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RESEARCH ARTICLE

ON d', D, G-METRIC SPACES

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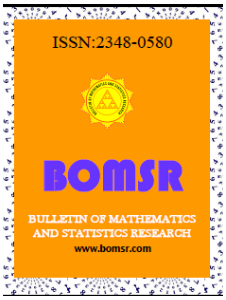

ABSTRACT
In this work, we point out the relationships between metric and generalized metric spaces d', D, G defined in literature illustrating with examples.

Keywords: d'-metric, D-metric, G-metric.

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1. INTRODUCTION

The notion of metric spaces is used as a basic concept in all other scientific fields as in Mathematics. During last 50 years, a generalization of metric spaces and to give a proper definition have become one of the major problem of topology. For this purpose, we give notions of d'-metric, D-metric and G-metric spaces as a generalization of metric spaces, and investigate some relationships between them. Using these relations, one can easily generalize some known results in metric spaces to generalized metric spaces.

Let us remind that a metric space is a set X together with a function d (called a metric or distance function) which assigns a positive real number d(x,y) to every pair $x,y \in X$ satisfying the axioms:

(M1) $d(x,y) \geq 0$,
(M2) $d(x,y) = 0$ iff $x = y$,
(M3) $d(x,y) = d(y,x)$,
(M4) $d(x,y) \leq d(x,z) + d(z,y)$.

As a basic example, take X to be any set. The discrete metric on the X is given by: $d(x,y) = 0$ if $x = y$ and $d(x,y) = 1$ otherwise. Then, this does define a metric, in which no distinct pair of points are "close". The fact that every pair is spread out is why this metric is called discrete.

2. GENERALIZED METRIC SPACES

In 1963, Gahler [1] defined d'-metric space as a generalization of metric space and gave basic properties of it [2,3] as follows:

Definition 2.1. A pair (X,d') is called a generalized metric space or shortly d'-metric space if X is an arbitrary set and $d': X\times X\times X \to [0,\infty)$ a function such that

($d'1$) For every pair $x,y\in X$, there exists a $z\in X$ such that $d'(x,y,z) \neq 0$,

($d'2$) Two of x,y and z are same then $d'(x,y,z) = 0$,

($d'3$) $d'(x,y,z) = d'(p(x,y,z))$ where p is a permutation function,

($d'4$) $d'(x,y,z) \leq d'(x,y,a) + d'(x,a,z) + d'(a,y,z)$.

d' is also called generalized metric on X.

Example 2.1. Define d':IR×IR×IR $\to [0,\infty)$ by $d'(x,y,z) = \begin{cases} 1, & x\neq y, y\neq z, x\neq z \\ 0, & other. \end{cases}$ Then d' is a generalized metric on IR.

d'1-2-3) are obvious. d'4) For every $x,y,z,a\in$IR, if $x\neq y$, $y\neq z$, $z\neq a$ then it holds since,

$d'(x,y,z) = 1 < 3 = d'(x,y,a) + d'(x,a,z) + d'(a,y,z)$. Also, other situations hold in similar way. So, d' is a generalized metric on IR. It is also called generalized discrete metric on IR.

In 1984, Dhage [4] defined D-metric space as a generalization of metric space modifying ($d'2$) in Definition 2.1 as follows:

Definition 2.2. A pair (X,D) is called a generalized metric space or shortly D-metric space if X is an arbitrary set and $D: X\times X\times X \to [0,\infty)$ a function such that, for all $x,y,z,a \in X$

(D1) $D(x,y,z) \geq 0$,

(D2) $D(x,y,z) = 0$ iff $x = y = z$,

(D3) $D(x,y,z) = D(p(x,y,z))$ where p is a permutation function,

(D4) $D(x,y,z) \leq D(x,y,a) + D(x,a,z) + D(a,y,z)$.

D is also called generalized metric on X.

Example 2.2. Define D:IR×IR×IR $\to [0,\infty)$ by $D(x,y,z) = \begin{cases} 0, & x=y=z, \\ 1, & other. \end{cases}$ Then D is a generalized metric on IR.

D1-2-3) are obvious. D4) For every $x,y,z,a\in$IR, if x=y, y=z, z=a then it holds since,

$D(x,y,z) = 0 = D(x,y,a) + D(x,a,z) + D(a,y,z)$. Also, other situations hold in similar way. So, D is a generalized metric on IR. It is also called generalized discrete metric on IR.

Example 2.3. Let (X,d) be a metric space. Define $D:X\times X\times X \to [0,\infty)$ by $D(x,y,z) = d(x,y) + d(y,z) + d(z,x)$. Then D is a generalized metric on X.

D1-2-3) are obvious. D4) For every $x,y,z,a\in X$ it holds since,

$$\begin{aligned} D(x,y,z) &= d(x,y) + d(y,z) + d(z,x) \\ &\leq d(x,a) + d(a,y) + d(y,a) + d(a,z) + d(z,a) + d(a,x) + d(x,x) \\ &\leq d(x,a) + d(a,y) + d(y,a) + d(a,z) + d(z,a) + d(a,x) + d(x,z) + d(z,x) \\ &\leq d(x,a) + d(a,y) + d(y,a) + d(a,z) + d(z,a) + d(a,x) + d(x,z) + d(x,y) + d(y,z) + d(z,x) \\ &\leq d(y,x) + d(y,a) + d(a,x) + d(x,a) + d(a,z) + d(x,z) + d(a,y) + d(y,z) + d(z,a) \\ &= D(x,y,a) + D(x,a,z) + D(a,y,z). \end{aligned}$$

So, D is a generalized metric on X. It is said to be also generalized standard metric on X. Hence, every metric space is a D-metric space but converse is only true under following certain conditions:

Theorem 2.1. Let (X,D) be a D-metric space. If, for every $x,y,z \in X$,

(i) $D(x,y,y) = D(y,x,x)$,

(ii) $D(x,y,y) \leq D(x,z,z) + D(z,y,y)$

then a function d defined by $d(x,y) = D(x,y,y)$ is a metric on X.

