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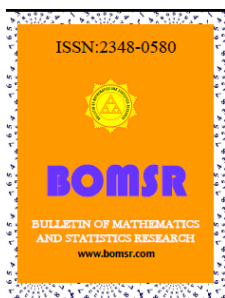


OPTIMAL CONVEX COMBINATION BOUNDS OF HARMONIC AND CONTRA-HARMONIC MEANS FOR NEUMAN-SÁNDOR MEAN

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ABSTRACT

In this paper, we present the least value a and the greatest value b such that the double inequality

$$aH(a,b) + (1-a)C(a,b) < M(a,b) < bH(a,b) + (1-b)C(a,b)$$

holds for all $a, b > 0$ with $a^1 b$. Here $H(a,b)$, $C(a,b)$ and $M(a,b)$ denote the harmonic, contra-harmonic and Neuman-Sándor means of two positive numbers a and b , respectively.

Keywords: Inequality, Neuman-Sándor mean, Harmonic mean, Contra-Harmonic mean

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1. INTRODUCTION

For $a, b > 0$ with $a^1 b$, the Neuman-Sándor mean $M(a,b)$ [1] was defined by

$$M(a,b) = \frac{a-b}{2 \sinh^{-1} [(a-b)/(a+b)]}, \tag{1.1}$$

where $\sinh^{-1} x = \log(x + \sqrt{1+x^2})$ is the inverse hyperbolic sine function.

Recently, the Neuman-Sándor mean has been the subject of intensive research. In particular, many remarkable inequalities for $M(a,b)$ can be found in the literature [1, 2]. Let $H(a,b) = (2ab)/(a+b)$,

$$G(a,b) = \sqrt{ab},$$

$$L(a,b) = (a-b)/(\log a - \log b), P(a,b) = (a-b)/(4 \tan^{-1} \sqrt{a/b} - \pi), A(a,b) = (a+b)/2,$$

$$T(a,b) = (a-b)/[2 \tan^{-1} (a-b)/(a+b)], Q(a,b) = \sqrt{(a^2 + b^2)/2} \text{ and } C(a,b) = (a^2 + b^2)/(a+b)$$

be the harmonic, geometric, logarithmic, first Seiffert, arithmetic, second Seiffert, quadratic and contra-harmonic mean of a and b , respectively. Then

$$\min\{a,b\} < H(a,b) < G(a,b) < L(a,b) < P(a,b) < A(a,b) < M(a,b) < T(a,b) < Q(a,b) < C(a,b) < \max\{a,b\} \tag{1.2}$$

hold for all $a, b > 0$ with $a^1 b$.

In [3], Neuman proved that the double inequalities

$$aQ(a,b) + (1-a)A(a,b) < M(a,b) < bQ(a,b) + (1-b)A(a,b) \tag{1.3}$$

and

$$lC(a,b) + (1-l)A(a,b) < M(a,b) < mC(a,b) + (1-m)A(a,b) \tag{1.4}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $a \in [1 - \log(1 + \sqrt{2})] / [(\sqrt{2} - 1)\log(1 + \sqrt{2})] = 0.3249L$, $b \in [1/3, 1/3]$,

$l \in [1 - \log(1 + \sqrt{2})] / \log(1 + \sqrt{2})$ and $m \in [1/6, 1/6]$.

In [4], Li etc showed that the double inequality

$$L_{p_0}(a,b) < M(a,b) < L_2(a,b) \tag{1.5}$$

holds for all $a, b > 0$ with $a \neq b$, where $L_p(a,b) = [(a^{p+1} - b^{p+1}) / (p+1)(a - b)]^{1/p}$ ($p \neq -1, 0$),

$L_0(a,b) = 1/e[(a^a)/b^b]^{1/(a-b)}$ and $L_{-1}(a,b) = (a - b) / (\log a - \log b)$ is the p -th generalized logarithmic mean of a and b , and $p_0 = 1.843L$ is the unique solution of the equation $(p+1)^{1/p} = \log(1 + \sqrt{2})$.

In [5], Chu etc proved that the double inequalities

$$a_1L(a,b) + (1-a_1)Q(a,b) < M(a,b) < b_1L(a,b) + (1-b_1)Q(a,b) \tag{1.6}$$

and

$$a_2L(a,b) + (1-a_2)C(a,b) < M(a,b) < b_2L(a,b) + (1-b_2)C(a,b) \tag{1.7}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $a_1 \in [2/5, 2/5]$, $b_1 \in [1 - 1/(\sqrt{2}\log(1 + \sqrt{2}))] = 0.1977L$, $a_2 \in [5/8, 5/8]$ and $b_2 \in [1 - 1/(2\log(1 + \sqrt{2}))] = 0.4327L$.

