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RESEARCH ARTICLE

OPTIMAL CONVEX COMBINATION BOUNDS OF HARMONIC AND CONTRA-HARMONIC MEANS FOR NEUMAN-SÁNDOR MEAN

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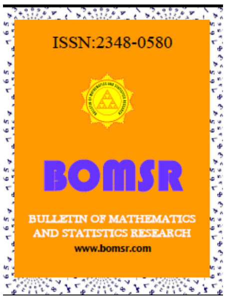

ABSTRACT

In this paper, we present the least value α and the greatest value β such that the double inequality

$\alpha H(a,b)+(1-\alpha)C(a,b)<M(a,b)<\beta H(a,b)+(1-\beta)C(a,b)$

holds for all $a,b>0$ with $a\neq b$. Here $H(a,b)$, $C(a,b)$ and $M(a,b)$ denote the harmonic, contra-harmonic and Neuman-Sándor means of two positive numbers a and b, respectively.

Keywords: Inequality, Neuman-Sándor mean, Harmonic mean, Contra-Harmonic mean

1. INTRODUCTION

For $a,b>0$ with $a\neq b$, the Neuman-Sándor mean $M(a,b)$ [1] was defined by

$$M(a,b)=\frac{a-b}{2\sinh^{-1}[(a-b)/(a+b)]}, \tag{1.1}$$

where $\sinh^{-1}x=\log(x+\sqrt{1+x^2})$ is the inverse hyperbolic sine function.

Recently, the Neuman-Sándor mean has been the subject of intensive research. In particular, many remarkable inequalities for $M(a,b)$ can be found in the literature [1, 2]. Let $H(a,b)=(2ab)/(a+b)$,

$G(a,b)=\sqrt{ab}$,

$L(a,b)=(a-b)/(\log a-\log b)$, $P(a,b)=(a-b)\big/(4\tan^{-1}\sqrt{a/b}-\pi)$, $A(a,b)=(a+b)/2$,

$T(a,b)=(a-b)/[2\tan^{-1}(a-b)/(a+b)]$, $Q(a,b)=\sqrt{(a^2+b^2)/2}$ and $C(a,b)=(a^2+b^2)\big/(a+b)$

be the harmonic, geometric, logarithmic,, first Seiffert, arithmetic, second Seiffert, quadratic and contra-harmonic mean of a and b, respectively. Then

$$\min\{a,b\}<H(a,b)<G(a,b)<L(a,b)<P(a,b)<A(a,b)<M(a,b)<T(a,b)<Q(a,b)<C(a,b)<\max(a,b) \tag{1.2}$$

hold for all $a,b>0$ with $a\neq b$.

In [3], Neuman proved that the double inequalities

$$\alpha Q(a,b)+(1-\alpha)A(a,b)<M(a,b)<\beta Q(a,b)+(1-\beta)A(a,b) \quad (1.3)$$

and

$$\lambda C(a,b)+(1-\lambda)A(a,b)<M(a,b)<\mu C(a,b)+(1-\mu)A(a,b) \quad (1.4)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leq[1-\log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})]=0.3249\cdots$, $\beta\geq 1/3$,
$\lambda\leq[1-\log(1+\sqrt{2})]/\log(1+\sqrt{2})$ and $\mu\geq 1/6$.

In [4], Li etc showed that the double inequality

$$L_{p_0}(a,b)<M(a,b)<L_2(a,b) \quad (1.5)$$

holds for all $a,b>0$ with $a\neq b$, where $L_p(a,b)=[(a^{p+1}-b^{p+1})/(p+1)(a-b)]^{1/p}$ $(p\neq -1,0)$, $L_0(a,b)=1/e[(a^a)/b^b]^{1/(a-b)}$ and $L_{-1}(a,b)=(a-b)/(\log a-\log b)$ is the p-th generalized logarithmic mean of a and b, and $p_0=1.843\cdots$ is the unique solution of the equation $(p+1)^{1/p}=\log(1+\sqrt{2})$.

In [5], Chu etc proved that the double inequalities

$$\alpha_1 L(a,b)+(1-\alpha_1)Q(a,b)<M(a,b)<\beta_1 L(a,b)+(1-\beta_1)Q(a,b) \quad (1.6)$$

and

$$\alpha_2 L(a,b)+(1-\alpha_2)C(a,b)<M(a,b)<\beta_2 L(a,b)+(1-\beta_2)C(a,b) \quad (1.7)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\geq 2/5$, $\beta_1\leq 1-1/[\sqrt{2}\log(1+\sqrt{2})]=0.1977\cdots$, $\alpha_2\geq 5/8$ and $\beta_2\leq 1-1/[2\log(1+\sqrt{2})]=0.4327\cdots$.

