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RESEARCH ARTICLE

ON GENERALIZED METRIC SPACES

SERVET KUTUKCU[1*], HANDE POSUL[2]

[1, 2]Department of Mathematics, Ondokuz Mayis University 55139 Kurupelit, Samsun, Turkey

*E-mail: skutukcu@omu.edu.tr

ABSTRACT

In this work, we point out the relationships between generalized metric spaces d’, D, G and generalized metric spaces D*, S defined in literature illustrating with examples.

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1. INTRODUCTION

The notion of metric spaces is used as a basic concept in all other scientific fields as in Mathematics. During last 50 years, a generalization of metric spaces and to give a proper definition have become one of the major problem of topology. For this purpose, we give notions of D^*-metric and S-metric spaces as a generalization of metric spaces, and investigate some relationships between metric spaces and generalized metric spaces d', D, G. Using these relations, one can easily generalize some known results in metric spaces to generalized metric spaces.

Let us remind that a metric space is a set X together with a function d (called a metric or distance function) which assigns a positive real number d(x,y) to every pair $x,y \in X$ satisfying the axioms:

(M1) $d(x,y) \geq 0$,

(M2) $d(x,y) = 0$ iff $x = y$,

(M3) $d(x,y) = d(y,x)$,

(M4) $d(x,y) \leq d(x,z) + d(z,y)$.

As a basic example, take X to be all real numbers IR and define $d(x,y) = |x-y|$ for all $x,y \in$ IR. Then, the function d defines a metric on IR, namely usual metric.

Definition 1.1 [1]. A pair (X,d') is called a generalized metric space or shortly d'-metric space if X is an arbitrary set and $d' : X \times X \times X \to [0,\infty)$ a function such that

($d'1$) For every pair $x,y \in X$, there exists a $z \in X$ such that $d'(x,y,z) \neq 0$,

($d'2$) Two of x,y and z are same then $d'(x,y,z) = 0$,

($d'3$) $d'(x,y,z) = d'(p(x,y,z))$ where p is a permutation function,

($d'4$) $d'(x,y,z) \leq d'(x,y,a) + d'(x,a,z) + d'(a,y,z)$.

d' is also called generalized metric on X.

Definition 1.2 [2]. A pair (X,D) is called a generalized metric space or shortly D-metric space if X is an arbitrary set and D: $X\times X\times X \to [0,\infty)$ a function such that, for all $x,y,z,a \in X$

(D1) $D(x,y,z) \geq 0$,

(D2) $D(x,y,z) = 0$ iff $x = y = z$,

(D3) $D(x,y,z) = D(p(x,y,z))$ where p is a permutation function,

(D4) $D(x,y,z) \leq D(x,y,a) + D(x,a,z) + D(a,y,z)$.

D is also called generalized metric on X.

Definition 1.3 [3,4]. A pair (X,G) is called a generalized metric space or shortly G-metric space if X is an arbitrary set and G: $X\times X\times X \to [0,\infty)$ a function such that, for all $x,y,z,a \in X$

(G1) $G(x,y,z) = 0$ iff $x = y = z$,

(G2) If $x \neq y$ then $G(x,y,y) > 0$,

(G3) If $z \neq y$ then $G(x,x,y) \leq G(x,y,z)$,

(G4) $G(x,y,z) = G(p(x,y,z))$ where p is a permutation function,

(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$.

2. GENERALIZED METRIC SPACES

In 2007, Sedgi et. all [6] defined D^*-metric space as a generalization of metric space as follows:

Definition 2.1. A pair (X,D^*) is called a generalized metric space or shortly D^*-metric space if X is an arbitrary set and $D^*: X\times X\times X \to [0,\infty)$ a function such that, for all $x,y,z,a \in X$

(D^*1) $D^*(x,y,z) \geq 0$,

(D^*2) $D^*(x,y,z) = 0$ iff $x = y = z$,

(D^*3) $D^*(x,y,z) = D^*(p(x,y,z))$ where p is a permutation function,

(D^*4) $D^*(x,y,z) \leq D^*(x,y,a) + D^*(a,z,z)$.

D^* is also called generalized metric on X.

Example 2.1. Define $D^*: IR^+\times IR^+\times IR^+ \to [0,\infty)$ by $D^*(x,y,z) = \begin{cases} 0, & x = y = z, \\ \max\{x,y,z\}, & other. \end{cases}$ Then D^* is a generalized metric on IR^+.

D^*1-2-3) are obvious. D^*4) For every $x,y,z,a \in IR^+$, if $x = y = z = a$ then it holds since,
$D^*(x,y,z) = 0 = D^*(x,y,a) + D^*(a,z,z)$. Also, other situations hold in similar way. So, D^* is a generalized metric on IR^+.

Example 2.2. Let (IR,d) be a usual metric space. Define $D^*: IR\times IR\times IR \to [0,\infty)$ by $D^*(x,y,z) = |x+y-2z| + |x+z-2y| + |y+z-2x|$. Then D^* is a generalized metric on IR.