Proof. M1-2) are obvious.

M3) Using (i), $d(x,y) = D(x,y,y) = D(y,x,x) = d(y,x)$,

M4) Using (ii), $d(x,y) = D(x,y,y) \leq D(x,z,z) + D(z,y,y) = d(x,z) + d(z,y)$.

This completes the proof.

In 2003, Mustafa and Sims [5,6] defined G-metric space as a generalization of metric space modifying Definition 2.1 as follows:

Definition 2.3. A pair (X,G) is called a generalized metric space or shortly G-metric space if X is an arbitrary set and $G: X\times X\times X \to [0,\infty)$ a function such that, for all $x,y,z,a \in X$

(G1) $G(x,y,z) = 0$ iff $x = y = z$,

(G2) If $x \neq y$ then $G(x,y,y) > 0$,

(G3) If $z \neq y$ then $G(x,x,y) \leq G(x,y,z)$,

(G4) $G(x,y,z) = G(p(x,y,z))$ where p is a permutation function,

(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$.

Example 2.4. Let X = {a,b} and define $G:X\times X\times X \to [0,\infty)$ by $G(a,a,a) = G(b,b,b) = 0$, $G(a,a,b) = G(a,b,a) = G(b,a,a) = 1$, $G(a,b,b) = G(b,a,b) = G(b,b,a) = 2$. Then G is a generalized metric on X.

G1-2-3-4) are obvious. G5) For every $x,y,z,a' \in X$ it holds since, take x = a, y = a and z = b,

if $a' = a$, then $1 = G(a,a,b) \leq G(a,a,a) + G(a,a,b) = 1$,

if $a' = b$, then $1 = G(a,a,b) \leq G(a,b,b) + G(b,a,b) = 4$.

Also, other situations hold in similar way. So, G is a generalized metric on X.

Example 2.5. Let (X,d) be a metric space. Define $G:X\times X\times X \to [0,\infty)$ by $D(x,y,z) = d(x,y) + d(y,z) + d(z,x)$. Then G is a generalized metric on X.

G1-2-3-4) are obvious. G5) For every $x,y,z,a\in X$ it holds since,

$$\begin{aligned} G(x,y,z) &= d(x,y) + d(y,z) + d(z,x) \\ &\leq d(x,a) + d(a,y) + d(y,z) + d(z,a) + d(a,x) + d(a,a) \\ &= d(x,a) + d(a,a) + d(a,x) + d(a,y) + d(y,z) + d(z,a) \\ &= G(x,a,a) + D(a,y,z). \end{aligned}$$

So, G is a generalized metric on X. It is said to be also generalized standard metric on X. Hence, every metric space is a G-metric space but converse is only true under following certain condition:

Theorem 2.2. Let (X,G) be a G-metric space. A function d defined by $d(x,y) = G(x,y,y) + G(x,x,y)$ is a metric on X.

Proof. M1-2) are obvious.

M3) $d(x,y) = G(x,y,y) + G(x,x,y) = G(y,x,x) + G(y,y,x) = d(y,x)$,

$$\begin{aligned} \text{M4) } d(x,y) &= G(x,y,y) + G(x,x,y) \\ &= G(x,y,y) + G(y,x,x) \\ &\leq G(x,z,z) + G(z,y,y) + G(y,z,z) + G(z,x,x) \\ &= G(x,z,z) + G(x,x,z) + G(z,y,y) + G(z,z,y) \\ &= d(x,z) + d(z,y). \end{aligned}$$

This completes the proof.

Remark 2.1. Every d'-metric space not need to be a G-metric space. One can easily deduce that d'-metric space given in Example 2.1 is not a G-metric space since (G2) is not satisfied.

Remark 2.2. Every G-metric space not need to be a d'-metric space. One can easily deduce that G-metric space given in Example 2.4 is not a d'-metric space since (d'2) is not satisfied.

Definition 2.4. (X,G) is said to be a symmetric G-metric space if G(x,y,y) = G(x,x,y) for all $x,y \in X$.

Remark 2.3. Definition 2.4 is a fundamental condition in proofs of fixed point theorems. Also, every symmetric G-metric space is G-metric space but converse is not true. One can easily deduce that G-metric space given in Example 2.4 is not a symmetric G-metric space since 2= G(a,b,b) ≠ G(a,a,b) = 1.

3. OPEN PROBLEMS

Problem 3.1. In [8], Vasuki and Veeramani defined fuzzification of metric spaces. What are fuzzifications of generalized d'-metric, D-metric and G-metric spaces.

Problem 3.2. In [7], Park defined intuitionistic fuzzification of metric spaces. What are intuitionistic fuzzifications of generalized d'-metric, D-metric and G-metric spaces.

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