition of G then it is not necessary that Υ_{tstd} (G) = χ (G) = Υ_t (G). We give the following example to justify this.

Example 3.12:

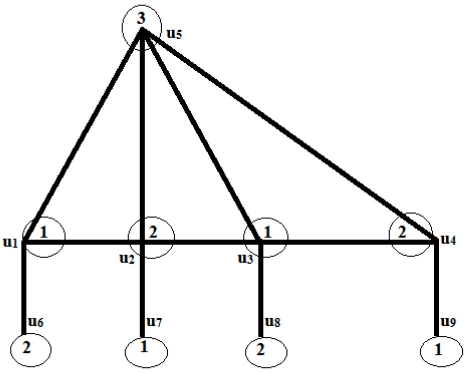


G

Fig. 5

Here $\{u_1, u_2, u_3\}$ is a Υ_t - set of G. Clearly it is a transversal of the defined χ - Partition of G. AlsoΥ_{tstd} (G) = 3 = Υ_t (G) $\neq \chi$ (G) = 2.

Example 3.13: For the below given graph G, χ (G) < Υ_t (G) but Υ_{tstd} (G) $\neq \Upsilon_t$ (G).

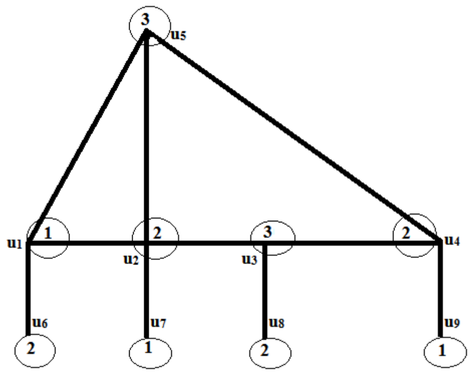


G

Fig. 6

χ (G) = 3, Υ_t (G) = 4 but Υ_{tstd} (G) = 5 $\neq \Upsilon_t$ (G).

Example 3.14: For the below given graph G, χ (G) < Υ_t (G) and Υ_{tstd} (G) = Υ_t (G).



G

Fig. 7

χ (G) = 3, Υ_t (G) = 4 and Υ_{tstd} (G) = 4 = Υ_t (G).

Theorem 3.15: Let G be a graph. Then Υ_{tstd} (G) = Υ_t (G) > χ (G) if and only if following two conditions holds:

(1) There exists a Υ_t - Set of G such that it is a transversal of some χ – Partition {V_1, V_2,...., V_χ} of G.

(2) For every Υ_t – Set D of G satisfying (1) , $|D \cap V_i|$ > 1 for some i $\in$ {1, 2, 3, ..., χ}.

Proof: Obvious.

Now we discuss upper bound of Υ_{tstd} in terms of Υ_t and χ. We provide different examples to justify our results.

Theorem 3.16: For every graph G, Υ_{tstd}(G) $\leq$ Υ_t (G) + $\chi(G)$ -2.

Proof: Let S be a Υ_t - set of G. Then S has at least two vertices that are adjacent and so they are in different color classes for every χ – Partition of G. Adding, to S , one vertex from each remaining χ - 2 color classes yields a Total Dominating Color Transversal Set of G. Hence Υ_{tstd}(G) $\leq$ Υ_t (G) + $\chi(G)$ - 2.

Remark 3.17: Above given upper bound is sharp. Following example 3.18 justifies this.

Example 3.18: Let G be a graph. If $\chi(G)$ = 2 or Υ_t (G) = 2 then Υ_{tstd}(G) = Υ_t (G) + $\chi(G)$ -2.

Proof: If $\chi(G) = 2$ thenΥ_{tstd}(G) = Υ_t (G) by theorem 3.2. So Υ_{tstd}(G) = Υ_t (G) + 2 – 2 = Υ_t (G) + $\chi(G)$ -2. If Υ_t (G) = 2 then Υ_{tstd}(G) = $\chi(G)$ by theorem 3.3. SoΥ_{tstd}(G) = $\chi(G)$+ 2 – 2 = Υ_t (G) + $\chi(G)$ -2.

Remark 3.19: There are graphs G for which $\chi(G)$ > 2 and Υ_t (G) > 2 but still Υ_{tstd}(G) = Υ_t (G) + $\chi(G)$ - 2. We give the following example 3.20 to justify this.

Example 3.20: Consider graph G as in Fig. 3. Υ_{tstd}(G) = 4, Υ_t (G) = 3 and $\chi(G) = 3$. Trivially Υ_{tstd}(G) = 4 = 3 + 3 - 2 = Υ_t (G) + $\chi(G)$ - 2.