D^*1-2-3) are obvious. D^*4) For every $x,y,z,a \in IR$ it holds since,

$$\begin{aligned} D^*(x,y,z) &= |x+y-2z| + |x+z-2y| + |y+z-2x| \\ &= |x+y-2z+2a-2a| + |x+z-2y+a-a| + |y+z-2x+a-a| \\ &\leq |x+y-2a| + |2a-2z| + |x+a-2y| + |z-a| + |a+y-2x| + |z-a| \\ &\leq D^*(x,y,a) + D^*(a,z,z). \end{aligned}$$

So, D^* is a generalized metric on IR.

Example 2.3. Let (X,d) be a metric space. Define $D^*: X\times X\times X \to [0,\infty)$ by $D^*(x,y,z) = d(x,y) + d(y,z) + d(z,x)$. Then D^* is a generalized metric on X.

D^*1-2-3) are obvious. D^*4) For every $x,y,z,a \in X$ it holds since,

$D^*(x,y,z) = d(x,y) + d(y,z) + d(z,x)$

≤ d(x,y) + d(y,a) + d(a,z) + d(z,a) + d(a,x) + d(z,z)
= d(x,y) + d(y,a) + d(a,x) + d(a,z) + d(z,z) + d(z,a)
= D^*(x,y,a) + D^*(a,z,z).

So, D^* is a generalized metric on X. It is said to be also generalized standard metric on X. Hence, every metric space is a D^*-metric space but converse is only true under following certain condition:

Theorem 2.1. Let (X,D^*) be a D^*-metric space. A function d defined by d(x,y) = D^*(x,y,y) + D^*(x,x,y) is a metric on X.

proof. M1-2) are obvious.

M3) d(x,y) = D^*(x,y,y) + D^*(x,x,y) = D^*(x,x,y) + D^*(x,y,y) = D^*(y,x,x) + D^*(y,y,x) = d(y,x),

M4) d(x,y) = D^*(x,y,y) + D^*(x,x,y) = D^*(y,y,x) + D^*(x,x,y)
≤ D^*(y,y,z) + D^*(z,x,x) + D^*(x,x,z) + D^*(z,y,y)
= D^*(y,y,z) + D^*(z,z,x) + D^*(x,x,z) + D^*(z,z,y)
= D^*(x,z,z) + D^*(x,x,z) + D^*(y,z,z) + D^*(y,y,z)
= d(x,z) + d(z,y).

This completes the proof.

Remark 2.1. Every (X,D^*) is a symmetric, i.e., D^*(x,y,y) = D^*(x,x,y) for all x,y∈X. It is a fundamental condition in proofs of fixed point theorems.

Remark 2.2. (X,d') generalized metric space does not need to be (X,D^*) generalized metric space. Define d':IR×IR×IR → [0,∞) by $d'(x,y,z)=\begin{cases}1, & x\neq y, y\neq z, x\neq z\\ 0, & other.\end{cases}$ Then d' is a generalized metric on IR.

d'1-2-3) are obvious. d'4) For every x,y,z,a∈IR, if x≠y, y≠z, z≠a then it holds since,

d'(x,y,z) = 1 < 3 = d'(x,y,a) + d'(x,a,z) + d'(a,y,z). Also, other situations hold in similar way. So, d' is a generalized metric on IR. But d' is not D^*-metric, if we take $x\neq y, y\neq z, x\neq z, a=y$ then 1 = d'(x,y,z) ≤ d'(x,y,a) + d'(a,z,z) = 0 and hence (D^*4) is not satisfied.

Remark 2.3. (X,D^*) generalized metric space does not need to be (X,d') generalized metric space. In Example 2.1, D^* is not d'-metric, if we take $x=y<z$ then D^*(z,y,z) = z and hence (d'2) is not satisfied.

Remark 2.4. (X,G) generalized metric space does not need to be (X,D^*) generalized metric space. Let X = {a,b} and define G:X×X×X → [0,∞) by G(a,a,a) = G(b,b,b) = 0, G(a,a,b) = G(a,b,a) = G(b,a,a) = 1, G(a,b,b) = G(b,a,b) = G(b,b,a) = 2. Then G is a generalized metric on X.

G1-2-3-4) are obvious. G5) For every x,y,z,a'∈X it holds since, take x = a, y = a and z = b,

if a' = a, then 1 = G(a,a,b) ≤ G(a,a,a) + G(a,a,b) = 1,

if a' = b, then 1 = G(a,a,b) ≤ G(a,b,b) + G(b,a,b) = 4.

Also, other situations hold in similar way. So, G is a generalized metric on X. But G is not D^*-metric, if we take $x=b, y=b, z=a,\ a'=b$ then 2 = G(b,b,a) ≤ G(b,b,b) + G(b,a,a) = 1 and hence (D^*4) is not satisfied.

Remark 2.5. (X,D^*) generalized metric space does not need to be (X,G) generalized metric space. In Example 2.2, D^* is not G-metric, if we take $x=5, y=-5, z=0$ then D^*(x,y,z) = 30, D^*(x,x,y) = 40 and hence (G3) is not satisfied.