The main purpose of this paper is to found the least value a and the greatest value b such that the double inequality

$$aH(a,b) + (1-a)C(a,b) < M(a,b) < bH(a,b) + (1-b)C(a,b)$$

holds for all $a, b > 0$ with $a \neq b$. All numerical computations are carried out using MATHEMATICAL software.

2. MAIN RESULTS

THEOREM 2.1. The double inequality

$$aH(a,b) + (1-a)C(a,b) < M(a,b) < bH(a,b) + (1-b)C(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $a \in [1 - 1/(2\log(1 + \sqrt{2}))] = 0.4327L$ and $b \in [5/12, 5/12]$.

Proof. Without loss of generality, we assume that $a > b > 0$.

Let $x = (a - b) / (a + b) \in (0, 1)$, and $l = 1 - 1/[2\log(1 + \sqrt{2})]$. Then

$$\frac{H(a,b)}{A(a,b)} = 1 - x^2, \frac{M(a,b)}{A(a,b)} = \frac{x}{\sinh^{-1} x}, \frac{C(a,b)}{A(a,b)} = 1 + x^2. \tag{2.1}$$

Firstly, we prove that

$$\frac{5}{12}H(a,b) + \frac{7}{12}C(a,b) > M(a,b), \tag{2.2}$$

Equation (2.1) lead to

$$\frac{\frac{5}{12}H(a,b) + \frac{7}{12}C(a,b) - M(a,b)}{A(a,b)} = \frac{x^2 + 6}{6\log(x + \sqrt{1 + x^2})} d(x), \tag{2.3}$$

where

$$d(x) = \log(x + \sqrt{1 + x^2}) - \frac{6x}{x^2 + 6}. \tag{2.4}$$

Simple computations yield

$$\lim_{x \rightarrow 0^+} d(x) = 0, \tag{2.5}$$

and

$$d(x) = \frac{1}{\sqrt{1+x^2}} - \frac{6(6-x^2)}{(x^2+6)^2}. \tag{2.6}$$

Because $(x^2+6)^4 - [6(6-x^2)\sqrt{1+x^2}]^2 = x^4[(x^2-6)^2+576] > 0$, so

$$d(x) > 0. \tag{2.7}$$

Therefore the inequality (2.2) follows from (2.3) and (2.5) together with (2.7).

Secondly, we prove that

$$lH(a,b) + (1-l)C(a,b) < M(a,b). \tag{2.8}$$

Equation (2.1) lead to

$$\begin{aligned} \frac{lH(a,b) + (1-l)Q(a,b) - M(a,b)}{A(a,b)} &= 1 + (1-2l)x^2 - \frac{x}{\log(x + \sqrt{1+x^2})} \\ &= \frac{1 + (1-2l)x^2}{\log(x + \sqrt{1+x^2})} D(x), \end{aligned} \tag{2.9}$$

where

$$D(x) = \log(x + \sqrt{1+x^2}) - \frac{x}{1 + (1-2l)x^2}. \tag{2.10}$$

Simple computations yield

$$\lim_{x \rightarrow 0^+} D(x) = 0, \tag{2.11}$$

$$\lim_{x \rightarrow 1^-} D(x) = \log(1 + \sqrt{2}) - \frac{1}{2(1-l)} = 0, \tag{2.12}$$

and

$$\begin{aligned} D(x) &= \frac{1}{\sqrt{1+x^2}} - \frac{1 + (1-2l)x^2}{[1 + (1-2l)x^2]^2} \\ &= \frac{1}{\sqrt{1+x^2} + \frac{1 + (1-2l)x^2}{[1 + (1-2l)x^2]^2}} \\ &= \frac{x^2 F(x)}{\frac{1}{\sqrt{1+x^2}} + \frac{1 + (1-2l)x^2}{[1 + (1-2l)x^2]^2} (1+x^2)[1 + (1-2l)x^2]^4}, \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} F(x) &= (16l^4 - 32l^3 + 24l^2 - 8l + 1)x^6 - (32l^3 - 44l^2 + 20l - 3)x^4 \\ &\quad + (20l^2 - 24l + 7)x^2 - 12l + 5. \end{aligned} \tag{2.14}$$