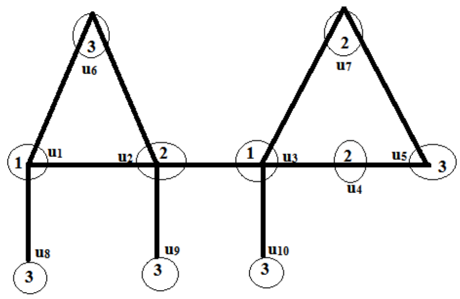
Theorem 3.21: Let G be a graph. Then Υ_{tstd}(G) = Υ_t (G) + $\chi(G)$ -2 if and only if every Υ_t - Set D of G is contained in the union of (some) two color classes of every χ – Partition of G.

Proof: Assume Υ_{tstd}(G) = Υ_t (G) + $\chi(G)$ -2. Suppose there exists a Υ_t - Set D of G that is not contained in the union of any two color classes of some χ – Partition {V_1, V_2, ..., V_χ} of G. Then D meets at least three color classes of {V_1, V_2, ..., V_χ}. Then adding, to D, one vertex from each remaining at most χ - 3 color classes, yields a Total Dominating Color Transversal set of G. Hence Υ_{tstd}(G) ≤ |D| + χ(G) – 3 < Υ_t (G) + $\chi(G)$ -2. So we get contradiction. Hence every Υ_t - Set D of G is contained in the union of (some) two color classes of every χ – Partition of G. Conversely assume that every Υ_t - Set D of G is contained in the union of (some) two color classes of every χ – Partition of G. Then adding, to D, one vertex from each remaining χ - 2 color classes yields Minimum Total Dominating Color Transversal Set of G. Therefore Υ_{tstd}(G) = Υ_t (G) + $\chi(G)$ -2.

Corollary 3.22: Let G be a graph. If there exists Υ_t - Set of G that meets at least k color classes of some χ – Partition of G then Υ_{tstd}(G) ≤ Υ_t (G) + $\chi(G)$ - k.

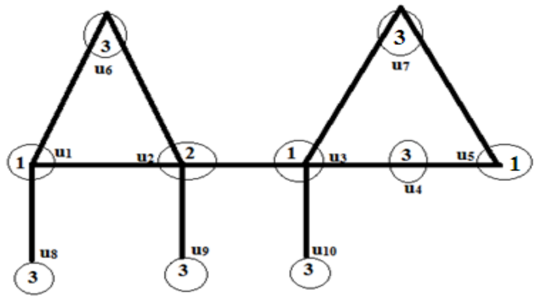
Remark 3.23: Let G be a graph. Every Υ_t - Set D of G is contained in the union of (some) two color classes of some χ – Partition of G does not imply Υ_{tstd}(G) = Υ_t (G) + $\chi(G)$ -2. Following example 3.24 justifies this.

Example 3.24:



G Fig.8

Υ_t (G) = 4 and $\chi(G)$ = 3. Graph G has exactly two Υ_t – Sets. They are D_1 = {u_1, u_2, u_3, u_4} and D_2 = {u_1, u_2, u_3, u_7}. Let {V_1, V_2, V_3} be a χ - Partition of G, as shown above in Fig. 8, with V_1 = {u_1, u_3}, V_2 = {u_2, u_4, u_7} and V_3 = {u_5, u_6, u_8, u_9, u_{10}}. Clearly $D_i \subset V_1 \cup V_2$. (i = 1, 2). But Υ_{tstd} (G) = 4 ≠ 5 = Υ_t (G) + χ(G) – 2 = 4 + 3 – 2, under the following χ - Partition of G.



G

Fig. 9

Theorem 3.25: Let G be a graph. If Υ_{tstd}(G) = Υ_t (G) + $\chi(G)$ -2 then every Υ_t - Set of G is contained in a Υ_{tstd} - Set of G.

Proof: Consider a Υ_t – Set D of G. We know by theorem 3.21 that every Υ_t – Set is contained in (some) two color classes of every χ - Partition of G. So by adding, to D, one vertex from each remaining χ -2 color classes yields a Total Dominating Color Transversal Set of G. Note that this set is minimum with this property. So every Υ_t - Set of G is contained in a Υ_{tstd} - Set of G.

Remark 3.26: Converse of above theorem 3.25 is not true in general. Below given example 2.37 justifies this.

Example 3.27: Consider the graph G given in Fig. 9. Graph G has exactly two Υ_t – Sets. They are D_1 = {u_1, u_2, u_3, u_4} and D_2 = {u_1, u_2, u_3, u_7}. Note that both are Υ_{tstd} - Sets of G. So we can say that both D_1 and D_2 are contained a Υ_{tstd} – Set of G. But $\Upsilon_{tstd}(G) = 4 \neq 5 = 4 + 3 - 2 = \Upsilon_t(G) + \chi(G) - 2$.