In 2012, Sedgi et. all [7] defined S-metric space as a generalization of metric space as follows:

Definition 2.2. A pair (X,S) is called a generalized metric space or shortly S-metric space if X is an arbitrary set and S: X×X×X → [0,∞) a function such that, for all x,y,z,a ∈ X

(S1) S(x,y,z) ≥ 0,

(S2) S(x,y,z) = 0 iff x = y = z,

(S4) S(x,y,z) ≤ S(x,x,a) + S(y,y,a) + S(z,z,a).

S is also called generalized metric on X.

Example 2.4. Let (IR,d) be a usual metric space. Define S:IR×IR×IR → [0,∞) by S(x,y,z) = ly+z-2xl + ly-zl. Then S is a generalized metric on IR.

S1-2) are obvious. S3) For every x,y,z,a∈IR it holds since,

S(x,y,z) = ly+z-2xl + ly-zl
 = ly+z-2x+2a-2al + ly-z+a-al
 ≤ l2a-2xl + ly+z-2al + ly-al + la-zl
 ≤ la-xl + la-xl + ly-al + lz-al + la-yl + la-zl
 ≤ lx+a-2xl + lx-al + la+y-2yl + ly-al + la+z-2zl + lz-al
 = S(x,x,a) + S(y,y,a) + S(z,z,a).

So, S is a generalized metric on IR.

Example 2.5. Let (X,d) be a metric space. Define S:X×X×X → [0,∞) by S(x,y,z) = d(x,y) + d(y,z) + d(z,x). Then S is a generalized metric on X.

S1-2) are obvious. S3) For every x,y,z,a∈X it holds since,

S(x,y,z) = d(x,y) + d(y,z) + d(z,x)
 ≤ d(x,a) + d(a,y) + d(y,a) + d(a,z) + d(z,a) + d(a,x)
 ≤ d(x,a) + d(a,y) + d(y,a) + d(a,z) + d(z,a) + d(a,x) + d(x,x) + d(y,y) + d(z,z)
 = S(x,x,a) + S(y,y,a) + S(z,z,a).

So, S is a generalized metric on X. It is said to be also generalized standard metric on X. Hence, every metric space is a S-metric space.

Theorem 2.2. Every (X,S) is a symmetric, i.e., S(x,x,y) = S(y,y,x) for all x,y∈X.

proof. Let (X,S) be a generalized metric space. Then, for all x,y∈X

S(x,x,y) ≤ S(x,x,x) + S(x,x,x) + S(y,y,x) = S(y,y,x),

S(y,y,x) ≤ S(y,y,y) + S(y,y,y) + S(x,x,y) = S(x,x,y).

This completes the proof.

Remark 2.6. (X,S) generalized metric space does not need to be (X,d$'$) generalized metric space. In Example 2.4, S is not d$'$-metric, if we take $x=1, y=2, z=3$ then then S(x,y,z) = 4, S(y,z,x) = 2 and hence (d$'$3) is not satisfied.

Remark 2.7. (X,S) generalized metric space does not need to be (X,D) generalized metric space. In Example 2.2, S is not D-metric, if we take $x=1, y=2, z=3$ then then S(x,y,z) = 4, S(y,z,x) = 2 and hence (D3) is not satisfied.

Theorem 2.3. Every (X,D*) generalized metric space is a S-metric space.

proof. Let (X,D*) be a generalized metric space. S1-2) is obvious. S3) For every x,y,z,a∈X it holds since,

D*(x,y,z) ≤ D*(x,y,a) + D*(a,z,z)
 = D*(a,x,y) + D*(a,z,z)
 ≤ D*(a,x,a) + D*(a,y,y) + D*(a,z,z)
 ≤ D*(x,x,a) + D*(y,y,a) + D*(z,z,a).

This completes the proof. But the converse is not true:

Example 2.6. In Example 2.4, if we take $x=1, y=2, z=3$ then then S(x,y,z) = 4, S(y,z,x) = 2 and hence (D*3) is not satisfied.

Remark 2.8. (X,G) generalized metric space does not need to be (X,S) generalized metric space. Let X = {a,b} and define G:X×X×X → [0,∞) by G(a,a,a) = G(b,b,b) = 0, G(a,a,b) = G(a,b,a) = G(b,a,a) = 1, G(a,b,b) = G(b,a,b) = G(b,b,a) = 2. Then, from Remark 2.4, G is a generalized metric on X. But G is not S-metric, if we take $x=a, y=b, z=b,\ a'=b$ then 2 = G(a,b,b) ≤ G(a,a,b) + G(b,b,b) + G(b,b,b) = 1 and hence (S3) is not satisfied.

Remark 2.9. (X,S) generalized metric space does not need to be (X,G) generalized metric space. In Example 2.4, S is not G-metric, if we take $x=2, y=0, z=1$ then then S(x,y,z) = 4, S(y,x,z) = 2 and hence (G4) is not satisfied.

3. OPEN PROBLEMS

Problem 3.1. In [8], Vasuki and Veeramani defined fuzzification of metric spaces. What are fuzzifications of generalized D^*-metric, S-metric and A-metric spaces.

Problem 3.2. In [5], Park defined intuitionistic fuzzification of metric spaces. What are intuitionistic fuzzifications of generalized D^*-metric, S-metric and A-metric spaces.

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