Let $x^2 = t, t \in (0,1)$. Then

$$\begin{aligned} F(x) &= (16l^4 - 32l^3 + 24l^2 - 8l + 1)t^3 - (32l^3 - 44l^2 + 20l - 3)t^2 + (20l^2 - 24l + 7)t - 12l + 5 \\ &= G(t). \end{aligned} \tag{2.15}$$

Simple calculations induce

$$\lim_{t \rightarrow 0^+} G(t) = 5 - 12l < 0, \tag{2.16}$$

$$\lim_{t \rightarrow 1^-} G(t) = 8(2l^4 - 8l^3 + 11l^2 - 8l + 2) = 0.1592L > 0, \tag{2.17}$$

$$G'(t) = 3(16l^4 - 32l^3 + 24l^2 - 8l + 1)t^2 - 2(32l^3 - 44l^2 + 20l - 3)t + (20l^2 - 24l + 7), \tag{2.18}$$

$$\lim_{t \rightarrow 1^-} G(t) = 4(12l^4 - 40l^3 + 45l^2 - 22l + 4) = 0.3440L > 0, \quad (2.19)$$

$$G(t) = 2[3(16l^4 - 32l^3 + 24l^2 - 8l + 1)t - (32l^3 - 44l^2 + 20l - 3)], \quad (2.20)$$

$$\lim_{t \rightarrow 1^-} G(t) = 4(24l^4 - 64l^3 + 58l^2 - 22l + 3) = -0.0147L < 0, \quad (2.21)$$

and

$$G(t) = 6(16l^4 - 32l^3 + 24l^2 - 8l + 1) = 0.0019L > 0. \quad (2.22)$$

From (2.22) we clearly see $G(t)$ is strictly increasing in $(0,1)$. (2.21) and the monotonicity of $G(t)$ imply $G(t) < 0$ for $t \in (0,1)$, hence $G(t)$ is strictly decreasing in $(0,1)$. (2.19) and the monotonicity of $G(t)$ show $G(t) > 0$ for $t \in (0,1)$, hence $G(t)$ is strictly increasing in $(0,1)$. It follows from (2.16) and (2.17) together with the monotonicity of $G(t)$ that there exists $t_0 \in (0,1)$ such that $G(t) < 0$ for $t \in (0, t_0)$ and $G(t) > 0$ for $t \in (t_0, 1)$. From (2.13) and (2.15) together with the signs of $G(t)$ we affirm $D(x) < 0$ for $x \in (0, \sqrt{t_0})$ and $D(x) > 0$ for $x \in (\sqrt{t_0}, 1)$, thus $D(x)$ is strictly decreasing in $(0, \sqrt{t_0})$ and strictly increasing in $(\sqrt{t_0}, 1)$. It follows from (2.11) and (2.12) together with the monotonicity of $D(x)$ that

$$D(x) < 0. \quad (2.23)$$

Therefore the inequality (2.8) follows from (2.9) and (2.23).

Finally we prove that $lH(a,b) + (1-l)C(a,b)$ is the best possible lower convex combination bound and $5/12H(a,b) + 7/12C(a,b)$ is the best possible upper convex combination bound of the harmonic and the contra-harmonic means for the Neuman-Sándor mean.

Equation (2.1) lead to

$$\frac{C(a,b) - M(a,b)}{C(a,b) - H(a,b)} = \frac{1 + x^2 - \frac{x}{\sinh^{-1} x}}{1 + x^2 - (1 - x^2)} = B(x). \quad (2.24)$$

From (2.24) one has

$$\lim_{x \rightarrow 1^-} B(x) = l, \quad (2.25)$$

and

$$\lim_{x \rightarrow 0^+} B(x) = \frac{5}{12}. \quad (2.26)$$

If $a < l$, then (2.24) and (2.25) lead to the conclusion that there exists $0 < d_1 < 1$ such that $aH(a,b) + (1-a)C(a,b) > M(a,b)$ for all $a, b > 0$ with $(a-b)/(a+b) \in (1-d_1, 1)$.

If $b > 5/12$, then (2.24) and (2.26) lead to the conclusion that there exists $0 < d_2 < 1$ such that $M(a,b) > bH(a,b) + (1-b)Q(a,b)$ for all $a, b > 0$ with $(a-b)/(a+b) \in (0, d_2)$.

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