Now we obtain an upper bound of Υ_{tstd} number for Perfect graphs. we assume that complement graph $\overline{G}$ have no isolated vertex.

Theorem 3.28 [8]: A graph G is Perfect if and only if $\overline{G}$ is Perfect.

Theorem 3.29: If G is a perfect graph then $\Upsilon_{tstd}(G) \leq \Upsilon_t(G) + \omega(G) - 2$.

Proof: Obvious as for a Perfect G, $\chi(G) = \omega(G)$ and by theorem 3.16.

Theorem 3.30: If G is a perfect graph then $\Upsilon_{tstd}(\overline{G}) \leq 2\,\omega(G) + \omega(\overline{G}) - 2$.

Proof: It is obvious to say that the theorem 3.16 is also true for a disconnected graph G.

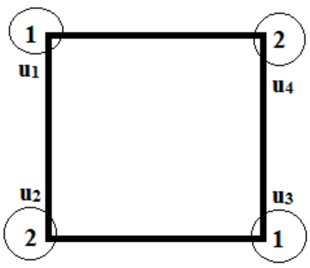
By theorem 3.16, $\Upsilon_{tstd}(\overline{G}) \leq \Upsilon_t(\overline{G}) + \chi(\overline{G}) - 2$. By theorem 3.28, $\chi(\overline{G}) = \omega(\overline{G})$. And hence $\Upsilon_{tstd}(\overline{G}) \leq \Upsilon_t(\overline{G}) + \omega(\overline{G}) - 2$.

Claim: $\Upsilon_t(\overline{G}) \leq 2\,\omega(G)$

Suppose $S \subset V$ is a maximum clique of G. Hence $|S| = \omega(G)$. Also S is a maximum independent set in $\overline{G}$. Therefore S is a dominating set in $\overline{G}$. If S is not a Total Dominating Set in $\overline{G}$ then S has at most $|S|$ isolates. As $\overline{G}$ is a graph with no isolated vertex, each vertex of S has adjacent vertex in $\overline{G}$. So adding at most $|S|$ vertices the resultant set becomes a Total Dominating set of $\overline{G}$. So $\Upsilon_t(\overline{G}) \leq 2|S| = 2\,\omega(G)$. Hence $\Upsilon_{tstd}(\overline{G}) \leq 2\,\omega(G) + \omega(\overline{G}) - 2$.

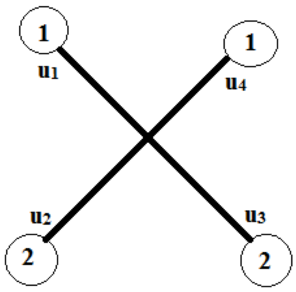
Remark 3.31: Bound of $\Upsilon_{tstd}(\overline{G})$ in theorem 3.30 is sharp. Following example 3.32 justifies this.

Example 3.32: We know that every bipartite graph is Perfect.



C_4

Fig. 10



$\overline{C_4}$

Fig. 11

$\Upsilon_{tstd}(\overline{C_4}) = 4$, $\omega(C_4) = 2$ and $\omega(\overline{C_4}) = 2$. So $\Upsilon_{tstd}(\overline{C_4}) = 4 = 2\omega(C_4) + \omega(\overline{C_4}) - 2$.

Theorem 3.33: If G is a Perfect graph then $\Upsilon_{tstd} = k - 2$, where k = min {$\Upsilon_t(G) + \omega(G)$, $2\omega(\overline{G}) + \omega(G)$}.

Proof: By theorem 3.16, $\Upsilon_{tstd}(G) \leq \Upsilon_t(G) + \chi(G) - 2 = \Upsilon_t(G) + \omega(G) - 2$ (1)

By theorem 3.30, $\Upsilon_{tstd}(\overline{G}) \leq 2\,\omega(G) + \omega(\overline{G}) - 2$ (2)

As G is perfect, $\overline{G}$ is also Perfect.(theorem 3.28). Hence applying inequality (2) to $\overline{G}$, we obtain the following inequality: $\Upsilon_{tstd}(G) = \Upsilon_{tstd}(\overline{\overline{G}}) \leq 2\,\omega(\overline{G}) + \omega(G) - 2$ (3)

From (1) and (3), $\Upsilon_{tstd} = k - 2$, where k = min {$\Upsilon_t(G) + \omega(G)$, $2\omega(\overline{G}) + \omega(G)$}.

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