The main purpose of this paper is to found the least value α and the greatest value β such that the double inequality

$$\alpha H(a,b)+(1-\alpha)C(a,b)<M(a,b)<\beta H(a,b)+(1-\beta)C(a,b)$$

holds for all $a,b>0$ with $a\neq b$. All numerical computations are carried out using MATHEMATICAL software.

2. MAIN RESULTS

THEOREM 2.1. The double inequality

$$\alpha H(a,b)+(1-\alpha)C(a,b)<M(a,b)<\beta H(a,b)+(1-\beta)C(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha\geq 1-1/[2\log(1+\sqrt{2})]=0.4327\cdots$ and $\beta\leq 5/12$.

Proof. Without loss of generality, we assume that $a>b>0$.

Let $x=(a-b)/(a+b)\in(0,1)$, and $\lambda=1-1/[2\log(1+\sqrt{2})]$. Then

$$\frac{H(a,b)}{A(a,b)}=1-x^2,\ \frac{M(a,b)}{A(a,b)}=\frac{x}{\sinh^{-1}x},\ \frac{C(a,b)}{A(a,b)}=1+x^2. \quad (2.1)$$

Firstly, we prove that

$$\frac{5}{12}H(a,b)+\frac{7}{12}C(a,b)>M(a,b), \quad (2.2)$$

Equation (2.1) lead to

$$\frac{\frac{5}{12}H(a,b)+\frac{7}{12}C(a,b)-M(a,b)}{A(a,b)}=\frac{x^2+6}{6\log\left(x+\sqrt{1+x^2}\right)}d(x), \quad (2.3)$$

where

$$d(x)=\log\left(x+\sqrt{1+x^2}\right)-\frac{6x}{x^2+6}. \quad (2.4)$$

Simple computations yield

$$\lim_{x\to 0^+} d(x) = 0, \tag{2.5}$$

and

$$d'(x) = \frac{1}{\sqrt{1+x^2}} - \frac{6(6-x^2)}{(x^2+6)^2}. \tag{2.6}$$

Because $(x^2+6)^4 - [6(6-x^2)\sqrt{1+x^2}]^2 = x^4[(x^2-6)^2+576] > 0$, so

$$d'(x) > 0. \tag{2.7}$$

Therefore the inequality (2.2) follows from (2.3) and (2.5) together with (2.7).

Secondly, we prove that

$$\lambda H(a,b) + (1-\lambda)C(a,b) < M(a,b). \tag{2.8}$$

Equation (2.1) lead to

$$\frac{\lambda H(a,b) + (1-\lambda)Q(a,b) - M(a,b)}{A(a,b)} = 1 + (1-2\lambda)x^2 - \frac{x}{\log\left(x+\sqrt{1+x^2}\right)}$$

$$= \frac{1+(1-2\lambda)x^2}{\log\left(x+\sqrt{1+x^2}\right)} D(x), \tag{2.9}$$

where

$$D(x) = \log\left(x+\sqrt{1+x^2}\right) - \frac{x}{1+(1-2\lambda)x^2}. \tag{2.10}$$

Simple computations yield

$$\lim_{x\to 0^+} D(x) = 0, \tag{2.11}$$

$$\lim_{x\to 1^-} D(x) = \log(1+\sqrt{2}) - \frac{1}{2(1-\lambda)} = 0, \tag{2.12}$$

and

$$D'(x) = \frac{1}{\sqrt{1+x^2}} - \frac{1-(1-2\lambda)x^2}{[1+(1-2\lambda)x^2]^2}$$

$$= \frac{1}{\frac{1}{\sqrt{1+x^2}} + \frac{1-(1-2\lambda)x^2}{[1+(1-2\lambda)x^2]^2}} \left\{ \frac{1}{1+x^2} - \frac{[1-(1-2\lambda)x^2]^2}{[1+(1-2\lambda)x^2]^4} \right\}$$

$$= \frac{x^2 F(x)}{\left\{ \frac{1}{\sqrt{1+x^2}} + \frac{1-(1-2\lambda)x^2}{[1+(1-2\lambda)x^2]^2} \right\}(1+x^2)[1+(1-2\lambda)x^2]^4}, \tag{2.13}$$

where

$$F(x) = (16\lambda^4 - 32\lambda^3 + 24\lambda^2 - 8\lambda + 1)x^6 - (32\lambda^3 - 44\lambda^2 + 20\lambda - 3)x^4$$
$$+ (20\lambda^2 - 24\lambda + 7)x^2 - 12\lambda + 5. \tag{2.14}$$

Let $x^2 = t, t \in (0,1)$. Then

$$F(x) = (16\lambda^4 - 32\lambda^3 + 24\lambda^2 - 8\lambda + 1)t^3 - (32\lambda^3 - 44\lambda^2 + 20\lambda - 3)t^2 + (20\lambda^2 - 24\lambda + 7)t - 12\lambda + 5$$
$$= G(t). \tag{2.15}$$

Simple calculations induce

$$\lim_{t\to 0^+} G(t) = 5 - 12\lambda < 0, \tag{2.16}$$

$$\lim_{t\to 1^-} G(t) = 8(2\lambda^4 - 8\lambda^3 + 11\lambda^2 - 8\lambda + 2) = 0.1592\cdots > 0, \tag{2.17}$$

$$G'(t) = 3(16\lambda^4 - 32\lambda^3 + 24\lambda^2 - 8\lambda + 1)t^2 - 2(32\lambda^3 - 44\lambda^2 + 20\lambda - 3)t + (20\lambda^2 - 24\lambda + 7), \tag{2.18}$$

$$\lim_{t\to 1^-} G'(t) = 4(12\lambda^4 - 40\lambda^3 + 45\lambda^2 - 22\lambda + 4) = 0.3440\cdots > 0, \tag{2.19}$$

$$G''(t) = 2[3(16\lambda^4 - 32\lambda^3 + 24\lambda^2 - 8\lambda + 1)t - (32\lambda^3 - 44\lambda^2 + 20\lambda - 3)], \tag{2.20}$$

$$\lim_{t\to 1^-} G''(t) = 4(24\lambda^4 - 64\lambda^3 + 58\lambda^2 - 22\lambda + 3) = -0.0147\cdots < 0, \tag{2.21}$$

and

$$G'''(t) = 6(16\lambda^4 - 32\lambda^3 + 24\lambda^2 - 8\lambda + 1) = 0.0019\cdots > 0. \tag{2.22}$$

From (2.22) we clearly see $G''(t)$ is strictly increasing in $(0,1)$. (2.21) and the monotonicity of $G''(t)$ imply $G''(t) < 0$ for $t \in (0,1)$, hence $G'(t)$ is strictly decreasing in $(0,1)$. (2.19) and the monotonicity of $G'(t)$ show $G'(t) > 0$ for $t \in (0,1)$, hence $G(t)$ is strictly increasing in $(0,1)$. It follows from (2.16) and (2.17) together with the monotonicity of $G(t)$ that there exists $t_0 \in (0,1)$ such that $G(t) < 0$ for $t \in (0,t_0)$ and $G(t) > 0$ for $t \in (t_0,1)$. From (2.13) and (2.15) together with the sighs of $G(t)$ we affirm $D'(x) < 0$ for $x \in (0,\sqrt{t_0})$ and $D'(x) > 0$ for $x \in (\sqrt{t_0},1)$, thus $D(x)$ is strictly decreasing in $(0,\sqrt{t_0})$ and strictly increasing in $(\sqrt{t_0},1)$. It follows from (2.11) and (2.12) together with the monotonicity of $D(x)$ that

$$D(x) < 0. \tag{2.23}$$

Therefore the inequality (2.8) follows from (2.9) and (2.23).

Finally we prove that $\lambda H(a,b) + (1-\lambda)C(a,b)$ is the best possible lower convex combination bound and $5/12\,H(a,b) + 7/12\,C(a,b)$ is the best possible upper convex combination bound of the harmonic and the contra-harmonic means for the Nueman-Sándor mean.

Equation (2.1) lead to

$$\frac{C(a,b) - M(a,b)}{C(a,b) - H(a,b)} = \frac{1 + x^2 - \dfrac{x}{\sinh^{-1} x}}{1 + x^2 - (1 - x^2)} = B(x). \tag{2.24}$$

From (2.24) one has

$$\lim_{x\to 1^-} B(x) = \lambda, \tag{2.25}$$

and

$$\lim_{x\to 0^+} B(x) = \frac{5}{12}. \tag{2.26}$$

If $\alpha < \lambda$, then (2.24) and (2.25) lead to the conclusion that there exists $0 < \delta_1 < 1$ such that $\alpha H(a,b) + (1-\alpha)C(a,b) > M(a,b)$ for all $a,b > 0$ with $(a-b)/(a+b) \in (1-\delta_1,1)$.

If $\beta > 5/12$, then (2.24) and (2.26) lead to the conclusion that there exists $0 < \delta_2 < 1$ such that $M(a,b) > \beta H(a,b) + (1-\beta)Q(a,b)$ for all $a,b > 0$ with $(a-b)/(a+b) \in (0,\delta_2)$.

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